Some Problems in Vectorial Calculus of Variations in L^∞

Birzhan Ayanbayev

Department of Mathematics

University of Reading

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Birzhan Ayanbayev

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Dedication

This thesis is dedicated to:

My parents,

My wife and son.

Abstract

This thesis is a collection of published and submitted papers. Each paper is the chapter of the thesis and new approach involves proving a pointwise characterisation of the vectorial infinity Laplacian in the Chapter 2, proving a geometric feature of the p – Harmonic and ∞ – Harmonic maps in the Chapter 3, finding an explicit ∞ – Harmonic functions in the Chapter 4, proving two distinct minimality principles for a general supremal first order functionals in the Chapter 5.

In Chapter 2 we introduce the joint paper with N.Katzourakis, which extends the result of [56]. Let $n, N \in \mathbb{N}$ with $\Omega \subseteq \mathbb{R}^n$ open. Given $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn})$, we consider the functional

$$\mathrm{E}_\infty(u, \mathcal{O}) \,:=\, \underset{\mathcal{O}}{\operatorname{ess\,sup}}\, \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega. \tag{1}$$

The associated PDE system which plays the role of Euler-Lagrange equations in L^∞ is

$$\left\{ \begin{aligned} \mathrm{H}_P(\cdot, u, \mathrm{D}u)\, \mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) \,&=\, 0, \\ \mathrm{H}(\cdot, u, \mathrm{D}u)\, [\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp \Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \,&=\, 0, \end{aligned} \right. \tag{2}$$

where $[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}$ denotes the orthogonal projection onto the orthogonal complement of the range $R(A) \subseteq \mathbb{R}^N$ of a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$. Herein we establish that generalised solutions to (2) can be characterised as local minimisers of (1) for appropriate classes of affine variations of the energy. Generalised solutions to (2) are understood as $\mathcal{D}$-solutions, a general framework recently introduced by N.Katzourakis in [57, 58].

In Chapter 3 we present the joint paper with N.Katzourakis and H.Abugirda in which we consider PDE system of vanishing normal projection of the Laplacian for C^2 maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$:

$$[\![\mathrm{D}u]\!]^\perp \Delta u = 0 \quad \text{in } \Omega.$$

This system has discontinuous coefficients and geometrically expresses the fact that the Laplacian is a vector field tangential to the image of the mapping. It arises as a constituent component of the p-Laplace system for all $p \in [2, \infty]$. For $p = \infty$, the ∞-Laplace system is the archetypal equation describing extrema of supremal functionals in vectorial Calculus of Variations in L^∞. Herein we show

that the image of a solution u is piecewise affine if either the rank of $\mathrm{D}u$ is equal to one or $n = 2$ and u has additively separated form. As a consequence we obtain corresponding flatness results for p-Harmonic maps for $p \in [2, \infty]$.

The aim of the Chapter 4 is to derive new explicit solutions to the ∞-Laplace equation, the fundamental PDE arising in Calculus of Variations in the space L^∞. These solutions obey certain symmetry conditions and are derived in arbitrary dimensions, containing as particular sub-cases the already known classes of two-dimensional infinity-harmonic functions.

Chapter 5 is the joint paper with N.Katzourakis. We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞-minimal maps introduced more recently by N.Katzourakis. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞-minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

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Chapter 1

Background and motivations

1.1 Introduction

Minimization problems have been studied by many mathematician for a different purposes. Most of their efforts were focused in study the relation between minimality conditions and partial differential equations (PDEs). One of the way to view the minimality as a variational approach, which is the core idea of Calculus of Variations. We introduce some fundamental methods of Calculus of Variations to solve possibly non-linear PDE, which for a simplicity we have in the following form

$$L[u] = 0, \tag{1.1.1}$$

where $L[u]$ is a given differential operator and u is the unknown. The equation (1.1.1) can be characterised as the minimiser of appropriate energy functional $\mathrm{E}[u]$ such that

$$\mathrm{E}'[u] = L[u].$$

The usefulness of this method that now we can proof existence of extremum points of the functional energy $\mathrm{E}[\cdot]$ and consequently the solution of (1.1.1). One of difficulties of described method that in general the minimiser of the functional might not be a classical solution of the PDE and the definition of *generalised solutions* is an issue. The generalised solutions that we are using in this thesis are not based either on the viscosity solutions which are playing an enormous role in nonlinear first and second order PDEs or on the maximum principle or on integration-by-parts which helps to "pass derivatives to the test functions". Instead we utilize a recent method that uses the combination of difference quotients and Young measures, for precise definition we refer to Section 2.2 and for more information about background material of this generalized solutions we refer to [24, 37, 41, 43, 67, 71, 77].

This thesis is a collection of papers with researcher's at the University of Reading, except Chapter 4, which is a single author paper. In all papers we study some problems of *Calculus of Variations in* L^∞ which are in a nutshell a minimisation problems with respect to the supremum norm and their corresponding PDEs with the method mentioned above. Section 1.2 and 1.3 of this chapter give more details on *Calculus of Variations in* L^∞ and organization of the thesis respectively.

1.2 Calculus of Variations in $\mathbf{L}^\infty$

Calculus of Variations in L^∞ has a long history started in the 1960s by G. Aronsson [4–8]. He considered the following variational problem for the supremal functional

$$E_\infty(u,\mathcal{O}) := \operatorname*{ess\,sup}_{x\in\mathcal{O}} |\mathrm{D}u(x)|, \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}),\ \mathcal{O} \Subset \Omega \subseteq \mathbb{R}^n \tag{1.2.1}$$

and introduced appropriate L^∞- notion of minimisers, namely we say the map $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R})$ is an absolute minimiser of (1.2.1) if for all $\mathcal{O} \Subset \Omega$ and all $\phi \in W^{1,\infty}_0(\mathcal{O},\mathbb{R})$ we have

$$E_\infty(u,\mathcal{O}) \le E_\infty(u+\phi,\mathcal{O}). \tag{1.2.2}$$

Also Aronsson derived the associated PDE

$$\Delta_\infty u := \mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0 \quad \text{in } \Omega, \tag{1.2.3}$$

where "$\otimes$" is a tensor product and ":" is the Euclidean scalar product. Equation (1.2.3) is playing the role of L^∞- analogue of the $Euler - Lagrange$ equation and called ∞-*Laplacian*. In particular case when $n = 2$ and $\Omega \subset \mathbb{R}^2$ is open bounded domain, $u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ is the most well-known Aronsson's solution of (1.2.3) which has a $C^{1,\frac{1}{3}}$ regularity and definition of the Hessian on the axes is an issue. However in [9] it was shown that $u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ is an absolute minimiser not only in Ω but in $\mathbb{R}^2$. This phenomena was later justified using *viscosity solutions*.

The study of vectorial absolute minimisers (i.e. when $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $N \ge 2$) started much more recently in early 2010s by N.Katzourakis in papers [49–59] where he found a new additional term which completes (1.2.3). The associated PDE to the functional

$$E_\infty(u,\mathcal{O}) := \operatorname*{ess\,sup}_{x\in\mathcal{O}} |\mathrm{D}u(x)|, \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N),\ \mathcal{O} \Subset \Omega \subseteq \mathbb{R}^n, \tag{1.2.4}$$

is the so called ∞-*Laplacian* system,

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2 [\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I}\Big) : \mathrm{D}^2 u = 0, \tag{1.2.5}$$

where $X \in \mathbb{R}^{N\times n}$, $[\![X]\!]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^\perp := \mathrm{Proj}_{\mathrm{R}(X)^\perp}.$$

Some difficulties of equation (1.2.5) are: the theory of viscosity solutions does not work for mappings, coefficients of full system are *discontinuous* and solutions are need to be C^2 to make a classical sense. However there is a method to reduce the regularity of solutions to C^1 using measures as parameters. For more details about this method we refer to Chapter 5.

For more details about explicit classical solutions of (1.2.3) we refer to the introduction of the paper presented in Chapter 4.

1.3 Organisation of thesis

The aim of the thesis is to find new classical solutions, derive necessary and sufficient conditions and describe a geometric properties of absolute minimisers. We have reached our goal by publishing and submitting papers in different fields of nonlinear PDEs. Each paper is presented in this thesis as chapter. Every chapter below is explained in the outline below.

Chapter 2 is the joint paper with N.Katzourakis. The estimated contribution is 50%. The paper has been accepted at Proceedings of the Royal Society of Edinburgh A (Mathematics). We study a pointwise characterisation of the PDE system of vectorial calculus of variations in L^∞. In this chapter we prove that generalized solution to PDE

$$\left\{\begin{array}{rl} \mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) &= 0,\\ \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big) &= 0, \end{array}\right.$$

can be characterized as local minimisers for appropriate classes of affine variations to the following energy

$$\mathrm{E}_\infty(u,\mathcal{O}) := \operatorname*{ess\,sup}_{\mathcal{O}} \mathrm{H}(\cdot,u,\mathrm{D}u), \quad u\in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N), \quad \mathcal{O}\Subset\Omega,$$

where $n,N\in\mathbb{N}$ with $\Omega\subseteq\mathbb{R}^n$ open, $\mathrm{H}\in C^2(\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn})$ is a given, $\mathrm{H}_P,\mathrm{H}_\eta,\mathrm{H}_x$ denotes the derivatives of $\mathrm{H}(x,\eta,P)$ with respect to the respective arguments and $[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}$. After an introduction involving a corollary which is a main result for classical solutions, we use Young measures and difference quotients to define a generalised solutions to fully nonlinear PDE introduced by N.Katzourakis. This generalized solutions are called $\mathcal{D}$-solutions. Then we prove two auxiliary lemmas that we will use to prove our main result. Finally we prove our main result for $\mathcal{D}$-solutions and corollary.

Chapter 3 presents the joint paper with N.Katzourakis and H. Abugirda [1]. The paper has been accepted at the Rocky Mountain Journal of Mathematics. We study rigidity and flatness of the image of certain classes of mappings having tangential Laplacian. The estimated percentage contribution is 30%. We start with a brief introduction bringing an attention to one of results of the paper [52]. As generalisation of this theorem we introduce our first result, let $\Omega\subseteq\mathbb{R}^n$ be an open set and $n, N \geq 1$ and $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ in Ω, satisfying that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections). After we show that this theorem is optimal by giving an example that system can not have affine image but only piecewise affine. Then we have next theorem as the consequence of our first main result which supplement one of the results in the paper [53]. Finally we end introduction by our second result which states, let $\Omega \subseteq \mathbb{R}^2$ be an open set and $N \geq 2$ let also $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a classical solution to the nonlinear system $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ in Ω, having the separated form $u(x, y) = f(x) - f(y)$, for some curve $f \in (W^{3,p} \cap C^2)(\mathbb{R}, \mathbb{R}^N)$ and some $p > 1$. Then, the image $u(\Omega)$ of the solution is contained in an at most countable union of affine planes in $\mathbb{R}^N$. At the end of the chapter we give proofs of our results.

Chapter 4 is the single author paper published in Journal of Elliptic and Parabolic Equations in June 2018. We study explicit ∞- harmonic functions in high dimensions. We begin our paper with an introduction contained two main results for smooth solutions. First result states, let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^2(\Omega)$ *separated ∞-harmonicfunction* of the ∞-Laplace equation in polar coordinates

$$u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 = 0$$

of the form $u(r, \theta) = f(r)g(\theta)$.

(i) Assume $|f(r)| = r^A$ and $|g(\theta)| = e^{B\theta}$, where A and B are any constants, then

$$A^2 - A + B^2 = 0$$

or

(ii) Assume $|f(r)| = r^A$ and $|g(\theta)| = |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t)dt}$, then G satisfies the following

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}} \arctan \frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}} \ln \left| \frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}} \right|, & \text{if } A^2 - A < 0, \end{cases}$$

where c is any constant, provided RHS is well defined.

or

(iii) Assume $|g(\theta)| = e^{B\theta}$ and $|f(r)| = |f(r_0)| e^{\int_{r_0}^{r} \frac{\Phi(t)}{t} dt}$, then Φ satisfies the following

$$\ln|t| + c = \begin{cases} \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| - \frac{1}{2}\frac{1}{\sqrt{B^2-\frac{1}{4}}}\arctan\frac{\Phi(t)-\frac{1}{2}}{\sqrt{B^2-\frac{1}{4}}}, & \text{if } B^2-\frac{1}{4}>0 \\ \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| + \frac{1}{2}\frac{1}{\Phi(t)-\frac{1}{2}}, & \text{if } B^2-\frac{1}{4}=0 \\ \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| - \frac{1}{4\sqrt{\frac{1}{4}-B^2}}\ln\left|\frac{\Phi(t)-\frac{1}{2}-\sqrt{\frac{1}{4}-B^2}}{\Phi(t)-\frac{1}{2}+\sqrt{\frac{1}{4}-B^2}}\right|, & \text{if } B^2-\frac{1}{4}<0, \end{cases}$$

where c is any constant, provided RHS is well defined.

Finally second result, let $n \geq 2$ and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $C^2(\Omega)$ *separated* ∞-*harmonic function* of the ∞-Laplace equation

$$\sum_{i,j=1}^{n} \mathrm{D}_i u\, \mathrm{D}_j u\, \mathrm{D}^2_{ij} u \;=\; 0.$$

Then

$$|f_i(x_i)| = |f_i(x_i^0)|\, e^{A_i(x_i - x_i^0)} \text{ for } 1 \leq i \neq j \leq n$$

and

$$|f_j(x_j)| = |f_j(x_j^0)|\, e^{\displaystyle\int_{x_j^0}^{x_j} F_j(t)\, dt},$$

where F_j satisfies

$$t + c = -\frac{1}{2\Big(\sum_{i\neq j} A_i^2\Big)^{1/2}} \arctan\frac{F_j(t)}{\Big(\sum_{i\neq j} A_i^2\Big)^{1/2}} + \frac{F_j(t)}{2\Big(\sum_{i\neq j} A_i^2 + F_j^2(t)\Big)}.$$

In the end we provide proofs for the results and numerical experiments.

Chapter 5 is the joint paper with N.Katzourakis. The estimated contribution is 50%. The paper has been accepted in the journal Applied Mathematics and Optimization. In this paper we prove vectorial variational principles in L^∞ and their characterisation through PDE systems. We start with an introduction involved our two main results.

First result "Variational Structure of Aronsson's system" says, let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a map in $C^2(\Omega;\mathbb{R}^N)$. Then:

(I) If u is a rank-one absolute minimiser for (1.1.1) on Ω (Definition 5.1.2(i)), then it solves

$$\mathrm{H}_P(\cdot, u, \mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) \;=\; 0 \;\text{ on } \Omega. \tag{1.3.1}$$

The converse statement is true if in addition H does not depend on $\eta \in \mathbb{R}^N$ and $\mathrm{H}_P(\cdot, \mathrm{D}u)$ has full rank on Ω.

(II) If u has ∞-minimal area for (1.1.1) on Ω (Definition 5.1.2(ii)), then it solves

$$\mathrm{H}(\cdot, u, \mathrm{D}u)\,[\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \;=\; 0 \;\text{ on } \Omega. \tag{1.3.2}$$

The converse statement is true if in addition for any $x \in \Omega$, $\mathrm{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^{N\times n}$.

(III) If u is ∞-minimal map for (1.1.1) on Ω, then it solves the (reduced) Aronsson system

$$\begin{aligned} \mathrm{A}_\infty u \,:=\, & \mathrm{H}_P(\cdot, u, \mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) \\ & + \mathrm{H}(\cdot, u, \mathrm{D}u)\,[\![\mathrm{H}_P(\cdot, u, \mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \,=\, 0. \end{aligned}$$

The converse statement is true if in addition H does not depend on $\eta \in \mathbb{R}^N$, $\mathrm{H}_P(\cdot, \mathrm{D}u)$ has full rank on Ω and for any $x \in \Omega$ $H(x, \cdot)$ is convex in $\mathbb{R}^{N\times n}$.

Second result "Divergence PDE characterisation of Absolute minimiser" says, let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a map in $C^1(\Omega; \mathbb{R}^N)$. Fix also $\mathcal{O} \Subset \Omega$ and consider the following statements:

(I) u is a vectorial minimiser of $\mathrm{E}_\infty(\cdot, \mathcal{O})$ in $C^1_u(\overline{\mathcal{O}}; \mathbb{R}^N)$[1].

(II) We have

$$\max_{\mathrm{Argmax}\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\}} \Big[\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\psi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\cdot\psi\Big] \,\geq\, 0,$$

for any $\psi \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$[1].

(III) For any $\psi \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$, there exists a non-empty compact set

$$\mathrm{K}_\psi \equiv \mathrm{K} \,\subseteq\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \,:\, \overline{\mathcal{O}}\big\} \tag{1.3.3}$$

such that,

$$\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\psi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\cdot\psi\Big)\Big|_{\mathrm{K}} \,=\, 0. \tag{1.3.4}$$

Then, (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III). If additionally $\mathrm{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$ for any fixed $x \in \Omega$, then (III) $\Longrightarrow$ (I) and all three statements are equivalent. Further, any of the statements above are deducible from the statement:

(IV) For any Radon probability measure[2] $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ satisfying

$$\mathrm{supp}(\sigma) \,\subseteq\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \,:\, \overline{\mathcal{O}}\big\}, \tag{1.3.5}$$

we have

$$-\,\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\sigma\big) \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\sigma \,=\, 0, \tag{1.3.6}$$

in the dual space $(C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N))^*$.

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[1] We say $u \in C^1_g(\overline{\mathcal{O}}; \mathbb{R}^N)$ if $u - g \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$, where $C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N) := \big\{\psi \in C^1(\mathbb{R}^n; \mathbb{R}^N) \,:\, \psi = 0 \text{ on } \partial\mathcal{O}\big\}$.

[2] A Radon measure is a Borel measure that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. See [42] for precise definition.

Finally, all statement are equivalent if $\mathrm{K} = \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \,:\, \overline{\mathcal{O}}\big\}$ in (III) (this happens for instance when the argmax is a singleton set).

The result above provides an interesting characterisation of Aronsson's concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (1.3.6) is

$$\int_{\overline{\mathcal{O}}} \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\psi \, + \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \psi\Big) \mathrm{d}\sigma \, = \, 0$$

for all $\psi \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$, where the ":" notation in the PDE symbolises the Euclidean (Frobenius) inner product in $\mathbb{R}^{N \times n}$.

After that, we give a corollary which is a combination of two results in the scalar case and for the classical solutions. Then we prove the maximum-minimum principle which generalises a corresponding result from [52] and the remark on Danskin's theorem and some of its consequences. Finally, we prove our main results using a lemma and the proof of the lemma ends chapter.

In Chapter 6 we discuss the conclusions and some future work.

Chapter 2

A Pointwise Characterisation of the PDE System of Vectorial Calculus of Variations in L^∞

2.1 Introduction

Calculus of Variations is the branch of Analysis which deals with the problem of finding and studying extrema of nonlinear functionals defined on certain infinite-dimensional topological vector spaces, as well as with describing these extrema through appropriate necessary and sufficient conditions. Such problems are called *variational* and are ubiquitous in nature, being also of paramount importance for other sciences such as Data Assimilation arising in the Earth sciences and Meteorology (see [23, 47]). In most applications, the functional one wishes to study models some kind of "energy" or "action".

Let $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn})$ be a given function, where $\Omega \subseteq \mathbb{R}^n$ is an open set and $n, N \in \mathbb{N}$. One of the most standard particular class of functionals of interest in Calculus of Variations has the form of

$$\mathrm{E}(u,\Omega) \,:=\, \int_\Omega \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big)\, \mathrm{d}x$$

defined on differentiable maps (i.e. vectorial functions) $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. In the above, $\mathbb{R}^{Nn}$ denotes the space of $N \times n$ matrices wherein the gradient matrix

$$\mathrm{D}u(x) \;=\; \big(\mathrm{D}_i u_\alpha(x)\big)_{i=1,\dots,n}^{\alpha=1,\dots,N} \;\in\; \mathbb{R}^{Nn}$$

of such maps is valued. We have also used the symbolisations $x = (x_1, ..., x_n)^\top$, $u = (u_1, ..., u_N)^\top$ and $\mathrm{D}_i \equiv \partial/\partial x_i$. Latin indices $i, j, k, ...$ will run in $\{1, ..., n\}$ and Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$, even if the range of summation is not explicitly mentioned. The simplest variational problem is to search for *minimisers* u of E, sought in a class $\mathscr{C}$ of differentiable maps u, subject to some kind of prescribed boundary condition on $\partial\Omega$ to avoid trivial minimisers. This means that any putative minimiser $u \in \mathscr{C}$, if it exists, should satisfy

$$\mathrm{E}(u,\Omega) \,\leq\, \mathrm{E}(v,\Omega), \quad \text{for all } v \in \mathscr{C} \text{ with } u = v \text{ on } \partial\Omega.$$

If such a minimiser exists, then the real function $t \mapsto \mathrm{E}(tv + (1-t)u)$ has a minimum at $t = 0$ and should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{E}\big(u + t(v-u)\big) \,=\, 0.$$

By the chain rule, this leads, at least *formally*, to the next necessary conditions, known as the *Euler-Lagrange system of Partial Differential Equations* (PDE):

$$\sum_i \mathrm{D}_i\Big(\mathrm{H}_{P_{\alpha i}}(\cdot,u,\mathrm{D}u)\Big) \,=\, \mathrm{H}_{\eta_\alpha}(\cdot,u,\mathrm{D}u), \quad \alpha = 1,\ldots,N.$$

In the above, the subscripts $\mathrm{H}_{P_{\alpha i}}, \mathrm{H}_{\eta_\alpha}$ denote the partial derivatives of H with respect to the respective variables $P_{\alpha i}$ and η_α. Further, since the integral is additive with respect to the domain on which we integrate, it can be easily seen that if u is a minimiser, then

$$\mathrm{E}(u,\mathcal{O}) \,\leq\, \mathrm{E}(v,\mathcal{O}), \quad \text{for all } v \in \mathscr{C} \text{ with } u = v \text{ on } \partial\mathcal{O},$$

where $\mathcal{O} \Subset \Omega$, namely $\overline{\mathcal{O}}$ is a compact subset of Ω. The above weaker condition still suffices to derive the Euler-Lagrange system and any putative u satisfying it is called an *absolute (or local) minimiser.*

The above discussion, although completely *formal*, nonetheless captures the quintessence of Calculus of Variations. However, one needs to use hardcore analytic tools to make rigorous the above formal reasoning. In particular, a central problem is that the minimisers are sought in a class of at most once differentiable maps, which the PDE is of second order and one has to devise a way to make sense of the PDE *weakly*, since second derivatives of u may not exist! Such objects are called *generalised solutions.* Finding a efficient concept of generalised solution which allows one to prove that such a generalised object in fact exists and study its properties is a highly nontrivial part of the problem. A particular relevant question of great interest is to identify conditions on H allowing to *characterise variationally* the PDE system in terms of the functional, namely to provide sufficient as well as necessary conditions.

In this paper we are interested in the variational characterisation of the PDE system arising as the analogue of the Euler-Lagrange equations when one considers vectorial minimisation problems for *supremal functionals* of the form

$$\mathrm{E}_\infty(u,\mathcal{O}) \,:=\, \operatorname*{ess\,sup}_{x\in\mathcal{O}} \mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big), \quad \mathcal{O} \Subset \Omega, \tag{2.1.1}$$

defined on maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. This is in the spirit of the above discussion, but for the modern class of functionals as in (2.1.1). The scalar case $N = 1$ first arose in the work of G. Aronsson in the 1960s [6, 7] who initiated the area of *Calculus of Variations in the space* L^∞. The field is fairly well-developed today and the relevant bibliography is vast. For a pedagogical introduction to the topic accessible to non-experts, we refer to [54].

The study of the vectorial case $N \geq 2$ started much more recently and the full system (2.1.2)-(2.1.4) first appeared in the paper [49] in the early 2010s and it is being studied quite systematically ever since (see [50],[53],[52],[51],[55], [57],[56], as well as the joint works of N.Katzourakis with Abugirda, Pryer, Croce and Pisante [2, 31, 63, 64]). The appropriate class of maps to place and study the functional is the Sobolev space $W^{1,\infty}(\Omega, \mathbb{R}^N)$ of L^∞ maps with L^∞ derivative defined a.e. on Ω (see e.g. [37]). The direct extension of the concept of absolute minimisers for (2.1.1) reads

$$\mathrm{E}_\infty(u, \mathcal{O}) \leq \mathrm{E}_\infty(u + \phi, \mathcal{O}), \qquad \mathcal{O} \Subset \Omega, \ \phi \in W^{1,\infty}_0(\mathcal{O}, \mathbb{R}^N)$$

and was introduced and studied by Aronsson in the context of the scalar case. The subscript nought means that $\phi = 0$ on $\partial\mathcal{O}$. The associated PDE system arising from (2.1.1) as a necessary condition is

$$\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2u) = 0 \quad \text{in } \Omega, \tag{2.1.2}$$

where

$$\mathcal{F}_\infty : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$$

is the Borel measurable map given by

$$\begin{aligned} \mathcal{F}_\infty(x, \eta, P, \mathbf{X}) := \; & \mathrm{H}_P(x, \eta, P)\Big(\mathrm{H}_P(x, \eta, P) : \mathbf{X} + \mathrm{H}_\eta(x, \eta, P)^\top P + H_x(x, \eta, P)\Big) \\ & + \mathrm{H}(x, \eta, P)\, [\![\mathrm{H}_P(x, \eta, P)]\!]^\perp \Big(\mathrm{H}_{PP}(x, \eta, P) : \mathbf{X} + \mathrm{H}_{P\eta}(x, \eta, P) : P \\ & + \mathrm{H}_{Px}(x, \eta, P) : \mathrm{I} - \mathrm{H}_\eta(x, \eta, P)\Big). \end{aligned} \tag{2.1.3}$$

In the above, $\mathbb{R}^{Nn^2}_s$ symbolises the space of symmetric tensors wherein the hessian of u is valued:

$$\mathrm{D}^2u(x) = \big(\mathrm{D}^2_{ij}u_\alpha(x)\big)^{\alpha=1,\dots,N}_{i,j=1,\dots,n} \in \mathbb{R}^{Nn^2}_s.$$

Further, $[\![A]\!]^\perp$ denotes the orthogonal projection onto the orthogonal complement of the range $R(A) \subseteq \mathbb{R}^N$ of a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}. \tag{2.1.4}$$

In index form, $\mathcal{F}_\infty$ reads

$$\begin{aligned}\mathcal{F}_\infty(x,\eta,\mathrm{P},\mathbf{X})_\alpha := \sum_i \mathrm{H}_{P_{\alpha i}}(x,\eta,P)\Bigg(&\sum_{\beta,j}\mathrm{H}_{P_{\beta j}}(x,\eta,P)\mathbf{X}_{\beta ij} + \sum_\beta \mathrm{H}_{\eta_\beta}(x,\eta,P)\,P_{\beta i}\\ &+ \mathrm{H}_{x_i}(x,\eta,P)\Bigg) \,+\, \mathrm{H}(x,\eta,P)\sum_\beta [\![\mathrm{H}_P(x,\eta,P)]\!]^\perp_{\alpha\beta}\,\cdot\\ &\cdot\Bigg(\sum_{i,j}\mathrm{H}_{P_{\alpha i}P_{\beta j}}(x,\eta,P)\,\mathbf{X}_{\beta ij} \,+\, \sum_i \mathrm{H}_{P_{\alpha i}\eta_\beta}(x,\eta,P)\,P_{\beta i}\\ &+ \sum_i \mathrm{H}_{P_{\alpha i}x_i}(x,\eta,P) \,-\, \mathrm{H}_{\eta_\beta}(x,\eta,P)\Bigg),\end{aligned}$$

where $\alpha = 1,\dots,N$. Note that, although H is C^2, the coefficient $[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp$ is *discontinuous* at points where the rank of $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ changes. Further, because of the perpendicularity of H_P and $[\![\mathrm{H}_P]\!]^\perp$ (that is $[\![\mathrm{H}_P]\!]^\perp\mathrm{H}_P = 0$), the system can be decoupled into the two independent systems

$$\left\{\begin{aligned} \mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) \;&=\; 0,\\ \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big) \;&=\; 0.\end{aligned}\right.$$

When $\mathrm{H}(x,\eta,P) = |P|^2$ (the Euclidean norm on $\mathbb{R}^{Nn}$ squared), the system (2.1.2)-(2.1.4) simplifies to the so-called ∞-*Laplacian*:

$$\Delta_\infty u \;:=\; \Big(\mathrm{D}u\otimes\mathrm{D}u + |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp\!\otimes \mathrm{I}\Big) : \mathrm{D}^2u \;=\; 0. \tag{2.1.5}$$

In this paper we are interested in the characterisation of appropriately defined generalised vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (2.1.2)-(2.1.4) in terms of the functional (2.1.1). It is well known even from classical scalar considerations for $N = 1$ that the solutions to (2.1.2)-(2.1.4) in general cannot be expected to be smooth. In the scalar case, generalised solutions are understood in the viscosity sense (see [28, 30, 54]). Since the viscosity theory does not work for (2.1.2)-(2.1.4) when $N \geq 2$, we will interpret solutions in the so-called $\mathcal{D}$-*sense*. This is a new concept of generalised solutions for fully nonlinear systems of very general applicability recently introduced in [57, 58].

Deferring temporarily the details of this new theory of $\mathcal{D}$-*solutions*, we stress the next purely vectorial peculiar occurrence: it is not yet known *whether Aronsson's variational notion is appropriate when* $\min\{n,N\} \geq 2$. In the model case of (2.1.5) and for C^2 solutions, the relevant notion of so-called ∞-Minimal maps allowing to *characterise variationally* solutions to (2.1.5) in term of $u \mapsto \|\mathrm{D}u\|_{L^\infty(\cdot)}$ was introduced in [52]. These findings are compatible with the early vectorial observations made in [17, 18], wherein the appropriate L^∞ quasi-convexity notion in the vectorial case is essentially different from its scalar counterpart. In the recent paper [56] a new characterisation has been discovered that allows to connect $\mathcal{D}$-

solutions of (2.1.5) to local minimisers of $u \mapsto \|\mathrm{D}u\|_{L^\infty(\cdot)}$ in terms of certain classes of *local affine variations*. This result offered new insights to the difficult problem of establishing connections of (2.1.1) to (2.1.2)-(2.1.4).

In this paper we generalise the results of [56], characterising general $\mathcal{D}$-solutions to (2.1.2)-(2.1.4) in terms of local affine variations of (2.1.1). Our main result is Theorem 2.3.2 that follows and asserts that $\mathcal{D}$-solutions to (2.1.2)-(2.1.4) in $C^1(\Omega, \mathbb{R}^N)$ can be characterised variationally in terms of (2.1.1). The a priori C^1 regularity assumed for our putative solutions is slightly higher than the generic membership in the space $W^{1,\infty}(\Omega, \mathbb{R}^N)$, but as a compensation we impose *no convexity* of any kind for the hamiltonian H for the derivation of the system.

In special case of classical solutions, our result reduces to the following corollary which shows the geometric nature of our characterisation[1]:

2.1.1 Corollary [C^2 solutions of $\mathcal{F}_\infty = 0$]

Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in C^2(\Omega, \mathbb{R}^N)$ and $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn})$. Then,

$$\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) \,=\, 0 \;\text{ in } \Omega \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathrm{E}_\infty(u, \mathcal{O}) \,\leq\, \mathrm{E}_\infty(u + A, \mathcal{O})\,, \\ \forall\, \mathcal{O} \Subset \Omega,\; \forall\, A \in \big(\mathcal{A}^{\parallel,\infty}_{\mathcal{O}} \cup \mathcal{A}^{\perp,\infty}_{\mathcal{O}}\big)(u). \end{array} \right.$$

Here $\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u), \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ are sets of affine maps given by

$$\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u) \;=\; \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \equiv 0,\; A(x) = 0 \text{ and exist } \xi \in \mathbb{R}^N \text{and} \\ x \in \mathcal{O}(u) \text{ s.t. the image of } A \text{ is parallel} \\ \text{to the tangent map of } \xi \mathrm{H}(\cdot, u, \mathrm{D}u) \text{ at } x \end{array} \right\},$$

$$\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u) \;=\; \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \,\equiv\, 0 \text{ and there exists } x \in \mathcal{O}(u) \text{ s.t. the} \\ \text{image of } A \text{ is normal to } \mathrm{H}_P(\cdot, u, \mathrm{D}u) \text{ at } x \\ \text{and } A^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) \text{ is divergenceless at } x \end{array} \right\}$$

and

$$\mathcal{O}(u) \;:=\; \mathrm{Argmax}\,\big\{ \mathrm{H}(\cdot, u, \mathrm{D}u) \,:\, \overline{\mathcal{O}} \big\}.$$

This paper is organised as follows. In Section 2.2 that follows we record all the basic facts needed regarding the concept of our $\mathcal{D}$-solutions, namely our notion of generalised solution required to make rigorous sense of (2.1.2)-(2.1.4). We also include a quick introduction to the analytic setup of so-called *Young measures*, on which $\mathcal{D}$-solutions are based. We also give two simple auxiliary results which are utilised in the proof of our variational characterisation. Finally, in Section 2.3 we state and prove our main result.

[1] We caution the reader that the statement of Corollary 2.1.1 sacrifices precision for the sake of clarity. The fully precise statement is that given in the main result, Theorem 2.3.2.

2.2 Young measures, $\mathcal{D}$-solutions and auxiliary results

2.2.1 Young Measures

Let $\Omega \subseteq \mathbb{R}^n$ be open and $\mathbb{K}$ a compact subset of some Euclidean space $\mathbb{R}^{Nn^2}$. The set of *Young measures* $\mathscr{Y}\big(\Omega,\mathbb{K}\big)$ forms a subset of the unit sphere of a certain L^∞ space of measure-valued maps and this provides its useful properties, including sequential weak* compactness. More precisely, $\mathscr{Y}\big(\Omega,\mathbb{K}\big)$ is defined as

$$\mathscr{Y}\big(\Omega,\mathbb{K}\big) := \Big\{\nu : \Omega \longrightarrow \mathscr{P}(\mathbb{K}) \,\Big|\, [\nu(\cdot)](\mathcal{U}) \in L^\infty(\Omega) \text{ for any open } \mathcal{U} \subseteq \mathbb{K}\Big\},$$

where $\mathscr{P}(\mathbb{K})$ is the set of Borel probability measures on $\mathbb{K}$. To see how it arises, consider the separable space $L^1\big(\Omega, C(\mathbb{K})\big)$ of Bochner integrable maps. This space contains Carathéodory functions $\Phi : \Omega \times \mathbb{K} \longrightarrow \mathbb{R}$ (namely functions for which $\Phi(\cdot, X)$ is measurable for all $X \in \mathbb{K}$ and $\Phi(x,\cdot)$ is continuous for a.e. $x \in \Omega$) which satisfy

$$\|\Phi\|_{L^1(\Omega,C(\mathbb{K}))} := \int_\Omega \big\|\Phi(x,\cdot)\big\|_{C^0(\mathbb{K})} \mathrm{d}x \,<\, \infty.$$

We refer e.g. to [35, 43, 77] and to [56, 57] for background material on these spaces. The dual space of this space is $L^\infty_{\mathrm{w}^*}\big(\Omega, \mathcal{M}(\mathbb{K})\big)$, namely

$$\Big(L^1\big(\Omega, C(\mathbb{K})\big)\Big)^* = L^\infty_{\mathrm{w}^*}\big(\Omega, \mathcal{M}(\mathbb{K})\big).$$

This dual Banach space consists of Radon measure-valued maps $\Omega \ni x \mapsto \nu(x) \in \mathcal{M}(\mathbb{K})$ which are weakly* measurable, in the sense that for any open set $\mathcal{U} \subseteq \mathbb{K}$, the function $x \mapsto [\nu(x)](\mathcal{U})$ is in $L^\infty(\Omega)$. The norm of the space is given by

$$\|\nu\|_{L^\infty_{\mathrm{w}^*}(\Omega,\mathcal{M}(\mathbb{K}))} := \operatorname*{ess\,sup}_{x\in\Omega} \|\nu(x)\|\,,$$

where "$\|\cdot\|$" denotes the total variation. It thus follows that

$$\mathscr{Y}\big(\Omega,\mathbb{K}\big) \,=\, \Big\{\nu \,\in\, L^\infty_{\mathrm{w}^*}\big(\Omega, \mathcal{M}(\mathbb{K})\big) \,:\, \nu(x) \in \mathscr{P}(\mathbb{K}), \text{ for a.e. } x \in \Omega\Big\}.$$

2.2.2 Remark [Properties of Young Measures]

We note the following facts about the set $\mathscr{Y}\big(\Omega,\mathbb{K}\big)$ (proofs can be found e.g. in [41]):

i) It is convex and sequentially compact in the weak* topology induced from $L^\infty_{\mathrm{w}^*}$.

ii) The set of measurable maps $V : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{K}$ can be identified with a subset of it via the embedding $V \mapsto \delta_V$, $\delta_V(x) := \delta_{V(x)}$.

iii) Let $V^i, V^\infty : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{K}$ be measurable maps, $i \in \mathbb{N}$. Then, up the passage to subsequences, the following equivalence holds true as $i \to \infty$: $V^i \longrightarrow V^\infty$ a.e. on Ω if and only if $\delta_{V^i} \overset{*}{\rightharpoondown} \delta_{V^\infty}$ in $\mathscr{Y}\big(\Omega, \mathbb{K}\big)$.

2.2.3 $\mathcal{D}$-solutions

We now give some rudimentary facts about generalised solutions which are required for the main result in this paper. For simplicity we will restrict the discussion to $n = 1$ for maps $u : \mathbb{R} \supseteq \Omega \longrightarrow \mathbb{R}^N$ with Ω an interval. The notion of $\mathcal{D}$-solutions is based on the probabilistic interpretation of limits of difference quotients by using *Young* measures. Unlike standard PDE approaches which utilise Young measures valued in Euclidean spaces (see e.g. [24, 37, 41, 43, 67, 71, 77]), $\mathcal{D}$-solutions are based on Young measures valued in the 1-point compactification $\overline{\mathbb{R}}^N := \mathbb{R}^N \cup \{\infty\}$ (which is isometric to the sphere $\mathbb{S}^N$). The motivation of the notion in the case of C^1 solutions to 2nd order fully nonlinear systems is the following: suppose temporarily $u \in C^2(\Omega, \mathbb{R}^N)$ is a solution to

$$\mathcal{F}\big(x, u(x), u'(x), u''(x)\big) \, = \, 0, \qquad x \in \Omega, \tag{2.2.1}$$

where $\mathcal{F} : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous. Let $\mathrm{D}^{1,h}$ be the usual difference quotient operator, i.e. $\mathrm{D}^{1,h}v(x) := \frac{1}{h}\big[v(x+h) - v(x)\big]$, $x \in \Omega$, $h \neq 0$. It follows that

$$\mathcal{F}\Big(x, u(x), u'(x), \lim_{h\to 0} \mathrm{D}^{1,h}u'(x)\Big) \, = \, 0, \qquad x \in \Omega. \tag{2.2.2}$$

Since $\mathcal{F}$ is continuous, (2.2.1) is equivalent to

$$\lim_{h\to 0} \mathcal{F}\Big(x, u(x), u'(x), \mathrm{D}^{1,h}u'(x)\Big) \, = \, 0, \qquad x \in \Omega. \tag{2.2.3}$$

The crucial observation is that *the limit in* (2.2.3) *may exist even if that of* (2.2.2) *does not*, whilst (2.2.3) makes sense for merely C^1 maps. In order to represent the limit in a convenient fashion, we need to view u'' and the difference quotients $\mathrm{D}^{1,h}u'$ as probability-valued maps from Ω to $\mathscr{P}\big(\overline{\mathbb{R}}^N\big)$, given by the respective Dirac masses $x \mapsto \delta_{\mathrm{D}^2u(x)}$ and $x \mapsto \delta_{\mathrm{D}^{1,h}u'(x)}$. The exact definition is as follows:

2.2.3.1 Definition [Diffuse Hessians]

Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be in $W^{1,\infty}(\Omega, \mathbb{R}^N)$. Let also $D^{1,h}$ denote the difference quotient operator, i.e. $\mathrm{D}^{1,h} := \big(\mathrm{D}^{1,h}_1, ..., \mathrm{D}^{1,h}_n\big)$ and $\mathrm{D}^{1,h}_i v := \frac{1}{h}\big[v(\cdot + he^i) - v\big]$, $h \neq 0$. The **diffuse hessians** $\mathcal{D}^2u$ of u are the subsequential weak* limits of the difference quotients of the gradient in the set of sphere-valued Young measures along infinitesimal sequences $(h_\nu)_{\nu=1}^\infty$ (i.e. $\lim\limits_{\nu\to\infty} h_\nu = 0$):

$$\delta_{\mathrm{D}^{1,h_{\nu_k}}\mathrm{D}u} \overset{*}{\rightharpoondown} \mathcal{D}^2u \quad \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \ \text{ as } k \to \infty.$$

The above means for any $\Phi \in L^1\big(\Omega, C(\mathbb{K})\big)$, we have

$$\int_\Omega \int_{\mathbb{K}} \Phi(x,X)\mathrm{d}[\delta_{\mathrm{D}^{1,h_{\nu_k}}\mathrm{D}u}](X)\,\mathrm{d}x \to \int_\Omega \int_{\mathbb{K}} \Phi(x,X)\mathrm{d}[\mathcal{D}^2u](X)\,\mathrm{d}x, \quad \text{as } k \to \infty.$$

Note that the set of Young measures is *sequentially weakly* compact* hence *every map as above possesses diffuse* 2*nd derivatives.*

2.2.3.2 Definition [$\mathcal{D}$-solutions to 2nd order systems]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{F} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$ a Borel measurable map which is continuous with respect to the last argument. Consider the PDE system

$$\mathcal{F}\big(\cdot, u, \mathrm{D}u, \mathrm{D}^2u\big) \;=\; 0 \quad \text{on } \Omega. \tag{2.2.4}$$

We say that the locally Lipschitz continuous map $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a **$\mathcal{D}$-solution of** (2.2.4) when for any diffuse hessian $\mathcal{D}^2u$ of u, we have

$$\sup_{\mathbf{X}_x \in \,\mathrm{supp}_*(\mathcal{D}^2u(x))} \Big|\mathcal{F}\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big)\Big| \;=\; 0, \qquad \text{a.e. } x \in \Omega. \tag{2.2.5}$$

Here "supp_*" symbolises the *reduced support* of a probability measure excluding infinity, namely $\mathrm{supp}_*(\vartheta) := \mathrm{supp}(\vartheta) \setminus \{\infty\}$ when $\vartheta \in \mathcal{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$.

We note that $\mathcal{D}$-solutions are readily compatible with strong/classical solutions: indeed, by Remark 2.2.2iii), if u happens to be twice weakly differentiable then we have $\mathcal{D}^2u(x) = \delta_{\mathrm{D}^2u(x)}$ for a.e. $x \in \Omega$ and the notion reduces to

$$\sup_{\mathbf{X}_x \in \,\mathrm{supp}(\delta_{\mathrm{D}^2u(x)})} \Big|\mathcal{F}\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big)\Big| \;=\; 0, \qquad \text{a.e. } x \in \Omega,$$

thus recovering strong/classical solutions because $\mathrm{supp}(\delta_{\mathrm{D}^2u(x)}) = \{\mathrm{D}^2u(x)\}$.

2.2.4 Two auxiliary lemmas

We now identify two simple technical results which are needed for our main result.

2.2.4.1 Lemma

Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $u \in C^1(\Omega, \mathbb{R}^N)$ and $\mathrm{H} \in C^2(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nn})$. Fix $\mathcal{O} \Subset \Omega$ and an affine map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$. We set

$$\mathcal{O}(u) \;:=\; \Big\{x \in \overline{\mathcal{O}} \;:\; \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big) \;=\; \mathrm{E}_\infty(u,\, \mathcal{O})\Big\}.$$

a) If we have $\mathrm{E}_\infty(u,\,\mathcal{O}) \leq \mathrm{E}_\infty(u+tA,\,\mathcal{O})$ for all $t>0$, it follows that

$$\max_{z\in\overline{\mathcal{O}}}\Big\{\mathrm{H}_P\big(z,u(z),\mathrm{D}u(z)\big):\mathrm{D}A(z)\;+\;\mathrm{H}_\eta\big(z,u(z),\mathrm{D}u(z)\big)\cdot A(z)\Big\}\;\geq\;0.$$

In the above ":" and "$\cdot$" denote the inner products in $\mathbb{R}^{Nn}$ and $\mathbb{R}^N$ respectively.

b) Let $x\in\mathcal{O}$ and $0<\varepsilon<\mathrm{dist}(x,\partial\mathcal{O})$. The set

$$\mathcal{O}_\varepsilon(x)\;:=\;\Big\{y\in\mathcal{O}\;:\;\mathrm{H}(y,u(y),\mathrm{D}u(y))\leq\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big)\Big\}^{\circ}\bigcap\mathbb{B}_\varepsilon(x)$$

(where "$(\cdot)^\circ$" denotes the interior) is open and compactly contained in $\mathcal{O}$, whilst

$$\mathrm{E}_\infty\big(u,\,\mathcal{O}_\varepsilon(x)\big)\;=\;\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big),$$

whenever $\mathcal{O}_\varepsilon(x)\neq\emptyset$.

Note: The proof does not use affinity of map A.

2.2.4.2 Proof of Lemma 2.2.4.1

a) Since $\mathrm{E}_\infty(u,\,\mathcal{O})\leq\mathrm{E}_\infty(u+tA,\,\mathcal{O})$, by Taylor-expanding H, we have

$$\begin{aligned}
0\;&\leq\;\max_{\overline{\mathcal{O}}}\,\mathrm{H}\big(\cdot,u+tA,\mathrm{D}u\;+t\,\mathrm{D}A\big)\;-\;\max_{\overline{\mathcal{O}}}\,\mathrm{H}(\cdot,u,\mathrm{D}u)\\
&=\;\max_{\overline{\mathcal{O}}}\Big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\;+\;t\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot A\;+\;t\,\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\\
&\qquad+\;O\big(t^2|A|^2\;+\;t^2|\mathrm{D}A|^2\big)\Big\}\;-\;\max_{\overline{\mathcal{O}}}\,\mathrm{H}(\cdot,u,\mathrm{D}u)\\
&\leq\;t\,\max_{\overline{\mathcal{O}}}\Big\{\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot A\;+\;\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}A\Big\}\;+\;O(t^2).
\end{aligned}$$

Consequently, by letting $t\to0$, we discover the desired inequality.

Item b) is a direct consequence of the definitions. $\square$

Next, we have the following simple consequence of Danskin's theorem [34]:

2.2.4.3 Lemma

Given an open set $\Omega\subseteq\mathbb{R}^n$, consider maps $u\in C^1(\Omega,\mathbb{R}^N)$ and $\mathrm{H}\in C^2(\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^{Nn})$ such that $\mathrm{H}(x,\cdot,\cdot)$ is jointly convex for any $x\in\Omega$, an affine map $A:\mathbb{R}^n\longrightarrow\mathbb{R}^N$ and $\mathcal{O}\Subset\Omega$. We define

$$r(\lambda)\;:=\;\mathrm{E}_\infty(u+\lambda A,\,\mathcal{O})\;-\;\mathrm{E}_\infty(u,\,\mathcal{O}),\qquad\lambda\geq0.$$

Let also $\mathcal{O}(u)$ be as in Lemma 2.2.4.1. Then, r is convex, $r(0) = 0$ and also it satisfies

$$\underline{\mathrm{D}}r(0^+) \;\geq\; \max_{\mathcal{O}(u)} \Big\{ \mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}A \;+\; \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot A \Big\},$$

where $\underline{\mathrm{D}}r(a^+) := \liminf\limits_{\lambda \to 0^+} \frac{r(a+\lambda) - r(a)}{\lambda}$ is the lower right Dini derivative of r at a.

2.2.4.4 Proof of Lemma 2.2.4.3

The result is deducible from Danskin's theorem (see [34]) but we prove it directly since the 1-sided version above is not given explicitly in the paper. By setting

$$R(\lambda, y) \;:= \mathrm{H}\Big(y\,,\; u(y) + \lambda A(y)\,,\; Du(y) + \;\lambda DA(y)\Big)$$

we have $r(\lambda) = \max_{y \in \overline{\mathcal{O}}} R(\lambda, y) - \max_{y \in \overline{\mathcal{O}}} R(0, y)$, whilst for any $\lambda \geq 0$ the maximum $\max_{y \in \overline{\mathcal{O}}} R(\lambda, y)$ is realised at (at least one) point $y^\lambda \in \overline{\mathcal{O}}$. Hence

$$\begin{aligned}
\frac{1}{\lambda}\big(r(\lambda) - r(0)\big) \;&=\; \frac{1}{\lambda}\Big[\max_{y \in \overline{\mathcal{O}}} R(\lambda, y) \;-\; \max_{y \in \overline{\mathcal{O}}} R(0, y)\Big] \\
&=\; \frac{1}{\lambda}\Big[R(\lambda, y^\lambda) \;-\; R(0, y^0)\Big] \\
&=\; \frac{1}{\lambda}\Big[\big(R(\lambda, y^\lambda) - R(\lambda, y^0)\big) \;+\; \big(R(\lambda, y^0) - R(0, y^0)\big)\Big]
\end{aligned}$$

and hence

$$\frac{1}{\lambda}\big(r(\lambda) - r(0)\big) \;\geq\; \frac{1}{\lambda}\big(R(\lambda, y^0) \;-\; R(0, y^0)\big),$$

where $y^0 \in \overline{\mathcal{O}}$ is any point such that $R(0, y^0) \;=\; \max_{\overline{\mathcal{O}}} R(0, \cdot)$. Hence, we have

$$\begin{aligned}
\underline{\mathrm{D}}r(0^+) &= \liminf_{\lambda \to 0^+} \frac{1}{\lambda}\big(r(\lambda) - r(0)\big) \\
&\geq \max_{y^0 \in \overline{\mathcal{O}}} \left\{ \liminf_{\lambda \to 0^+} \frac{1}{\lambda}\Big(R(\lambda, y^0) - R(0, y^0)\Big) \right\} \\
&= \max_{y \in \mathcal{O}(u)} \left\{ \liminf_{\lambda \to 0^+} \frac{1}{\lambda}\Big(R(\lambda, y) - R(0, y)\Big) \right\} \\
&= \max_{\mathcal{O}(u)} \left\{ \liminf_{\lambda \to 0^+} \frac{1}{\lambda}\Big(\mathrm{H}\big(\,\cdot\,, u + \lambda A, \mathrm{D}u + \lambda \mathrm{D}A\big) - \mathrm{H}\big(\cdot, u, \mathrm{D}u\big)\Big) \right\} \\
&= \max_{\mathcal{O}(u)} \left\{ \liminf_{\lambda \to 0^+} \frac{1}{\lambda}\Big(\mathrm{H}(\cdot, u, \mathrm{D}u) \;+\; \lambda\, \mathrm{H}_\eta\big(\cdot, u, \mathrm{D}u\big) \cdot A \;+\; \lambda\, \mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}A \right. \\
&\qquad \left. +\; O\big(|\lambda \mathrm{D}A|^2 + |\lambda A|^2\big) \;-\; \mathrm{H}(\cdot, u, \mathrm{D}u)\Big) \right\}
\end{aligned}$$

and the desired inequality has been established. Finally by convexity of H we have for any $x, y > 0$ and any $t \in [0,1]$

$$
\begin{aligned}
r\big(tx+(1-t)y\big) &:= \mathrm{E}_\infty\big(t(u+xA)+(1-t)(u+yA),\, \mathcal{O}\big) \,-\, \mathrm{E}_\infty(u,\, \mathcal{O}) \\
&\le t\,\mathrm{E}_\infty(u+xA,\, \mathcal{O}) + (1-t)\,\mathrm{E}_\infty(u+yA,\, \mathcal{O}) - \mathrm{E}_\infty(u,\, \mathcal{O}) \\
&\le tr(x) + (1-t)r(y)
\end{aligned}
$$

□

Let us record the next simple inequality which follows from the definitions of lower right Dini derivative, in the case that $\mathrm{H}(x,\cdot,\cdot)$ is jointly convex for any $x \in \Omega$. This is

$$
r(\lambda) \,-\, r(0) \,\ge\, \underline{\mathrm{D}}r(0^+)\,\lambda, \tag{2.2.6}
$$

for all $\lambda \ge 0$.

2.3 The main result of the Chapter 2

Now we proceed to the main result of the paper, the variational characterisation of $\mathcal{D}$-solutions to the PDE system (2.1.2) in terms of appropriate variations of the energy functional (2.1.1). We recall that the Borel mapping $\mathcal{F}_\infty : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}_s^{Nn^2} \longrightarrow \mathbb{R}^N$ is given by (2.1.3)-(2.1.4) and $\Omega \subseteq \mathbb{R}^n$ is a fixed open set.

2.3.1 Notational simplifications and perpendicularity considerations.

We begin by rewriting $\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) = 0$ in a more malleable fashion (see (2.1.3)). We define the maps

$$
\mathcal{F}_\infty^\perp(x,\eta,P,\mathbf{X}) \,:=\, \mathrm{H}_{PP}(x,\eta,P) : \mathbf{X} + \mathrm{H}_{P\eta}(x,\eta,P) : P + \mathrm{H}_{Px}(x,\eta,P) : \mathrm{I}, \tag{2.3.1}
$$

$$
\mathcal{F}_\infty^\parallel(x,\eta,P,\mathbf{X}) \,:=\, \mathrm{H}_P(x,\eta,P) : \mathbf{X} \,+\, \mathrm{H}_\eta(x,\eta,P)^\top P \,+\, H_x(x,\eta,P) \tag{2.3.2}
$$

and these are abbreviations of

$$
\begin{aligned}
\mathcal{F}_\infty^\perp(x,\eta,P,\mathbf{X})_\alpha \,&=\, \sum_{\beta,i,j} \mathrm{H}_{P_{\alpha i}P_{\beta j}}(x,\eta,P)\,\mathbf{X}_{\beta ij} \,+\, \sum_{\beta,i} \mathrm{H}_{P_{\alpha i}\eta_\beta}(x,\eta,P)\,P_{\beta i} \\
&\quad + \sum_i \mathrm{H}_{P_{\alpha i}x_i}(x,\eta,P)\ , \\
\mathcal{F}_\infty^\parallel(x,\eta,P,\mathbf{X})_i \,&=\, \sum_{\beta,j} \mathrm{H}_{P_{\beta j}}(x,\eta,P)\mathbf{X}_{\beta ij} + \sum_\beta \mathrm{H}_{\eta_\beta}(x,\eta,P)\,P_{\beta i} \,+\, H_{x_i}(x,\eta,P).
\end{aligned}
$$

Note that $\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X}) \in \mathbb{R}^N$, whilst $\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X}) \in \mathbb{R}^n$. By utilising (2.3.1)-(2.3.2), we can now express (2.1.3) as

$$\begin{aligned}\mathcal{F}_{\infty}(x,\eta,P,\mathbf{X}) \,:=\, &\mathrm{H}_P(x,\eta,P)\,\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X}) + \mathrm{H}(x,\eta,P)\,\cdot \\ &\cdot\,[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X}) \,-\, \mathrm{H}_{\eta}(x,\eta,P)\Big).\end{aligned}$$

Further, recall that in view of (2.1.4), $[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}$ is the projection on the orthogonal complement of $R(\mathrm{H}_P(x,\eta,P))$. Hence, by the orthogonality of $[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\cdot$ $\cdot\big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X}) - \mathrm{H}_{\eta}(x,\eta,P)\big)$ and $\mathrm{H}_P(x,\eta,P)\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X})$, we have

$$\mathcal{F}_{\infty}(x,\eta,P,\mathbf{X}) = 0, \quad \text{for some } (x,\eta,P,\mathbf{X}) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn^2}_s,$$

if and only if

$$\left\{\begin{aligned} \mathrm{H}_P(x,\eta,P)\,\mathcal{F}^{\parallel}_{\infty}(x,\eta,P,\mathbf{X}) \;&=\; 0,\\ \mathrm{H}(x,\eta,P)\,[\![\mathrm{H}_P(x,\eta,P)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}(x,\eta,P,\mathbf{X}) - \mathrm{H}_{\eta}(x,\eta,P)\Big) \;&=\; 0.\end{aligned}\right.$$

Finally, for the sake of clarity we state and prove our characterisation below only in the case of C^1 solutions, but due to its pointwise nature, the result holds true for piecewise C^1 solutions with obvious adaptations which we refrain from providing. We will assume that the Hamiltonian H satisfies

$$\big\{\mathrm{H}_P(x,\eta,\cdot) = 0\big\} \,\subseteq\, \big\{\mathrm{H}(x,\eta,\cdot) = 0\big\}, \quad (x,\eta)\in\Omega\times\mathbb{R}^N. \tag{2.3.3}$$

We will also suppose that the next set has vanishing measure

$$\Big|\Big\{x\in\Omega\,:\,\mathbb{B}_{r_x}(x)\bigcap\big\{h > h(x)\big\} \text{ is dense in } \mathbb{B}_{r_x}(x)\Big\}\Big| \,=\, 0, \tag{2.3.4}$$

where $r_x \equiv \mathrm{dist}(x,\partial\Omega)$ and $h \equiv \mathrm{H}(\cdot,u,\mathrm{D}u)$. This assumption is natural, in the sense that it is satisfied by all know examples of explicit solutions (see [55, 63–66]). It is trivially satisfied if h has no strict local minima in the domain.

Lets examine three examples for conditions (2.3.3) and (2.3.4). For all examples $\mathrm{H}(x,\eta,P) = |P|^2$. Clearly (2.3.3) is satisfied. Remains to show that actually condition (2.3.4) holds for our three examples.

Example 2.3.1. *$u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ is well-known explicit solution and let $\Omega = [-1,1]^2$. The function $h \equiv |\mathrm{D}u|^2 = \frac{16}{9}\big(|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}}\big)$ has only one point of local minimum at origin which means set $\{h > h(0)\}\cap\mathbb{B}_{r_0}(0)$ is the dense in the ball $\mathbb{B}_{r_0}(0)$. Let a point (x,y) be different from origin then it easy to check that set $\{h > h(x,y)\}\cap\mathbb{B}_{r_{x,y}}(x,y)$ is not a dense in the $\mathbb{B}_{r_{x,y}}(x,y)$ and as the result we have (2.3.4).*

Example 2.3.2. *Let $\Omega = [0.1,1]^2$ and $u(x,y) = \sqrt{x^2+y^2}$ is the conic solution. The function $h \equiv |\mathrm{D}u|^2 \equiv 1$ for any point of Ω. So clearly (2.3.4) is satisfied.*

Example 2.3.3. *Using notation $e^{it} = (\cos t, \sin t)$ we have vectorial solution $u(x,y) = e^{ix} - e^{iy}$ on $\Omega = [-1,1]^2$ which is Eikonal, namely $|\mathrm{D}u|^2 = |D_x u|^2 + |\mathrm{D}_y u|^2 \equiv 2$. So clearly (2.3.4) is satisfied.*

Our main result is as follows:

2.3.2 Theorem [Variational characterisation of the PDE system arising in L^∞]

Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in C^1(\Omega, \mathbb{R}^N)$ and $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn})$ a function satisfying (2.3.3) and suppose that (2.3.4) holds. Then:

(A) We have

$$\mathcal{F}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) = 0 \ \text{ in } \Omega,$$

in the $\mathcal{D}$-sense, if and only if

$$\mathrm{E}_\infty(u, \mathcal{O}) \le \mathrm{E}_\infty(u + A, \mathcal{O}), \qquad \forall\, \mathcal{O} \Subset \Omega,\ \forall\, A \in \mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u) \bigcup \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

For the sufficiency of the PDE for the variational problem we require that $\mathrm{H}(x,\cdot,\cdot)$ be convex. In the above, the sets $\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u), \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ consist, for any $\mathcal{O} \Subset \Omega$, by affine mappings as follows:

$$\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u) := \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \equiv 0,\ A(x) = 0 \text{ for } x \in \mathcal{O}(u), \ \&\, \text{exist } \xi \in \mathbb{R}^N, \\ \mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big) \ \&\ \mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2 u(x)\big) \\ \text{s.t. } :\ \mathrm{D}A \equiv \xi \otimes \mathcal{F}^{\parallel}_\infty\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) \end{array} \right\} \bigcup$$
$$\big\{A : \mathbb{R}^n \to \mathbb{R}^N, A \equiv const\big\}$$

and

$$\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u) := \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} \mathrm{D}^2 A \equiv 0 \ \&\text{ there exist } x \in \mathcal{O}(u),\ \mathcal{D}^2 u \\ \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big) \ \&\ \mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2 u(x)\big) \\ \text{s.t. } :\ A(x) \in R\Big(\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big)\Big)^{\perp} \\ \&\ \mathrm{D}A \in \mathscr{L}\big(x, A(x), \mathbf{X}_x\big) \end{array} \right\} \bigcup$$
$$\big\{A : \mathbb{R}^n \to \mathbb{R}^N, A \equiv const\big\}$$

where $\mathcal{O}(u)$ defined in lemma 2.2.4.1 and $\mathscr{L}\big(x, \eta, \mathbf{X}\big)$ is an affine space of $N \times n$ matrices, defined as

$$\mathscr{L}\big(x, \eta, \mathbf{X}\big) := \begin{cases} \Big\{ Q \in \mathbb{R}^{Nn} \Big| \mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big) : Q = -\eta \cdot \mathcal{F}^{\perp}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}\big) \Big\}, & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \neq 0, & \\ \{0\}, \qquad\qquad\qquad\qquad\qquad \text{if } \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) = 0, & \end{cases}$$

for any $(x, \eta, \mathbf{X}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn^2}_s$.

(B) In view of the mutual perpendicularity of the two components of $\mathcal{F}_\infty$ (see (2.3.1)-(2.3.2)), (A) is a consequence of the following particular results:

$$\mathrm{H}_P(\cdot, u, \mathrm{D}u)\, \mathcal{F}^{\parallel}_\infty(\cdot, u, \mathrm{D}u, \mathrm{D}^2 u) = 0 \ \text{ in } \Omega,$$

in the $\mathcal{D}$-sense, if and only if

$$\mathrm{E}_\infty(u,\,\mathcal{O})\;\leq\;\mathrm{E}_\infty(u+A,\,\mathcal{O}),\quad \forall\,\mathcal{O}\Subset\Omega,\;\forall\,A\in\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u)$$

and also

$$\mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_\infty(\cdot,u,\mathrm{D}u,\mathrm{D}^2u)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\;=\;0\;\text{ in }\Omega,$$

in the $\mathcal{D}$-sense, if and only if

$$\mathrm{E}_\infty(u,\,\mathcal{O})\;\leq\;\mathrm{E}_\infty(u+A,\,\mathcal{O}),\quad \forall\,\mathcal{O}\Subset\Omega,\;\forall\,A\in\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

We note that in the special case of C^2 solutions, Corollary 2.1.1 describes the way that classical solutions $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ to (2.1.2)-(2.1.4) are characterised.

2.3.3 Remark [About pointwise properties of C^1 $\mathcal{D}$-solutions]

Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be a $\mathcal{D}$-solution to (2.1.2)-(2.1.4) in $C^1(\Omega,\mathbb{R}^N)$. By Definition 2.2.3.2, this means that for any $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$,

$$\mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\;=\;0,\quad \text{a.e. }x\in\Omega,\;\forall\,\mathbf{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big).$$

By Definition 2.2.3.1, every diffuse hessian of a putative solution is defined a.e. on Ω as a weakly* measurable probability valued map $\mathbb{R}^n\supseteq\Omega\longrightarrow\mathscr{P}\big(\mathbb{R}^{Nn^2}_s\cup\{\infty\}\big)$. Let $\Omega\ni x\mapsto\mathbf{O}_x\in\mathbb{R}^{Nn^2}_s$ be any selection of elements of the zero level sets

$$\Big\{\mathbf{X}\in\mathbb{R}^{Nn^2}_s\;:\;\mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}\big)\;=\;0\Big\}.$$

By modifying each diffuse hessian on a Lebesgue nullset and choosing the representative which is redefined as $\mathcal{D}^2u(x)=\delta_{\mathbf{O}_x}$ for a negligible set of x's, we may assume that $\mathcal{D}^2u(x)$ exists for all $x\in\Omega$. Further, given that $\mathrm{D}u(x)$ exists for all $x\in\Omega$, by perhaps a further re-definition on a Lebesgue nullset, it follows that u is $\mathcal{D}$-solution to (2.1.2)-(2.1.4) if and only if for (any such representative of) any diffuse hessian

$$\mathcal{F}_\infty\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\;=\;0,\quad \forall\,x\in\Omega,\;\forall\,\mathbf{X}_x\in\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big).$$

Note that at points $x\in\Omega$ for which $\mathcal{D}^2u(x)=\delta_{\{\infty\}}$ and hence $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)=\emptyset$, the solution criterion is understood as being trivially satisfied.

2.3.4 Proof of Theorem 2.3.2

It suffices to establish only (B), since (A) is a consequence of it. Suppose that for any $\mathcal{O}\Subset\Omega$ and any $A\in\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$ we have $E_\infty(u,\mathcal{O})\leq\mathrm{E}_\infty(u+A,\mathcal{O})$. Fix a diffuse hessian $\mathcal{D}^2u\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, a point $x\in\overline{\mathcal{O}}$ such that $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)\neq\emptyset$

and an $\mathbf{X}_x \in \operatorname{supp}_*\big(\mathcal{D}^2 u(x)\big)$. In view of (2.3.1), if $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) = 0$, then, by our assumption on the level sets of H, we have $\mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big) = 0$ as well and as a consequence we readily obtain

$$
\begin{aligned}
&\mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big)\, [\![\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big)]\!]^\perp \cdot \\
&\cdot \Big(\mathcal{F}_\infty^\perp\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) - \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big)\Big) = 0
\end{aligned}
\tag{2.3.5}
$$

is clearly satisfied at x. If $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \neq 0$, then we select any direction normal to the range of $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \in \mathbb{R}^{Nn}$, that is

$$
n_x \in R\Big(\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big)\Big)^\perp \subseteq \mathbb{R}^N
$$

which means $n_x^\top \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) = 0$. Of course it may happen that the linear map $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) : \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ is surjective and then only the trivial $n_x = 0$ exists. In such an event, the equality (2.3.5) above is satisfied at x because $[\![\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big)]\!]^\perp = 0$. Hence, we may assume $n_x \neq 0$. Further, fix any matrix N_x in the affine space $\mathscr{L}(x, n_x, \mathbf{X}_x) \subseteq \mathbb{R}^{Nn}$. By the definition of $\mathscr{L}(x, n_x, \mathbf{X}_x)$, we have

$$
\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) : N_x \;=\; -n_x \cdot \mathcal{F}_\infty^\perp\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big).
$$

Consider the affine map defined by

$$
A(z) \;:=\; n_x \;+\; N_x(z - x), \quad z \in \mathbb{R}^n.
$$

We remark that $tA \in \mathcal{A}_{\mathcal{O}}^{\perp,\infty}(u)$ for any $t \in \mathbb{R}$. Indeed, this is a consequence of our choices and the next homogeneity property of the space $\mathscr{L}(x, \eta, \mathbf{X})$:

$$
\mathscr{L}(x, t\eta, \mathbf{X}) \;=\; t\,\mathscr{L}(x, \eta, \mathbf{X}), \quad t \in \mathbb{R}.
$$

Let $\varepsilon > 0$ be small, fix $x \in \Omega$ and let us choose as $\mathcal{O}$ the domain $\mathcal{O}_\varepsilon(x)$ defined in Lemma 2.2.4.1b). Our assumption (2.3.4) implies that $\mathcal{O}_\varepsilon(x) \neq \emptyset$ for a.e. $x \in \Omega$. In view of the above considerations, we have

$$
\mathrm{E}_\infty\big(u, \mathcal{O}_\varepsilon(x)\big) \;\leq\; \mathrm{E}_\infty\big(u + tA, \mathcal{O}_\varepsilon(x)\big).
$$

By applying Lemma 2.2.4.1a), we have

$$
\begin{aligned}
0 \;\leq\; & \max_{z \in \mathcal{O}_\varepsilon(x)} \Big\{ \mathrm{H}_P\big(z, u(z), \mathrm{D}u(z)\big) : \mathrm{D}A(z) \;+\; \mathrm{H}_\eta\big(z, u(z), \mathrm{D}u(z)\big) \cdot A(z) \Big\} \\
\xrightarrow{\varepsilon \to 0}\; & \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) : N_x \;+\; \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big) \cdot n_x \\
=\; & -n_x \cdot \Big(\mathcal{F}_\infty^\perp\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) - \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big)\Big).
\end{aligned}
$$

As a result, we have

$$
n_x \cdot \Big(\mathcal{F}_\infty^\perp\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) - \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big)\Big) \leq 0
$$

for any direction $n_x \perp R\Big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\Big)$ and by the arbitrariness of n_x, we deduce that

$$[\![\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)]\!]^{\perp}\Big(\mathcal{F}^{\perp}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\,-\,\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\Big)\;=\;0,$$

for any $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $x\in\Omega$ and $\mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$, as desired.

For the tangential component of the system we argue similarly. Suppose that for any $\mathcal{O} \Subset \Omega$ and any $A \in \mathcal{A}^{\|,\infty}_{\mathcal{O}}(u)$ we have $\mathrm{E}_\infty(u,\mathcal{O}) \le \mathrm{E}_\infty(u+A,\mathcal{O})$. Fix $x\in\overline{\mathcal{O}}$, a diffuse hessian $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$ such that $\mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)\neq\emptyset$, a point $\mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$ and $\xi\in\mathbb{R}^N$. Recalling (2.3.2), we define the affine map

$$A(z)\;:=\;\xi\otimes\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\cdot(z-x),\qquad z\in\mathbb{R}^n.$$

Fix $\varepsilon>0$ small, $x\in\Omega$ and choose as $\mathcal{O}$ the domain $\mathcal{O}_\varepsilon(x)$ of Lemma 2.2.4.1b). Then, $tA \in \mathcal{A}^{\|,\infty}_{\mathcal{O}_\varepsilon(x)}(u)$ for any $t\in\mathbb{R}$. Consequently, in view our the above we have

$$\mathrm{E}_\infty\big(u,\mathcal{O}_\varepsilon(x)\big)\;\le\;\mathrm{E}_\infty\big(u+tA,\mathcal{O}_\varepsilon(x)\big)$$

and by applying Lemma 2.2.4.1a), this yields

$$\begin{aligned}0\;\le\;&\max_{z\in\overline{\mathcal{O}_\varepsilon(x)}}\Big\{\mathrm{H}_P\big(z,u(z),\mathrm{D}u(z)\big):\mathrm{D}A(z)\;+\;\mathrm{H}_\eta\big(z,u(z),\mathrm{D}u(z)\big)\cdot A(z)\Big\}\\ \xrightarrow{\varepsilon\to 0}\;&\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):\Big(\xi\otimes\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big).\end{aligned}$$

Hence,

$$\xi\cdot\Big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\,\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)\;\ge\;0,$$

for any $\xi\in\mathbb{R}^N$. By the arbitrariness of ξ we infer that

$$\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\,\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)=0$$

for any $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $x\in\Omega$ and $\mathbf{X}_x \in \mathrm{supp}_*\big(\mathcal{D}^2u(x)\big)$, as desired.

Conversely, let us fix $\mathcal{O}\Subset\Omega$, $x\in\mathcal{O}(u)$, $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^2}_s\big)$, $\mathbf{X}_x\in\mathrm{supp}_*(\mathcal{D}^2u(x))$ and $\xi\in\mathbb{R}^N$ corresponding to a map $A\in\mathcal{A}^{\|,\infty}_{\mathcal{O}}(u)$. Let r be the function of Lemma 2.2.4.3. By applying Lemma 2.2.4.3 to the above setting, we have

$$\begin{aligned}\underline{\mathrm{D}}r(0^+)\;&\ge\;\max_{y\in\mathcal{O}(u)}\Big\{\mathrm{H}_P(y,u(y),\mathrm{D}u(y)):\mathrm{D}A(y)\;+\;\mathrm{H}_\eta(y,u(y),\mathrm{D}u(y))\cdot A(y)\Big\}\\ &\ge\;\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):\mathrm{D}A(x)\;+\;\mathrm{H}_\eta\big(x,u(x),\mathrm{D}u(x)\big)\cdot A(x)\\ &=\;\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big):\Big(\xi\otimes\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)\\ &=\;\xi\cdot\Big(\mathrm{H}_P\big(x,u(x),\mathrm{D}u(x)\big)\mathcal{F}^{\|}_{\infty}\big(x,u(x),\mathrm{D}u(x),\mathbf{X}_x\big)\Big)\end{aligned}$$

and hence $\underline{\mathrm{D}}r(0^+)\ge 0$ because u is a $\mathcal{D}$-solution. Due to the fact that $r(0)=0$

and r is convex, by inequality (2.2.6) we have $r(t) \geq 0$ for all $t \geq 0$. Therefore,

$$\mathrm{E}_\infty(u,\, \mathcal{O}) \,\leq\, \mathrm{E}_\infty(u + A,\, \mathcal{O}), \quad \forall\, \mathcal{O} \Subset \Omega,\; \forall\, A \in \mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u).$$

The case of $A \in \mathcal{A}^{\perp,\infty}_{\mathcal{O}}$ is completely analogous. Fix $\mathcal{D}^2 u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$, $\mathcal{O} \Subset \Omega$, $x \in \mathcal{O}(u)$, $\mathbf{X}_x \in \mathrm{supp}_*(\mathcal{D}^2 u(x))$ and an A with $A(x) \perp R\big(\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big)\big)$ and $\mathrm{D}A \in \mathscr{L}\big(x, A(x), \mathbf{X}_x\big)$. By applying Lemma 2.2.4.3 again, we have

$$\begin{aligned}
\underline{\mathrm{D}}r(0^+) \,\geq\, &\max_{y \in \mathcal{O}(u)} \Big\{ \mathrm{H}_P(y, u(y), \mathrm{D}u(y)) : \mathrm{D}A(y) \;+\; \mathrm{H}_\eta(y, u(y), \mathrm{D}u(y)) \cdot A(y) \Big\} \\
\geq\, &\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}A(x) \;+\; \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big) \cdot A(x).
\end{aligned}$$

If $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \neq 0$, then by the definition of $\mathscr{L}\big(x, A(x), \mathbf{X}_x\big)$ we have

$$\begin{aligned}
\underline{\mathrm{D}}r(0^+) \geq\; &\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}A(x) \;+\; \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big) \cdot A(x) \\
=\; &-A(x) \cdot \Big(\mathcal{F}^{\perp}_\infty\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) \;-\; \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big)\Big) \\
=\; &-A(x)^\top [\![\mathrm{H}_P(x, u(x), \mathrm{D}u(x))]\!]^\perp \Big(\mathcal{F}^{\perp}_\infty\big(x, u(x), \mathrm{D}u(x), \mathbf{X}_x\big) \\
&\quad - \mathrm{H}_\eta\big(x, u(x), \mathrm{D}u(x)\big)\Big)
\end{aligned}$$

and hence $\underline{\mathrm{D}}r(0^+) \geq 0$ because u is a $\mathcal{D}$-solution on Ω. If $\mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) = 0$, then again $\underline{\mathrm{D}}r(0^+) \geq 0$ because $A(x) = 0$. In either cases, by inequality (2.2.6) we obtain $r(t) \geq 0$ for all $t \geq 0$ and hence

$$\mathrm{E}_\infty(u,\, \mathcal{O}) \,\leq\, \mathrm{E}_\infty(u + A,\, \mathcal{O}), \quad \forall\, \mathcal{O} \Subset \Omega,\; \forall\, A \in \mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u).$$

The theorem has been established. □

2.3.5 Proof of Corollary 2.1.1

If $u \in C^2(\Omega, \mathbb{R}^N)$, then by Lemma 2.2.2 any diffuse hessian of u satisfies $\mathcal{D}^2 u(x) = \delta_{\mathrm{D}^2 u(x)}$ for a.e. $x \in \Omega$. By Remark 2.3.3, we may assume this happens for all $x \in \Omega$. Therefore, the reduced support of $\mathcal{D}^2 u(x)$ is the singleton set $\{\delta_{\mathrm{D}^2 u(x)}\}$. Hence, for $\mathcal{A}^{\parallel,\infty}_{\mathcal{O}}(u)$, we have that any possible affine map A satisfies $\mathrm{D}A \equiv \mathrm{D}\big(\xi \mathrm{H}\big(x, u(x), \mathrm{D}u(x)\big)\big)$ and $A(x) = 0$. In the case of $\mathcal{A}^{\perp,\infty}_{\mathcal{O}}(u)$, we have that any possible affine map A satisfies

$$A(x)^\top \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \,=\, 0\,, \quad \mathrm{D}A \in \mathscr{L}\big(x, A(x), \mathrm{D}^2 u(x)\big),$$

which gives

$$\begin{aligned}
\mathrm{D}A(x) : \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \;=\; &-A(x) \cdot \Big(\mathrm{H}_{PP}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}^2 u(x) \,+ \\
&+ \mathrm{H}_{P\eta}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{D}u(x) \;+\; \mathrm{H}_{Px}\big(x, u(x), \mathrm{D}u(x)\big) : \mathrm{I}\Big) \\
=\; &-A(x) \cdot \mathrm{Div}\big(\mathrm{H}_P\big(\cdot, u, \mathrm{D}u\big)\big)(x).
\end{aligned}$$

As a consequence, the divergence $\mathrm{Div}\big(A^\top \mathrm{H}_P\big(\cdot, u, \mathrm{D}u\big)\big)(x)$ vanishes because

$$\mathrm{D}A(x) : \mathrm{H}_P\big(x, u(x), \mathrm{D}u(x)\big) \;+\; A(x) \cdot \mathrm{Div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big)(x) \;=\; 0.$$

The corollary has been established. $\square$

Chapter 3

Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian

3.1 Introduction

Suppose that n, N are integers and Ω an open subset of $\mathbb{R}^n$. In this paper we study geometric aspects of the image $u(\Omega) \subseteq \mathbb{R}^N$ of certain classes of C^2 vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to the following nonlinear degenerate elliptic PDE system:

$$[\![\mathrm{D}u]\!]^\perp \Delta u = 0 \quad \text{in } \Omega. \tag{3.1.1}$$

Here, for the map u with components $(u_1, ..., u_N)^\top$ the notation $\mathrm{D}u$ symbolises the gradient matrix

$$\mathrm{D}u(x) = \big(\mathrm{D}_i u_\alpha(x)\big)_{i=1...n}^{\alpha=1...N} \in \mathbb{R}^{N\times n}\ , \quad \mathrm{D}_i \equiv \partial/\partial x_i,$$

Δu stands for the Laplacian

$$\Delta u(x) = \sum_{i=1}^{n} \mathrm{D}^2_{ii} u(x) \in \mathbb{R}^N$$

and for any $X \in \mathbb{R}^{N\times n}$, $[\![X]\!]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^\perp := \mathrm{Proj}_{\mathrm{R}(X)^\perp}.$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [32, 38]. Note that, since the rank is a discontinuous function, the map $[\![\,\cdot\,]\!]^\perp$ is discontinuous on $\mathbb{R}^{N\times n}$; therefore, the PDE system (3.1.1) has *discontinuous coefficients*. The geometric meaning of (3.1.1) is that *the Laplacian vector field Δu is tangential to the image $u(\Omega)$* and hence (3.1.1) is equivalent to the next statement: there exists a vector field

$$\mathrm{A} : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$$

such that

$$\Delta u = \mathrm{D}u\, \mathrm{A} \quad \text{in } \Omega.$$

As we show later, the vector field is generally discontinuous (Lemma 3.2.1).

Our interest in (3.1.1) stems from the fact that it is a constituent component of the p-Laplace PDE system for all $p \in [2, \infty]$. Further, contrary perhaps to appearances, (3.1.1) is in itself a *variational PDE system* but in a non-obvious way. Deferring temporarily the specifics of how exactly (3.1.1) arises and what is the variational principle associated with it, let us recall that, for $p \in [2, \infty)$, the celebrated p-Laplacian is the divergence system

$$\Delta_p u := \mathrm{Div}\big(|\mathrm{D}u|^{p-2}\mathrm{D}u\big) = 0 \quad \text{in } \Omega \tag{3.1.2}$$

and comprises the Euler-Lagrange equation which describes extrema of the model p-Dirichlet integral functional

$$E_p(u) := \int_\Omega |\mathrm{D}u|^p, \quad u \in W^{1,p}(\Omega, \mathbb{R}^N), \tag{3.1.3}$$

in conventional vectorial Calculus of Variations. Above and subsequently, for any $X \in \mathbb{R}^{N\times n}$, the notation $|X|$ symbolises its Euclidean (Frobenius) norm:

$$|X| = \left(\sum_{\alpha=1}^{N}\sum_{i=1}^{n}(\mathbf{X}_{\alpha i})^2\right)^{1/2}.$$

The pair (3.1.2)-(3.1.3) is of paramount important in applications and has been studied exhaustively. The extremal case of $p \to \infty$ in (3.1.2)-(3.1.3) is much more modern and intriguing, in that totally new phenomena arise which are not present in the scalar case. It turns out that one then obtains the following nondivergence PDE system

$$\Delta_\infty u := \Big(\mathrm{D}u \otimes \mathrm{D}u + |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp \otimes \mathrm{I}\Big) : \mathrm{D}^2 u = 0 \quad \text{in } \Omega, \tag{3.1.4}$$

which is known as the ∞-Laplacian. In index from, (3.1.4) reads

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}\Big(\mathrm{D}_i u_\alpha\, \mathrm{D}_j u_\beta + |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^\perp_{\alpha\beta}\,\delta_{ij}\Big)\mathrm{D}^2_{ij}u_\beta \;=\; 0, \quad \alpha = 1, ..., N.$$

The system (3.1.4) plays the role of the Euler-Lagrange equation and arises in connexion with variational problems for the supremal functional

$$E_\infty(u, \mathcal{O}) := \|\mathrm{D}u\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^N),\ \mathcal{O} \Subset \Omega. \tag{3.1.5}$$

The scalar case of $N = 1$ in (3.1.4)-(3.1.5) was pioneered by G. Aronsson in the 1960s [4–8] who initiated the field of Calculus of Variations in L^∞, namely the study of supremal functionals and of their associated equations describing critical points. Since then, the field has developed tremendously and there is an extensive relevant literature (see e.g. [16–19, 21, 26, 46, 72, 73] and the lecture notes [15, 28, 54]). In particular, although vectorial supremal functionals began to be explored early enough, the ∞-Laplace system (3.1.4) which describes the necessary critical conditions in L^∞ in the vectorial case $N \geq 2$ first arose in the early 2010s in [49]. The area is now developing very rapidly due to both the mathematical significance as well as the importance for applications in several areas (see [2, 14, 31, 36, 63], [50, 52, 53, 56–58]).

In this paper we focus on the C^2 case and establish the geometric rigidity and flatness of the images of solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to the nonlinear system (3.1.1), under the assumption that either $\mathrm{D}u$ has rank at most 1, or that $n = 2$ and u has an additively separated form, see (3.1.6). As a consequence, we obtain corresponding flatness results for the images of solutions to (3.1.2) and (3.1.4). Both aforementioned classes of solutions furnish particular examples which provide substantial intuition for the behaviour of general extremal maps in Calculus of Variations in L^∞, see e.g. [9, 10, 28, 50, 53, 54, 63] where solutions of this form have been studied. Obtaining further information for the still largely mysterious behaviour of ∞-Harmonic maps is perhaps the greatest driving force to isolate and study the particular nonlinear system (3.1.1). For example, it is not yet know to what extend the possible discontinuities of the coefficients relates to the failure of absolute minimality.

It is also worth clarifying that, although as it is well-known the Dirichlet problem over a bounded domain may not in general be solvable for the ∞-Laplacian not even in the scalar-valued case, if one does not prescribe boundary values (and we do not in this paper) it can be demonstrated that infinitely many non-trivial classical solutions do exist, in particular of the form arising in this paper (see for instance the explicit constructions of C^2 solutions in [50]). Therefore, the results herein are non-void and numerous solutions as those exhibited herein do exist.

Let us note that the rank-one case includes the scalar and the one-dimensional case (i.e. when $\min\{n, N\} = 1$), although in the case of $N = 1$ (in which the single ∞-Laplacian reduces to $\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0$) (3.1.1) has no bearing since it vanishes identically at any non-critical point.

The effect of (3.1.1) *to the flatness of the image can be seen through the L^∞ variational principle introduced in [52]*, wherein it was shown that solutions to (3.1.1) of constant rank can be characterised as those having minimal area with respect to (3.1.3)-(3.1.5). More precisely, therein the following result was proved:

3.1.1 Theorem [cf. [52, Theorem 2.7, Lemma 2.2]]

Given $N \geq n \geq 1$, let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a C^2 immersion defined on the open set Ω (more generally u can be a map with constant rank of its gradient on Ω). Then, the following statements are equivalent:

1. The map u solves the PDE system (3.1.1) on Ω.

2. For all $p \in [2, \infty]$, for all compactly supported domains $\mathcal{O} \Subset \Omega$ and all C^1 vector fields $\nu : \mathcal{O} \longrightarrow \mathbb{R}^N$ which are normal to the image $u(\mathcal{O}) \subseteq \mathbb{R}^N$ (without requiring to vanish on $\partial\mathcal{O}$), namely those for which $\nu = [\![\mathrm{D}u]\!]^\perp \nu$ in $\mathcal{O}$, we have
$$\|\mathrm{D}u\|_{L^p(\mathcal{O})} \leq \|\mathrm{D}u + \mathrm{D}\nu\|_{L^p(\mathcal{O})}.$$

3. The same statement as in item (2) holds, but only for *some* $p \in [2, \infty]$.

If in addition $p < \infty$ in (2)-(3), then we may further restrict the class of normal vector fields to those satisfying $\nu|_{\partial\mathcal{O}} = 0$ (see Figure 1).

In the paper [52], it was also shown that in the conformal class, (3.1.1) *expresses the vanishing of the mean curvature vector of* $u(\Omega)$.

The effect of (3.1.1) to the flatness of the image can be easily seen in the case of $n = 1 \leq N$ as follows: since
$$[\![u']\!]^\perp u'' = 0 \quad \text{in } \Omega \subseteq \mathbb{R}$$
and in one dimension we have
$$[\![u']\!]^\perp = \begin{cases} \mathrm{I} - \dfrac{u' \otimes u'}{|u'|^2}, & \text{on } \{u' \neq 0\}, \\ \mathrm{I}, & \text{on } \{u' = 0\}, \end{cases}$$
we therefore infer that $u'' = fu'$ on the open set $\{u' \neq 0\} \subseteq \mathbb{R}$ for some function f, readily yielding after an integration that $u(\Omega)$ is necessarily contained in a piecewise polygonal line of $\mathbb{R}^N$. As a generalisation of this fact, our first main result herein is the following:

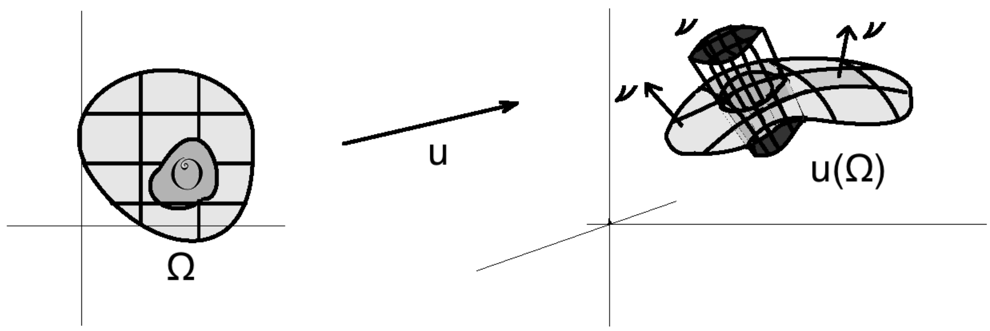


Figure 1. Illustration of the variational principle characterising (3.1.1).

3.1.2 Theorem [Rigidity and flatness of rank-one maps with tangential Laplacian]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system (3.1.1) in Ω, satisfying that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

Let us note that the rank-one assumption for $\mathrm{D}u$ is equivalent to the existence of two vector fields $\xi : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ and $a : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$ such that $\mathrm{D}u = \xi \otimes a$ in Ω.

Example 3.1.3 below shows that Theorem 3.1.2 is optimal and in general rank-one solutions to the system (3.1.1) can not have affine image but only piecewise affine.

3.1.3 Example

Consider the C^2 rank-one map $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$u(x,y) = \begin{cases} (-x^4, x^4), & x \leq 0,\ y \in \mathbb{R}, \\ (+x^4, x^4), & x > 0,\ y \in \mathbb{R}. \end{cases}$$

Then, $u = \nu \circ f$ with $\nu : \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $\nu(t) = (t, |t|)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x,y) = \mathrm{sgn}(x)x^4$ (see Figure 2).

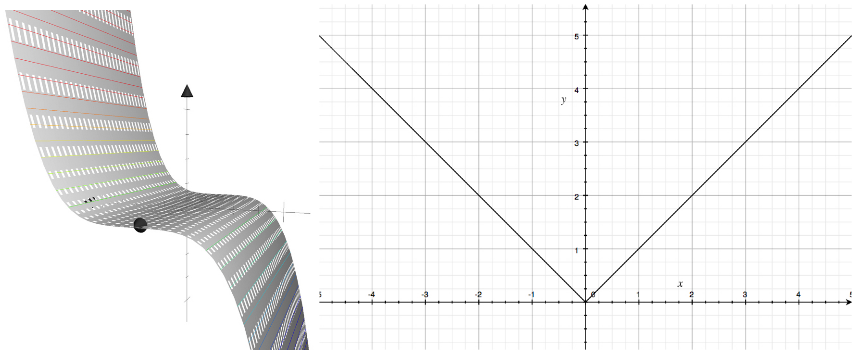


Figure 2. The graph of the function f and the image of the curve ν comprising u.

It follows that u solves (3.1.1) on $\mathbb{R}^2$: indeed, Δu is a non-vanishing vector field on $\{x \neq 0\}$, being tangential to the image thereon since it is parallel to the derivative $\nu'(t) = (1, \pm 1)$ for $t \neq 0$. On the other hand, on $\{x = 0\}$ we have that $\Delta u = 0$. However, the image $u(\mathbb{R}^2)$ of u is piecewise affine but not affine and equals $\nu(\mathbb{R})$. Note that (3.1.1) is undetermined, especially without the requirement of boundary conditions. Therefore, the point of this example is to show that the solutions in general do not have affine image, although some of them may do, for instance the trivial affine ones.

As a consequence of Theorem 3.1.2, we obtain the next result regarding the rigidity of p-Harmonic maps for $p \in [2, \infty)$ which complements one of the results in the paper [53] wherein the case $p = \infty$ was considered.

3.1.4 Corollary [Rigidity of p-Harmonic maps, cf. [53]]

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a p-Harmonic map in Ω for some $p \in [2, \infty)$, that is u solves (3.1.2). Suppose that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(\mathrm{D}u) \leq 1 \quad \text{in } \Omega.$$

Then, the same result as in Theorem 3.1.2 is true.

In addition, there exists a partition of Ω to at most countably many Borel sets, where each set of the partition is a non-empty open set with a (perhaps empty) boundary portion, such that, on each of these, u can be represented as

$$u = \nu \circ f.$$

Here, f is a scalar C^2 p-Harmonic function (for the respective $p \in [2, \infty)$), defined on an open neighbourhood of the Borel set, whilst $\nu : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Lipschitz curve which is twice differentiable and with unit speed on the image of f.

Now we move on to discuss our second main result which concerns the rigidity of solutions $u : \mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (3.1.1) for $N \geq 2$, having the additively separated form

$$u(x, y) \ = \ f(x) - f(y) \tag{3.1.6}$$

for some curve $f : \mathbb{R} \longrightarrow \mathbb{R}^N$. Solutions of this form are very important in relation to the ∞-Laplacian. If $N = 1$, all ∞-Harmonic functions of this form after a normalisation reduce to the so-called Aronsson solution on $\mathbb{R}^2$

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

which is the standard explicit example of a non-C^2 ∞-Harmonic function with conjectured optimal regularity. In the vectorial case, the family of separated solutions is quite large. For $N = 2$, a large class of such vectorial solutions was constructed in [50] and is given by

$$u(x, y) \ = \int_x^y \big(\cos(K(t)), \sin(K(t))\big)\, \mathrm{d}t$$

with K a function in $C^1(\mathbb{R})$ satisfying certain general conditions. The simplest non-trivial example of an ∞-Harmonic map with this form (defined on the strip $\{|x - y| < \pi/4\} \subseteq \mathbb{R}^2$) is given by the choice $K(t) = t$. Our second main result asserts that solutions of separated form to (3.1.1) have images which are piecewise affine, contained in a union of intersecting planes of $\mathbb{R}^N$. More precisely, we have:

3.1.5 Theorem [Rigidity and flatness of maps with tangential Laplacian in separated form]

Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let also $N \geq 2$. Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a classical solution to the nonlinear system (3.1.1) in Ω, having the separated form $u(x, y) = f(x) - f(y)$, for some curve $f \in (W^{3,p} \cap C^2)(\mathbb{R}, \mathbb{R}^N)$ and some $p > 1$.

Then, the image $u(\Omega)$ of the solution is contained in an at most countable union of affine planes in $\mathbb{R}^N$.

In addition, the proof of Theorem 3.1.5 shows that every connected component of the set $\{\mathrm{rk}(\mathrm{D}u) = 2\}$ is contained entirely in an affine plane and every connected component of the set $\{\mathrm{rk}(\mathrm{D}u) \leq 1\}$ is contained entirely in an affine line.

Note that our result is trivial in the case that $N = n = 2$ since the codimension $N - n$ vanishes. Additionally, due to the regularity of the solutions, if a C^2 mapping has piecewise affine image, then second derivatives must vanish when first derivatives vanish at the "breaking points". Further, one might also restrict their attention to domains of rectangular shape, since any map with separated form can be automatically extended to the smallest rectangle containing the domain.

Also, herein we consider only the illustrative case of $n = 2 < N$ and do not discuss more general situations, since numerical evidence obtained in [63] suggests that Theorem 3.1.5 does not hold in general for solutions in non-separated form.

In this paper we try to keep the exposition as simple as possible and therefore we refrain from discussing generalised solutions to (3.1.1) and (3.1.4) (or (3.1.2)). We confine ourselves to merely mentioning that in the scalar case, ∞-Harmonic functions are understood in the viscosity sense of Crandall-Ishii-Lions (see e.g. [28, 54]), whilst in the vectorial case a new candidate theory for systems has been proposed in [58] which has already borne significant fruit in [14, 31, 56–58, 63].

We now expound on how exactly the nonlinear system (3.1.1) arises from (3.1.2) and (3.1.4). By expanding the derivatives in (3.1.2) and normalising, we arrive at

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u + \frac{|\mathrm{D}u|^2}{p-2} \Delta u = 0. \tag{3.1.7}$$

For any $X \in \mathbb{R}^{N \times n}$, let $[\![X]\!]^{\parallel}$ denote the orthogonal projection on the range of the linear map $X : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$[\![X]\!]^{\parallel} := \mathrm{Proj}_{\mathrm{R}(X)}. \tag{3.1.8}$$

Since the identity of $\mathbb{R}^N$ splits as $\mathrm{I} = [\![\mathrm{D}u]\!]^{\parallel} + [\![\mathrm{D}u]\!]^{\perp}$, by expanding Δu with respect to these projections,

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \,+\, \frac{|\mathrm{D}u|^2}{p-2}[\![\mathrm{D}u]\!]^{\parallel}\Delta u = -\frac{|\mathrm{D}u|^2}{p-2}[\![\mathrm{D}u]\!]^{\perp}\Delta u.$$

The mutual perpendicularity of the vector fields of the left and right hand side leads via a renormalisation argument (see e.g. [49, 52, 53]) to the equivalence of the p-Laplacian with the pair of systems

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u \,+\, \frac{|\mathrm{D}u|^2}{p-2}[\![\mathrm{D}u]\!]^{\parallel}\Delta u = 0\ , \quad |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^{\perp}\Delta u = 0. \tag{3.1.9}$$

The ∞-Laplacian corresponds to the limiting case of (3.1.9) as $p \to \infty$, which takes the form

$$\mathrm{D}u \otimes \mathrm{D}u : \mathrm{D}^2 u = 0\ , \quad |\mathrm{D}u|^2[\![\mathrm{D}u]\!]^{\perp}\Delta u = 0. \tag{3.1.10}$$

Hence, the ∞-Laplacian (3.1.4) actually consists of the two independent systems in (3.1.10) above. The system $|\mathrm{D}u|^2[\![\mathrm{D}u]\!]^{\perp}\Delta u = 0$ is, at least on $\{\mathrm{D}u \neq 0\}$, equivalent to (3.1.1). Note that in the scalar case of $N = 1$ as well as in the case of submersion solutions (for $N \leq n$), the second system trivialises.

We conclude the introduction with a geometric interpretation of the nonlinear system (3.1.1), which can be expressed in a more geometric language as follows:[1] Suppose that $u(\Omega)$ is a C^2 manifold and let $\mathbf{A}(u)$ denote its second fundamental form. Then

$$[\![\mathrm{D}u]\!]^{\perp}\Delta u \,=\, -\,\mathrm{tr}\,\mathbf{A}(u)(\mathrm{D}u, \mathrm{D}u).$$

The tangential part $[\![\mathrm{D}u]\!]^{\parallel}\Delta u$ of the Laplacian is commonly called the *tension field* in the theory of *Harmonic maps* and is symbolised by $\tau(u)$ (see e.g. [69]). Hence, we have the orthogonal decomposition

$$\Delta u \,=\, \tau(u) \,-\, \mathrm{tr}\,\mathbf{A}(u)(\mathrm{D}u, \mathrm{D}u).$$

Therefore, in the case of higher regularity of the image of u, we obtain that the nonlinear system

$$\Delta u \,=\, \tau(u) \quad \text{in } \Omega, \tag{3.1.11}$$

is a further geometric reformulation of our PDE system (3.1.1).

3.2 Proofs

In this section we prove the results of this paper. Before delving into that, we present a result of independent interest in which we represent explicitly the vector field A arising in the parametric system $\Delta u = \mathrm{D}u\,\mathrm{A}$, in the illustrative case of $n = 2$.

We will be using the symbolisations "cof", "det" and "rk" to denote the cofactor

[1] This fact has been brought to our attention by Roger Moser.

matrix, the determinant function and the rank of a matrix, respectively.

3.2.1 Lemma [Representation of A]

Let $u \in C^2(\Omega, \mathbb{R}^N)$ be given, $\Omega \subseteq \mathbb{R}^2$ open, $N \geq 2$. The following are equivalent:

1. The map u is a solution to the PDE system (3.1.1).
2. There exists a vector field $\mathrm{A} : \mathbb{R}^2 \supseteq \Omega \longrightarrow \mathbb{R}^N$ such that

$$\Delta u = \mathrm{D}u \, \mathrm{A} \quad \text{in } \Omega.$$

In (2), as A one might choose

$$\bar{\mathrm{A}} := \begin{cases} \dfrac{\mathrm{cof}\big(\mathrm{D}u^\top \mathrm{D}u\big)^\top}{\det\big(\mathrm{D}u^\top \mathrm{D}u\big)} (\mathrm{D}u)^\top \Delta u, & \text{on } \{\mathrm{rk}(\mathrm{D}u) = 2\}, \\ (\Delta u)^\top \dfrac{\mathrm{D}u \, \mathrm{D}u^\top}{|\mathrm{D}u \, \mathrm{D}u^\top|^2} \mathrm{D}u, & \text{on } \{\mathrm{rk}(\mathrm{D}u) = 1\}, \\ 0, & \text{on } \{\mathrm{rk}(\mathrm{D}u) = 0\}. \end{cases}$$

A is uniquely determined on $\{\mathrm{rk}(\mathrm{D}u) = 2\}$ but not on $\{\mathrm{rk}(\mathrm{D}u) < 2\}$ and any other A has the form $\bar{\mathrm{A}} + V$, where $V(x)$ lies in the nullspace of $\mathrm{D}u(x)$, $x \in \Omega$.

3.2.2 Proof of Lemma 3.2.1

The equivalence between (1)-(2) is immediate, therefore it suffices to show that $\bar{\mathrm{A}}$ satisfies $\Delta u = \mathrm{D}u \, \bar{\mathrm{A}}$ and is unique on $\{\mathrm{rk}(\mathrm{D}u) = 2\}$. Let A be as in (2). On $\{\mathrm{rk}(\mathrm{D}u) = 2\}$, the 2×2 matrix-valued map $\mathrm{D}u^\top \mathrm{D}u$ is invertible and

$$\big(\mathrm{D}u^\top \mathrm{D}u\big)^{-1} = \frac{\mathrm{cof}\big(\mathrm{D}u^\top \mathrm{D}u\big)^\top}{\det\big(\mathrm{D}u^\top \mathrm{D}u\big)}.$$

Since $\mathrm{D}u^\top \Delta u = \mathrm{D}u^\top \mathrm{D}u \, A$, we obtain that $A = \bar{A}$.

The claim being obvious for $\{\mathrm{rk}(\mathrm{D}u) = 0\} = \{\mathrm{D}u = 0\}$, it suffices to consider only the set $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ in order to conclude. Thereon we have that $\mathrm{D}u$ can be written as

$$\mathrm{D}u = \xi \otimes a, \quad \text{in } \{\mathrm{rk}(\mathrm{D}u) = 1\},$$

for some non-vanishing vector fields ξ and a. By replacing ξ with $\xi|a|$ and a with $a/|a|$, we may assume $|a| \equiv 1$ throughout $\{\mathrm{rk}(\mathrm{D}u) = 1\}$. If $\Delta u = \mathrm{D}u \, \mathrm{A}$, we have $\Delta u = (\xi \otimes a)\mathrm{A}$ and since any component of A which is orthogonal to a is annihilated, we may replace A by λa for some function λ. Therefore,

$$\Delta u = (\xi \otimes a) \, \mathrm{A} = (\xi \otimes a)(\lambda a) = \xi \lambda |a|^2 = \lambda \xi$$

and hence $\xi \cdot \Delta u = \lambda|\xi|^2$ and also $\xi^\top \mathrm{D}u = a|\xi|^2$. On the other hand, since

$$\mathrm{D}u\, \mathrm{D}u^\top = (\xi \otimes a)(a \otimes \xi) = \xi \otimes \xi, \qquad \big|\mathrm{D}u\, \mathrm{D}u^\top\big| = |\xi|^2$$

we infer that

$$\mathrm{A} = \lambda a = \left(\frac{\Delta u \cdot \xi}{|\xi^2|}\right)\left(\frac{\xi^\top \mathrm{D}u}{|\xi^2|}\right) = \frac{\Delta u^\top (\xi \otimes \xi)\, \mathrm{D}u}{|\xi \otimes \xi|^2} = (\Delta u)^\top \frac{\mathrm{D}u\, \mathrm{D}u^\top}{|\mathrm{D}u\, \mathrm{D}u^\top|^2} \mathrm{D}u,$$

as claimed. $\square$

We now continue with the proof of the main results.

The main analytical tool needed in the proof of Theorem 3.1.2 is the next rigidity theorem for maps whose gradient has rank at most one. It was established in [53] and we recall it below for the convenience of the reader and only in the case needed in this paper.

3.2.3 Theorem [Rigidity of Rank-One maps, cf. [53]]

Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set and u is in $C^2(\Omega, \mathbb{R}^N)$. Then, the following are equivalent:

(i) The map u satisfies that $\mathrm{rk}(Du) \leq 1$ on Ω. Equivalently, there exist vector fields $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $a : \Omega \longrightarrow \mathbb{R}^n$ with $a \in C^1(\Omega, \mathbb{R}^n)$ and $\xi \in C^1(\Omega \setminus \{a = 0\}, \mathbb{R}^N)$ such that

$$\mathrm{D}u = \xi \otimes a, \quad \text{on } \Omega.$$

(ii) There exists Borel subset $\{B_i\}_{i \in \mathbb{N}}$ of Ω such that

$$\Omega = \bigcup_{i=1}^{\infty} B_i$$

and each B_i equals a non-empty connected open set with a (possibly empty) boundary portion, functions $\{f_i\}_{i \in \mathbb{N}} \in C^2(\Omega)$ and curves $\{\nu_i\}_{i \in \mathbb{N}} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ such that, on each B_i the map u has the form

$$u = \nu_i \circ f_i, \quad \text{on } B_i. \tag{3.2.1}$$

Moreover, $|\nu_i'| \equiv 1$ on the interval $f_i(B_i)$, $\nu_i' \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and ν_i'' exists everywhere on $f_i(B_i)$, interpreted as 1-sided derivative on $\partial f_i(B_i)$ (if $f_i(B_i)$ is not open). Also,

$$\left\{ \begin{array}{rcll} \mathrm{D}u &=& (\nu_i' \circ f_i) \otimes \mathrm{D}f_i\ , & \text{on } B_i, \\ \mathrm{D}^2 u &=& (\nu_i'' \circ f_i) \otimes \mathrm{D}f_i \otimes \mathrm{D}f_i\, +\, (\nu_i' \circ f_i) \otimes \mathrm{D}^2 f_i\ , & \text{on } B_i. \end{array} \right. \tag{3.2.2}$$

In addition, the local functions $(f_i)_1^\infty$ extend to a global function $f \in C^2(\Omega)$ with the same properties as above if Ω is contractible (namely, homotopically equivalent to a point).

We may now prove our first main result.

3.2.4 Proof of Theorem 3.1.2

Suppose that $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ is a solution to the nonlinear system (3.1.1) in $C^2(\Omega, \mathbb{R}^N)$ which in addition satisfies that $\mathrm{rk}(\mathrm{D}u) \leq 1$ in Ω. Since $\{\mathrm{D}u = 0\}$ is closed, necessarily its complement in Ω which is $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ is open.

By invoking Theorem 3.2.3, we have that there exists a partition of the open subset $\{\mathrm{rk}(\mathrm{D}u) = 1\}$ to countably many Borel sets $(B_i)_1^\infty$ with respective functions $(f_i)_1^\infty$ and curves $(\nu_i)_1^\infty$ as in the statement such that (3.2.1)-(3.2.2) hold true and in addition

$$\mathrm{D}f_i \neq 0 \quad \text{on } B_i,\ i \in \mathbb{N}.$$

Consequently, on each B_i we have

$$\begin{aligned}
[\![\mathrm{D}u]\!]^\perp \,&=\, [\![(\nu_i' \circ f_i) \otimes \mathrm{D}f_i]\!]^\perp \,=\, \mathrm{I} \,-\, \frac{(\nu_i' \circ f_i) \otimes (\nu_i' \circ f_i)}{|\nu_i' \circ f_i|^2},\\
\Delta u \,&=\, (\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 \,+\, (\nu_i' \circ f_i)\Delta f_i.
\end{aligned}$$

Hence, (3.1.1) becomes

$$\left[\mathrm{I} \,-\, \frac{(\nu_i' \circ f_i) \otimes (\nu_i' \circ f_i)}{|\nu_i' \circ f_i|^2}\right]\Big((\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 \,+\, (\nu_i' \circ f_i)\Delta f_i\Big) = 0,$$

on B_i. Since $|\nu_i|^2 \equiv 1$ on $f_i(B_i)$, we have that ν_i'' is orthogonal to ν_i' thereon and therefore the above equation reduces to

$$(\nu_i'' \circ f_i)|\mathrm{D}f_i|^2 = 0 \quad \text{on } B_i,\ i \in \mathbb{N}.$$

Therefore, ν_i is affine on the interval $f_i(B_i) \subseteq \mathbb{R}$ and as a result $u(B_i) = \nu_i(f_i(B_i))$ is contained in an affine line of $\mathbb{R}^N$, for each $i \in \mathbb{N}$. On the other hand, since

$$u(\Omega) \,=\, u\big(\{\mathrm{D}u = 0\}\big) \bigcup_{i \in \mathbb{N}} u(B_i)$$

and u is constant on each connected component of the interior of $\{\mathrm{D}u = 0\}$, the conclusion ensues by the regularity of u because $u\big(\{\mathrm{D}u = 0\}\big)$ is also contained in the previous union of affine lines. The result ensues. $\square$

Now we establish Corollary 3.1.4 by following similar lines to those of the respective result in [53].

3.2.5 Proof of Corollary 3.1.4

Suppose u is as in the statement of the corollary. By Theorem 3.2.3, there exists, a partition of Ω to Borel sets $\{B_i\}_{i\in\mathbb{N}}$, functions $f_i \in C^2(\Omega)$ and Lipschitz curves $\{\nu_i\}_{i\in\mathbb{N}} : \mathbb{R} \longrightarrow \mathbb{R}^N$ with $|\nu_i'| \equiv 1$ on $f_i(B_i)$, $|\nu_i'| \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and twice differentiable on $f_i(B_i)$, such that $u|_{B_i} = \nu_i \circ f_i$ and (3.2.2) holds as well. Since on each B_i we have

$$|\mathrm{D}u| = \big|(\nu_i' \circ f_i) \otimes \mathrm{D}f_i\big| = |\mathrm{D}f_i|,$$

by (3.1.7) and the above, we obtain

$$\begin{aligned}\big((\nu_i' \circ f_i) \otimes \mathrm{D}f_i\big) \otimes \big((\nu_i' \circ f_i) \otimes \mathrm{D}f\big) : &\Big[(\nu_i'' \circ f_i) \otimes \mathrm{D}f_i \otimes \mathrm{D}f_i \,+\, (\nu_i' \circ f_i) \otimes \mathrm{D}^2 f_i\Big] \\ &+\, \frac{|\mathrm{D}f_i|^2}{p-2}\Big\{(\nu_i' \circ f_i)\,\Delta f_i \,+\, (\nu_i'' \circ f)|\mathrm{D}f_i|^2\Big\} \;=\; 0,\end{aligned}$$

on B_i. Since ν_i'' is orthogonal to ν_i' and also ν_i' has unit length, the above reduces to

$$(\nu_i' \circ f_i)\Big[\mathrm{D}f_i \otimes Df_i : \mathrm{D}^2 f_i \,+\, \frac{|\mathrm{D}f_i|^2}{p-2}\,\Delta f_i\Big] \,+\, \frac{1}{p-2}(\nu_i'' \circ f_i)|\mathrm{D}f_i|^4 \;=\; 0,$$

on B_i. Again by orthogonality, the above is equivalent to the pair of independent systems

$$(\nu_i' \circ f_i)\Big[\mathrm{D}f_i \otimes Df_i : \mathrm{D}^2 f_i \,+\, \frac{|\mathrm{D}f_i|^2}{p-2}\,\Delta f_i\Big] \;=\; 0\;, \quad (\nu_i'' \circ f_i)|\mathrm{D}f_i|^4 \;=\; 0,$$

on B_i. Since $|\nu_i'| \equiv 1$ of $f_i(B_i)$, it follows that $\Delta_p f_i = 0$ on B_i and since $(B_i)_1^\infty$ is a partition of Ω of the form described in the statement, the result ensues by invoking Theorem 3.1.2. $\square$

We may now prove our second main result.

3.2.6 Proof of Theorem 3.1.5

The system $[\![\mathrm{D}u]\!]^\perp \Delta u = 0$ is equivalent to

$$\Delta u = \mathrm{D}u \cdot A \tag{3.2.3}$$

for a vector field A with components a, b. Then (3.2.3) can be rewritten as

$$f''(x) - f''(y) \;=\; a(x,y)f'(x) - b(x,y)f'(y). \tag{3.2.4}$$

The choices $(x,y) = (z, z+t)$ and $(x,y) = (z+t, z)$ in (3.2.4) yield the equations

$$f''(z) - f''(z+t) \;=\; a(z,z+t)f'(z) - b(z,z+t)f'(z+t) \tag{3.2.5}$$

and

$$f''(z+t) - f''(z) \;=\; a(z+t,z)f'(z+t) - b(z+t,z)f'(z) \tag{3.2.6}$$

respectively. Let f_α denote the α-component of f, $\alpha = 1, ..., N$.. By subtracting (3.2.5) from (3.2.6) we get for $t \neq 0$ that

$$2\,\frac{f''_\alpha(z+t) - f''_\alpha(z)}{t} \;=\; \Big(a(z+t,z) \,+\, b(z,z+t)\Big)\frac{f'_\alpha(z+t)-f'_\alpha(z)}{t} + f'_\alpha(z)\left(\frac{a(z+t,z)-a(z,z+t)}{t} \,+\, \frac{b(z,z+t)-b(z+t,z)}{t}\right) \tag{3.2.7}$$

for $\alpha = 1, ..., N$. On the set $\{f'_\alpha = 0\}$, equation (3.2.7) becomes

$$2f'''_\alpha(z) = \big(\bar a(z,z) \,+\, \bar b(z,z)\big) f''_\alpha(z) \tag{3.2.8}$$

as $t \to 0$. Note also that $\{f'_\alpha = 0\}$ is closed and its complement $\{f'_\alpha \neq 0\}$ is open. Now let us set

$$C_\alpha(z,t) \;:=\; \frac{a(z+t,z)-a(z,z+t)}{t} \,+\, \frac{b(z,z+t)-b(z+t,z)}{t}.$$

On $\{f'_\alpha \neq 0\}$, (3.2.7) yields that

$$C_\alpha(z,t) = \frac{1}{f'_\alpha(z)}\left[2\frac{f''_\alpha(z+t)-f''_\alpha(z)}{t} - \Big(a(z{+}t,z)+b(z,z{+}t)\Big)\frac{f'_\alpha(z+t)-f'_\alpha(z)}{t}\right].$$

Fix an index $\alpha \in \{1, ..., N\}$, $\delta > 0$, an infinitesimal sequence $(t_m)_1^\infty$ and consider the inner δ-neighbourhood $\mathcal{O}_\delta$ of the set $\{f'_\alpha \neq 0\}$, namely

$$\mathcal{O}_\delta \;:=\; \Big\{x \in \mathbb{R}^n \;:\; f'_\alpha(x) \neq 0 \;\text{ and }\; \mathrm{dist}\big(x, \partial\{f'_\alpha \neq 0\}\big) > \delta\Big\}.$$

Then for any fixed $\delta > 0$ small, there exists a constant $c_\delta > 0$ such that along the sequence $t_m \to 0$ we have

$$\begin{aligned}
\big\|C_\alpha(\cdot,t_m)\big\|_{L^p(\mathcal{O}_\delta)} &\leq 2\left\|\frac{1}{f'_\alpha(\cdot)}\frac{f''_\alpha(\cdot+t_m)-f''_\alpha(\cdot)}{t_m}\right\|_{L^p(\mathcal{O}_\delta)} \\
&\quad + \|a+b\|_{L^\infty(\Omega)}\left\|\frac{1}{f'_\alpha(\cdot)}\frac{f'_\alpha(\cdot+t_m)-f'_\alpha(\cdot)}{t_m}\right\|_{L^p(\mathcal{O}_\delta)} \\
&\leq \frac{1}{c_\delta}\left(2\|f'''_\alpha\|_{L^p(\mathcal{O}_\delta)} + \|a+b\|_{L^\infty(\Omega)}\|f''_\alpha\|_{L^p(\mathcal{O}_\delta)}\right) \\
&\leq \frac{1}{c_\delta}\left(2\|f'''\|_{L^p(\mathbb{R})} + \|a+b\|_{L^\infty(\Omega)}\|f''\|_{L^p(\mathbb{R})}\right).
\end{aligned} \tag{3.2.9}$$

Note that the right hand side of the above estimate is bounded uniformly in $m \in \mathbb{N}$ as $f''' \in L^p(\mathbb{R}, \mathbb{R}^N)$ and $f' \in C^1(\mathbb{R}, \mathbb{R}^N)$. By letting $\delta \to 0$ and using a standard diagonal argument, (3.2.9) implies that there exists a function $\bar C_\alpha$ such that

$$C_\alpha(\cdot,t_m) \longrightarrow \bar C_\alpha \quad \text{in} \;\; L^p_{loc}\big(\{f'_\alpha \neq 0\}\big),$$

as $m \to \infty$ along a subsequence of indices $(m_k)_1^\infty$. As a result, (3.2.7) becomes

$$2f_\alpha'''(z) = \big(\bar{a}(z,z) + \bar{b}(z,z)\big) f_\alpha''(z) \;+\; f_\alpha'(z)\bar{C}_\alpha(z) \quad \text{on} \quad \{f_\alpha' \neq 0\}, \tag{3.2.10}$$

for any $\alpha = 1, ..., N$. Combining equations (3.2.8) and (3.2.10), we infer that there exist measurable functions $A, B : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f''' \;=\; Af' \,+\, Bf'' \qquad \text{a.e. on } \mathbb{R}. \tag{3.2.11}$$

The goal in now to show that (3.2.11) implies that the torsion of the curve f vanishes, at least on a union of subintervals of $\mathbb{R}$. The idea to project on three-dimensional subspaces of $\mathbb{R}^N$ in order to utilise standard ideas of elementary differential geometry of curves.

To this end, let $P_3 : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the orthogonal projection on a $3D$ subspace $V_3 \equiv P_3(\mathbb{R}^N)$ of $\mathbb{R}^N$. The choice of 3-dimensional subspaces owes to the fact that we would like to use the classical formulas of differential geometry of curves in the Euclidean space. Then, $P_3 f : \mathbb{R} \longrightarrow V_3 \cong \mathbb{R}^3$ is a curve in $\mathbb{R}^3$, which is C^2. By (3.2.11) we have,

$$(P_3 f)''' = A(P_3 f)' + B(P_3 f)'' \quad \text{ a.e. on } \mathbb{R}.$$

Let $''\times''$ denote the cross (exterior) product in $\mathbb{R}^3$. Then, the curvature of $P_3 f$ is given by

$$\kappa = |(P_3 f)' \times (P_3 f)''|$$

and, on $\{\kappa \neq 0\}$, the torsion is given by

$$\tau = \frac{[(P_3 f)' \times (P_3 f)''] \cdot (P_3 f)'''}{|(P_3 f)' \times (P_3 f)''|^2}.$$

Note that $\{\kappa \neq 0\}$ is open, as $P_3 f$ is C^2. Then, we have:

- On the topological interior $\mathrm{int}(\{\kappa = 0\}), P_3 f$ is contained in an affine line of V_3.

- On the topological interior $\mathrm{int}(\{\kappa \neq 0\}), P_3 f$ is planar and hence contained in affine plane of V_3.

Since $\partial(\{\kappa = 0\})$ is nowhere dense, it follows that $f(\partial(\{\kappa = 0\}))$ is contained in the boundary of an affine plane or an affine line. Hence, we have that, for any projection $P_3 f$ on a $3D$ subspace of $\mathbb{R}^N$, the projected curve is contained in an at most countable union of affine planes and lines. Therefore, the same is true for f itself by elementary analytic geometry: if all 3-dimensional projections of the image set in the space R^N for $n \geq 3$ are planes or lines, the same is true for the image itself. The conclusion follows. $\square$

Chapter 4

Explicit ∞-harmonic functions in high dimensions

4.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in C^2(\Omega)$ a continuous twice differentiable function. In this paper we study the existence of solutions to the PDE

$$\Delta_\infty u := \sum_{i,j=1}^{n} \mathrm{D}_i u \, \mathrm{D}_j u \, \mathrm{D}^2_{ij} u = 0 \tag{4.1.1}$$

of the form

$$u(x) = \prod_{i=1}^{n} f_i(x_i),$$

where f_i are possibly non-linear for $1 \leq i \leq n$, and $x = (x_1, ..., x_n)^\top$, $x \in \Omega$. Solutions of this form are called *separated* ∞-$harmonic functions$. In the above $\mathrm{D}_i \equiv \frac{\partial}{\partial x_i}$ and $\mathrm{D}^2_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}$. The equation (4.1.1) is called ∞-Laplacian (being a special case of the so-called more general the Aronsson equation) and it arises in Calculus of Variations in L^∞ as the analogue of the *Euler-Lagrange* equation of the functional

$$\mathrm{E}_\infty(u, \mathcal{O}) := ||\mathrm{D}u||_{L^\infty(\mathcal{O})}, \qquad \mathcal{O} \Subset \Omega, \qquad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}).$$

These objects first arose in the work of G. Aronsson in the 1960s (see [6],[7]) and nowadays this is an active field of research for vectorial case $N \geq 2$ for $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ which has begun much more recently in 2010s (see e.g. [49]). Since then, the field has been developed enormously by N. Katzourakis in the series of papers ([50–53, 55–59]) and also in collaboration with the author, Abugirda, Croce, Manfredi, Moser, Parini, Pisante and Pryer ([14], [2], [31], [60], [61], [62], [63–65]). A standard difficulty of (4.1.1) is that it is nondivergence form equation and since in general smooth solutions do not exist, the definition of generalised solutions is an issue. To this end, the theory of viscosity solutions of Crandall-Ishii-Lions is utilised (see e.g. [54]).

In this paper all the *separated* ∞-*harmonicfunctions* are found for $n = 2$ in polar coordinates and for all $n \geq 2$ in cartesian coordinates. Some of these new solutions derived herein coincide with previously known classes of solutions. For instance, the well-known G. Aronsson's solution $u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ which has a $C^{1,1/3}$ regularity, described in Remark 4.2.2. Also M.-F. Bidaut-Veron, M. Garcia Huidobro and L. Veron have found solutions in [20] which coincide with first two solutions of the theorem 4.1.1. In addition I.L. Freire, A. C. Faleiros have found solutions of (4.1.1) in [44], but only one of their non-trivial solutions coincides with a particular case of Theorem 4.1.2 when $A = 1$. There may exist other additional solutions but this topic is not discussed herein.

The main results of this paper are contained in the following theorems.

4.1.1 Theorem [Separated two - dimensional ∞ - harmonic functions in polar coordinates]

Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^2(\Omega)$ *separated* ∞-*harmonicfunction* of the ∞-Laplace equation in polar coordinates

$$u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 = 0 \tag{4.1.2}$$

of the form $u(r,\theta) = f(r)g(\theta)$.

(i) Assume $|f(r)| = r^A$ and $|g(\theta)| = e^{B\theta}$, where A and B are any constants, then

$$A^2 - A + B^2 = 0$$

or

(ii) Assume $|f(r)| = r^A$ and $|g(\theta)| = |g(\theta_0)|e^{\int_{\theta_0}^{\theta} G(t)dt}$, then G satisfies the following

$$t + c = \begin{cases} -\arctan\frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}}\arctan\frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan\frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}}\ln\left|\frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}}\right|, & \text{if } A^2 - A < 0, \end{cases}$$

where c is any constant, provided RHS is well defined.

or

(iii) Assume $|g(\theta)| = e^{B\theta}$ and $|f(r)| = |f(r_0)|e^{\int_{r_0}^{r}\frac{\Phi(t)}{t}dt}$, where Φ satisfies the following

$$\ln|t| + c = \begin{cases} \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| - \frac{1}{2}\frac{1}{\sqrt{B^2-\frac{1}{4}}}\arctan\frac{\Phi(t)-\frac{1}{2}}{\sqrt{B^2-\frac{1}{4}}}, & \text{if } B^2-\frac{1}{4}>0 \\ \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| + \frac{1}{2}\frac{1}{\Phi(t)-\frac{1}{2}}, & \text{if } B^2-\frac{1}{4}=0 \\ \frac{1}{2}\ln\left|\frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2}\right| - \frac{1}{4\sqrt{\frac{1}{4}-B^2}}\ln\left|\frac{\Phi(t)-\frac{1}{2}-\sqrt{\frac{1}{4}-B^2}}{\Phi(t)-\frac{1}{2}+\sqrt{\frac{1}{4}-B^2}}\right|, & \text{if } B^2-\frac{1}{4}<0, \end{cases}$$

where c is any constant, provided RHS is well defined.

4.1.2 Theorem [Separated two-dimensional ∞ - harmonic functions]

Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^2(\Omega)$ *separated* ∞-*harmonicfunction* of the ∞-Laplace equation

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \tag{4.1.3}$$

of the form $u(x,y) = f(x)g(y)$. Then, one of the following holds: either

(i) $|f(x)| = |f(x_0)|e^{A(x-x_0)}$ and $|g(y)| = |g(y_0)|e^{\int_{y_0}^{y} G(t)\,dt}$, where G satisfies

$$t + c = \begin{cases} \frac{1}{G(t)}, & \text{if } A = 0 \\ -\frac{1}{2A}\arctan\frac{G(t)}{A} + \frac{G(t)}{2\left(A^2+G^2(t)\right)}, & \text{otherwise} \end{cases}$$

or

(ii) $|f(x)| = |f(x_0)|e^{\int_{x_0}^{x} F(t)\,dt}$ and $|g(y)| = |g(y_0)|e^{B(y-y_0)}$, where F satisfies

$$t + c = \begin{cases} \frac{1}{F(t)}, & \text{if } B = 0 \\ -\frac{1}{2B}\arctan\frac{F(t)}{B} + \frac{F(t)}{2\left(B^2+F^2(t)\right)}, & \text{otherwise.} \end{cases}$$

4.1.3 Theorem [Separated n-dimensional ∞ - harmonic functions]

Let $n \geq 2$ and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $C^2(\Omega)$ *separated* ∞-*harmonicfunction* of the ∞-Laplace equation

$$\sum_{i,j=1}^{n} \mathrm{D}_i u\, \mathrm{D}_j u\, \mathrm{D}^2_{ij} u \;=\; 0. \tag{4.1.4}$$

Then

$$|f_i(x_i)| = |f_i(x_i^0)|\, e^{A_i(x_i - x_i^0)} \text{ for } 1 \leq i \neq j \leq n$$

and

$$|f_j(x_j)| = |f_j(x_j^0)|\, e^{\displaystyle\int_{x_j^0}^{x_j} F_j(t)\, dt},$$

where F_j satisfies

$$t + c = -\frac{1}{2\Big(\sum_{i\neq j} A_i^2\Big)^{1/2}} \arctan \frac{F_j(t)}{\Big(\sum_{i\neq j} A_i^2\Big)^{1/2}} + \frac{F_j(t)}{2\Big(\sum_{i\neq j} A_i^2 + F_j^2(t)\Big)}.$$

4.2 Proofs of main results

In this section we prove our main results. The general idea of our method, which is essentially the same for all our proofs, is to use a substitution to derive a "better" PDE. Then, we take any points from the domain which are different only in one component put them to the "better" PDE and subtract these two equations from each other to get a "new" PDE.

4.2.1 Proof of Theorem 4.1.1

We can assume that $u \neq 0$ since if u is a solution then $u + c$ is also a solution then the equation (4.1.2) can be written as

$$\frac{u_r^2}{u^2}\frac{u_{rr}}{u} + \frac{2}{r^2}\frac{u_r}{u}\frac{u_\theta}{u}\frac{u_{r\theta}}{u} + \frac{1}{r^4}\frac{u_\theta^2}{u^2}\frac{u_{\theta\theta}}{u} - \frac{1}{r^3}\frac{u_r}{u}\frac{u_\theta^2}{u^2} = 0. \tag{4.2.1}$$

Let $F = \frac{u_r}{u}$ and $G = \frac{u_\theta}{u}$, then $F_r + F^2 = \frac{u_{rr}}{u}$, $G_\theta + G^2 = \frac{u_{\theta\theta}}{u}$ and $\frac{1}{2}F_\theta + \frac{1}{2}G_r + FG = \frac{u_{r\theta}}{u}$. Note that $u(r,\theta) = f(r)g(\theta)$, hence F does not depend on θ, since $F(r,\theta) = \frac{f'(r)}{f(r)}$. Analogously G does not depend on r, since $G(r,\theta) = \frac{g'(\theta)}{g(\theta)}$. Thus (4.2.1) becomes

$$F^2F_r + F^4 + \frac{2}{r^2}F^2G^2 + \frac{1}{r^4}G^2G_\theta + \frac{1}{r^4}G^4 - \frac{1}{r^3}FG^2 = 0. \tag{4.2.2}$$

Set $\Phi = Fr$, then $r\Phi_r - \Phi = F_r r^2$. Multiplying (4.2.2) by r^4, we have

$$(\Phi^2 + G^2)(\Phi^2 + G^2 - \Phi) + r\Phi^2\Phi_r + G^2G_\theta = 0. \tag{4.2.3}$$

We have the following 4 cases for the functions Φ and G:
Case (A) Φ and G are constant functions.
Case (B) Φ is constant and G is non-constant functions.
Case (C) Φ is non-constant and G is constant functions.
Case (D) Φ and G are non-constant functions.

Case (A) Let $\Phi \equiv A$ and $G \equiv B$, then (4.2.3) gives $A \equiv B \equiv 0$ or $A^2 - A + B^2 = 0$ which can be rewritten as $(A - \frac{1}{2})^2 + B^2 = \frac{1}{4}$ and as the consequent of substitutions $f(r) = r^A$ and $g(\theta) = e^{B\theta}$ up to a constants.

Case (B) Let $\Phi \equiv A$, then G is a non-constant function satisfying (4.2.3)

$$(A^2 + G^2)(A^2 + G^2 - A) + G^2 G_\theta = 0. \tag{4.2.4}$$

Therefore

$$(A^2 + G^2)(A^2 + G^2 - A) = -G^2 \frac{dG}{d\theta}.$$

Consequently

$$\begin{aligned}\int d\theta &= \int \frac{-G^2}{(A^2 + G^2)(A^2 + G^2 - A)}\, dG \\ &= \int \frac{-A}{A^2 + G^2}\, dG - \int \frac{1 - A}{A^2 - A + G^2}\, dG.\end{aligned} \tag{4.2.5}$$

Hence

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2 - A}} \arctan \frac{G(t)}{\sqrt{A^2 - A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A - A^2}} \ln \left| \frac{G(t) - \sqrt{A - A^2}}{G(t) + \sqrt{A - A^2}} \right|, & \text{if } A^2 - A < 0. \end{cases}$$

Case (C) Let $G \equiv B$, then Φ is a non-constant function satisfying (4.2.3)

$$(\Phi^2 + B^2)(\Phi^2 + B^2 - \Phi) + r\Phi^2 \Phi_r = 0. \tag{4.2.6}$$

Therefore

$$(\Phi^2 + B^2)(\Phi^2 + B^2 - \Phi) = -r\Phi^2 \frac{d\Phi}{dr}.$$

Consequently

$$\begin{aligned}\int \frac{1}{r}\, dr &= \int \frac{-\Phi^2}{(\Phi^2 + B^2)(\Phi^2 - \Phi + B^2)}\, d\Phi \\ &= \int \frac{\Phi}{\Phi^2 + B^2}\, d\Phi - \int \frac{\Phi - \frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}}\, d\Phi - \int \frac{\frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}}\, d\Phi.\end{aligned} \tag{4.2.7}$$

Hence

$$\ln|t| + c = \begin{cases} \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{2} \frac{1}{\sqrt{B^2 - \frac{1}{4}}} \arctan \frac{\Phi(t) - \frac{1}{2}}{\sqrt{B^2 - \frac{1}{4}}}, & \text{if } B^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & \text{if } B^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{4\sqrt{\frac{1}{4} - B^2}} \ln \left| \frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - B^2}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - B^2}} \right|, & \text{if } B^2 - \frac{1}{4} < 0. \end{cases}$$

Case (D) Let Φ and G be non-constant functions, then there exist $r_1 \neq r_2$ and $\theta_1 \neq \theta_2$ such that $\Phi(r_1) \neq \Phi(r_2)$ and $G(\theta_1) \neq G(\theta_2)$ satisfying (4.2.3). Thus

$$r_1\Phi(r_1)^2\Phi_r(r_1) - \Phi^3(r_1) + \Phi(r_1)^4 + 2\Phi(r_1)^2G(\theta)^2 + G(\theta)^2G_\theta(\theta) + G(\theta)^4 - \Phi(r_1)G(\theta)^2 = 0 \tag{4.2.8}$$

$$r_2\Phi(r_2)^2\Phi_r(r_2) - \Phi^3(r_2) + \Phi(r_2)^4 + 2\Phi(r_2)^2G(\theta)^2 + G(\theta)^2G_\theta(\theta) + G(\theta)^4 - \Phi(r_2)G(\theta)^2 = 0. \tag{4.2.9}$$

Subtracting (4.2.8) and (4.2.9) we get for any θ

$$\begin{aligned} G^2(\theta)(\Phi(r_1) - \Phi(r_2))(2(\Phi(r_1) + \Phi(r_2)) - 1) = r_2\Phi^2(r_2)\Phi_r(r_2) - \Phi(r_2)^3 + \Phi(r_2)^4 \\ - r_1\Phi^2(r_1)\Phi_r(r_1) + \Phi(r_1)^3 - \Phi(r_1)^4. \end{aligned} \tag{4.2.10}$$

Let's consider two cases.

Case (I) If there exists $r_1 \neq r_2$ such that $2(\Phi(r_1) + \Phi(r_2)) - 1 \neq 0$, then (4.2.10) gives that $G^2(\theta)$ is a constant, hence $G(\theta)$ is a step function.

Case (II) For any $r_1 \neq r_2$ we have $2(\Phi(r_1) + \Phi(r_2)) - 1 = 0$, hence $\Phi(r)$ is a step function.

For both cases we have a contradiction to $C^{1,\alpha}$ regularity for ∞-Harmonic functions in two dimensions (see [39], [75]), since $\Phi(r) = \frac{1}{r}\frac{u_r}{u}$ and $G(\theta) = \frac{u_\theta}{u}$ have to have at least $C^{0,\alpha}$ regularity.

Finally integrating $\frac{f'}{f} = \frac{\Phi}{r}$, $\frac{g'}{g} = G$ and substituting we get $|f(r)| = |f(r_0)|e^{\int_{r_0}^{r}\frac{\Phi(t)}{t}dt}$ and $|g(\theta)| = |g(\theta_0)|e^{\int_{\theta_0}^{\theta}G(t)dt}$, which completes the proof. $\square$

4.2.2 Remark [The Arronson solution]

Let $A = \frac{4}{3}$ in the Theorem 4.1.1ii, then $A^2 - A > 0$ and function G satisfies

$$t + c = -\arctan\frac{3}{4}G(t) + \frac{1}{2}\arctan\frac{3}{2}G(t),$$

which can be rewritten as

$$27G^3(t) + 54G^2(t)\tan 2(t+c) + 32\tan 2(t+c) = 0.$$

Solving a third degree equation with respect to $G(t)$, we get

$$G(t) = -\frac{4}{3}\frac{\tan^{\frac{1}{3}}(t+c) + \tan^{\frac{5}{3}}(t+c) + \tan(t+c)}{1 - \tan^2(t+c)}.$$

Therefore

$$\int G(t)\,dt = \ln\left|\frac{\left(1 - \tan^{\frac{2}{3}}(t+c)\right)\left(1 + \tan^{\frac{2}{3}}(t+c)\right)^{\frac{1}{3}}}{\left(\tan^{\frac{4}{3}}(t+c) - \tan^{\frac{2}{3}}(t+c) + 1\right)^{\frac{2}{3}}}\right| + c.$$

Hence

$$
\begin{aligned}
e^{\int_{\theta_0}^{\theta} G(t)\,dt} &= \frac{|1-\tan^{\frac{2}{3}}(\theta+c)||1+\tan^{\frac{2}{3}}(\theta+c)|^{\frac{1}{3}}}{|\tan^{\frac{4}{3}}(\theta+c)-\tan^{\frac{2}{3}}(\theta+c)+1|^{\frac{2}{3}}}\cdot c(\theta_0)\\
&= \frac{|1-\tan^{\frac{4}{3}}(\theta+c)|}{|1+\tan^{2}(\theta+c)|^{\frac{2}{3}}}\cdot c(\theta_0)\\
&= \left|\cos^{\frac{4}{3}}(\theta+c)-\sin^{\frac{4}{3}}(\theta+c)\right|\cdot c(\theta_0).
\end{aligned}
$$

We can ignore $c(\theta_0)$ since if $c_1 u + c_2$ is a solution then u is also a solution.

Finally

$$
\begin{aligned}
|g(\theta)| &= |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t)} dt\\
&= |g(\theta_0)|\big(\left|\cos^{\frac{4}{3}}(\theta+c)-\sin^{\frac{4}{3}}(\theta+c)\right|\big),\\
|f(r)| &= r^{\frac{4}{3}}.
\end{aligned}
$$

Thus, one of the possible solutions is

$$
\begin{aligned}
u(r,\theta) &= f(r)g(\theta)\\
&= r^{\frac{4}{3}}\big(\cos^{\frac{4}{3}}(\theta+c)-\sin^{\frac{4}{3}}(\theta+c)\big).\\
u(x,y) &= |x|^{\frac{4}{3}}-|y|^{\frac{4}{3}}.
\end{aligned}
$$

4.2.3 Remark [The Aronsson solution]

Let $A=-\frac{1}{3}$ in the Theorem 4.1.1ii, then $A^2-A>0$ and function G satisfies

$$
t+c = \arctan 3G(t) - 2\arctan\frac{3}{2}G(t).
$$

Carrying out a similar series of calculations as in Remark 4.2.2 we can find that possible solutions are $f(r)=r^{-\frac{1}{3}}$ and $g(\theta)=\cos^{\frac{4}{3}}(\frac{\theta+c}{2})-\sin^{\frac{4}{3}}(\frac{\theta+c}{2})$, hence $u(r,\theta)=r^{-\frac{1}{3}}\big(\cos^{\frac{4}{3}}(\frac{\theta+c}{2})-\sin^{\frac{4}{3}}(\frac{\theta+c}{2})\big)$ is the solution of the ∞-Laplace equation which was described in [10] as $u(r,\theta)=r^{-\frac{1}{3}}g(\theta)$, where

$$
g(\theta)=\frac{\cos t}{(1+3\cos^2 t)^{\frac{2}{3}}},\qquad \theta = t-2\arctan\left(\frac{\tan t}{2}\right),\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.
$$

The key fact these two solutions are identically equal is $\tan\frac{\theta}{2}=-\tan^3\frac{t}{2}$.

4.2.4 Proof of Theorem 4.1.2

It is a particular case of the Theorem 4.1.3. □

4.2.5 Proof of Theorem 4.1.3

We can assume that $u \neq 0$ since if u is a solution then $u+c$ is also a solution then equation (4.1.4) can be written as

$$\sum_{i,j=1}^{n} \frac{\mathrm{D}_i u}{u}\frac{\mathrm{D}_j u}{u}\frac{\mathrm{D}_{ij}^2 u}{u} = 0. \tag{4.2.11}$$

Let $F_i = \frac{\mathrm{D}_i u}{u}$ then $\mathrm{D}_i F_i + F_i^2 = \frac{\mathrm{D}_{ii} u}{u}$ and $F_i F_j = \frac{\mathrm{D}_{ij} u}{u}$. Thus (4.2.11) becomes

$$\left(\sum_{i=1}^{n} F_i^2(x_i)\right)^2 + \sum_{i=1}^{n} F_i^2(x_i) D_i F_i(x_i) = 0. \tag{4.2.12}$$

Since $u(x) = \prod_{i=1}^{n} f_i(x_i)$, then F_i depends only on x_i, consequently

$$\mathrm{D}_i F_i(x_i) = F_i'(x_i).$$

Set x^1, $x^2 \in \Omega$ such that $x^1 = (x_1, x_2, ..., x_j^1, ..., x_n)$ and $x^2 = (x_1, x_2, ..., x_j^2, ..., x_n)$, where $x_j^1 \neq x_j^2$ in (4.2.12) and subtract these two equations. We find

$$\begin{aligned}\Big(F_j^2(x_j^1) - F_j^2(x_j^2)\Big)\Big(2\sum_{i\neq j} F_i^2(x_i) + 2F_j^2(x_j^1) + 2F_j^2(x_j^1)\Big) + \\ F_j^2(x_j^1)F_j'(x_j^1) - F_j^2(x_j^2)F_j'(x_j^2) = 0,\end{aligned}$$

assuming $F_j^2(x_j^1) \neq F_j^2(x_j^2)$, we have

$$2\sum_{i\neq j} F_i^2(x_i) = -\frac{F_j^2(x_j^1)F_j'(x_j^1) - F_j^2(x_j^2)F_j'(x_j^2)}{F_j^2(x_j^1) - F_j^2(x_j^2)} - 2F_j^2(x_j^1) - 2F_j^2(x_j^1). \tag{4.2.13}$$

LHS of (4.2.13) does not depend on x_j^1 and x_j^2 so

$$\sum_{i\neq j} F_i^2(x_i) \equiv c$$

for all x_i. Then $F_i(x_i) = A_i$, where A_i is a constant for all $1 \leq i \neq j \leq n$ and hence $|f_i(x_i)| = |f_i(x_i^0)|\, e^{A_i(x_i - x_i^0)}$. Thus (4.2.12) gives

$$\left(\sum_{i\neq j} A_i^2 + F_j^2(x_j)\right)^2 + F_j(x_j)^2 F_j'(x_j) = 0,$$

consequently

$$dx_j = -\frac{F_j^2}{\left(\sum_{i\neq j} A_i^2 + F_j^2\right)^2} dF_j,$$

hence $|f_j(x_j)| = |f_j(x_j^0)|\, e^{\int_{x_j^0}^{x_j} F_j(t)\,dt}$, where $F_j(t)$ satisfies

$$t + c = -\frac{1}{2\sqrt{\sum_{i\neq j} A_i^2}} \arctan \frac{F_j(t)}{\sqrt{\sum_{i\neq j} A_i^2}} + \frac{F_j(t)}{2\left(\sum_{i\neq j} A_i^2 + F_j^2(t)\right)}, \text{ if } \sum_{i\neq j} A_i^2 \neq 0.$$

Otherwise (i.e. if $\sum_{i\neq j} A_i^2 = 0$)

$$F_j^4(x_j) + F_j^2(x_j) F_j'(x_j) = 0,$$

so

$$F_j^2(x_j) + F_j'(x_j) = 0, \text{ since we assume } F_j^2(x_j^1) \neq F_j^2(x_j^2). \tag{4.2.14}$$

Solving (4.2.14) we get $F_j(x_j) = \frac{1}{x_j + c}$. Hence $|f_i(x_i)| = c_i$ for all $i \neq j$ and $|f_j(x_j)| = c_j(|x_j + c|)$, where c and c_i are constants for all $1 \leq i \leq n$.

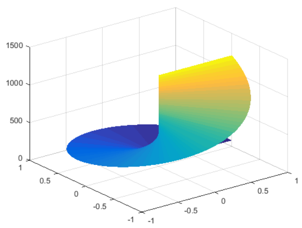
If there is no j such that $F_j^2(x_j^1) \neq F_j^2(x_j^2)$ then $F_j^2(x_j) \equiv c_j$ for all $1 \leq j \leq n$ and (4.2.12) gives that $c_j = 0$ for all $1 \leq j \leq n$. So $|f_i(x_i)| = C_i$, where C_i is constant for all i.

□

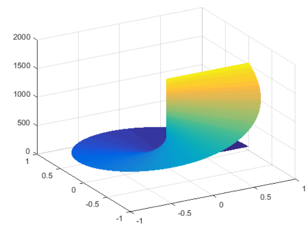
4.3 Numerical approximations of ∞ - harmonic functions

In this section we illustrate the ∞-Harmonic functions derived earlier, depending on the parameter(s)(see Figure 4.1 - 4.4). The results illustrate that we may have a family of solutions depending on the 2π-interval even if the parameter(s) is/are fixed. For example: the solution on Figure 4.2h is a combination of those in Figure 4.2i and Figure 4.2j when θ belongs to 1st and 2nd 2π- interval of the domain respectively. Colours are linear colour scaled from minimum to maximum.

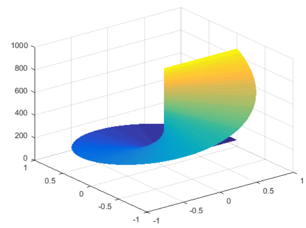
Figure 4.1: The approximation to u of the Theorem 4.1.1 i, depending on the parameters A and B.



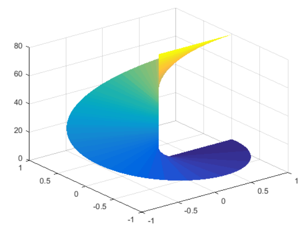
(a) $A = 0.25, B = 0.433$, min $= 79$, max $= 1209$



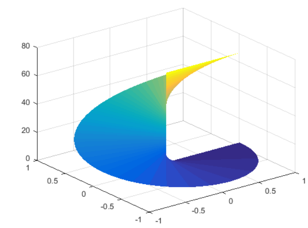
(b) $A = 0.5, B = 0.5$, min $= 66$, max $= 1538$



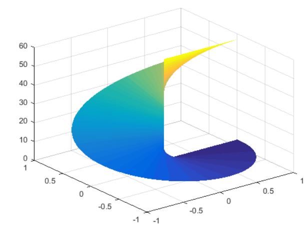
(c) $A = 0.75, B = 0.433$, min $= 57$, max $= 866$



(d) $A = 0.25, B = -0.433$, min $= 5$, max $= 79$

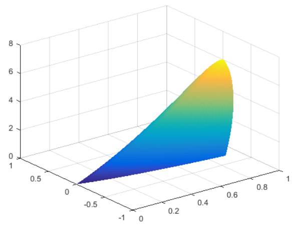


(e) $A = 0.5, B = -0.5$, min $= 2$, max $= 66$

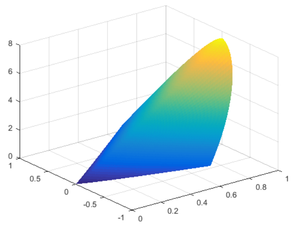


(f) $A = 0.75, B = -0.433$, min $= 3$, max $= 57$

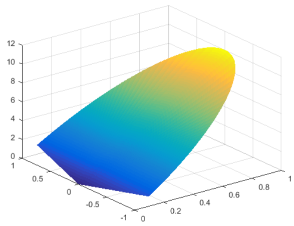
Figure 4.2: The approximation to u of the Theorem 4.1.1 ii, depending on the parameter A.



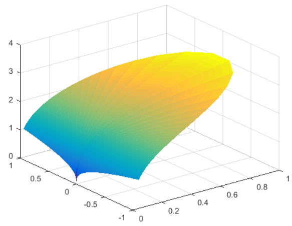
(a) $A = 4/3$, min $= 0$, max $= 6$



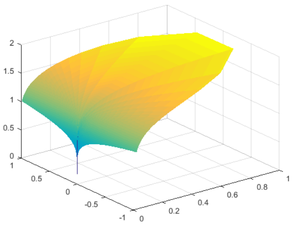
(b) $A = 1.15$, min $= 0$, max $= 7$



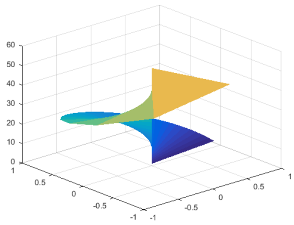
(c) $A = 1$, min $= 0$, max $= 10$



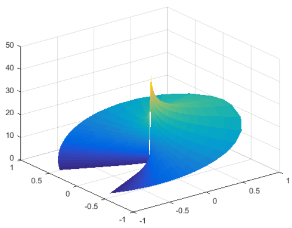
(d) $A = 1/3$, min $= 0$, max $= 3$



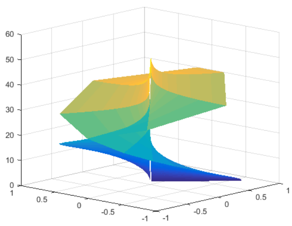
(e) $A = 0.15$, min $= 0$, max $= 1.8$



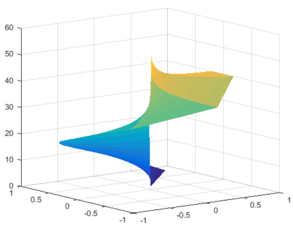
(f) $A = 0$, min $= 1$, max $= 39$



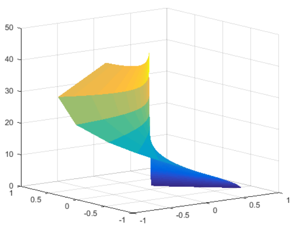
(g) $A = -0.15$, min $= 1$, max $= 41$



(h) $A = -0.05$, min $= 1$, max $= 50$

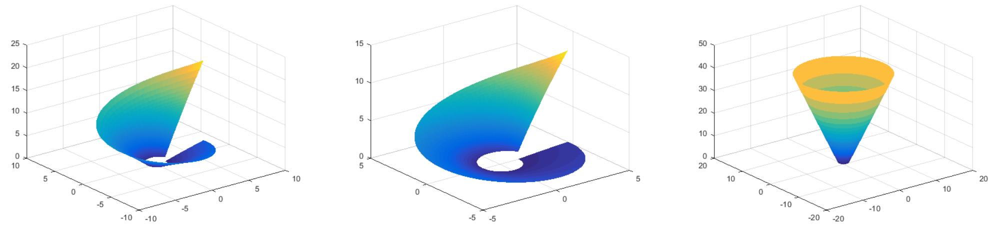


(i) $A = -0.05$, min $= 1$, max $= 50$



(j) $A = -0.05$, min $= 1$, max $= 48$

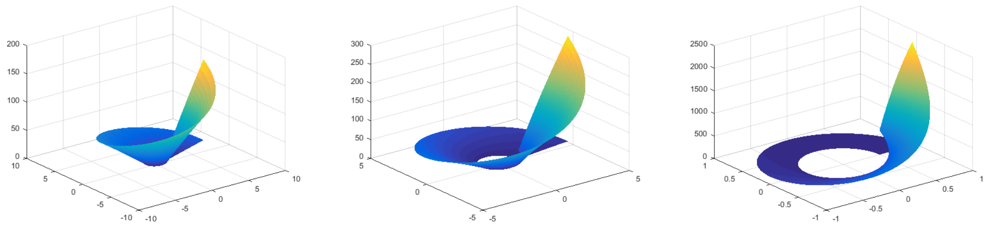
Figure 4.3: The approximation to u of the Theorem 4.1.1 iii, depending on the parameter B.



(a) $B = -1/3$, min $= 0.12$, max $= 20$

(b) $B = -1/2$, min $= 0.04$, max $= 12$

(c) $B = 0$, min $= 1$, max $= 40$

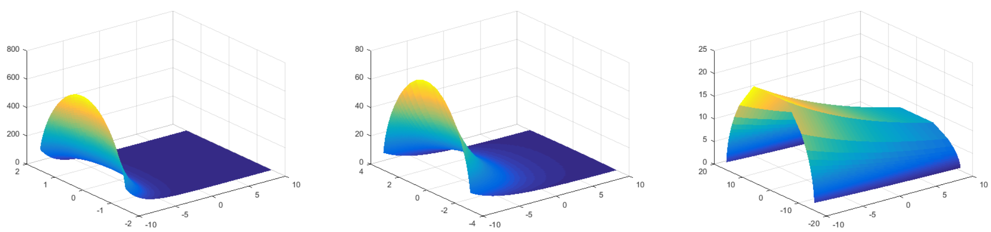


(d) $B = 1/3$, min $= 1$, max $= 163.58$

(e) $B = 1/2$, min $= 1$, max $= 293.26$

(f) $B = 1$, min $= 1$, max $= 2304$

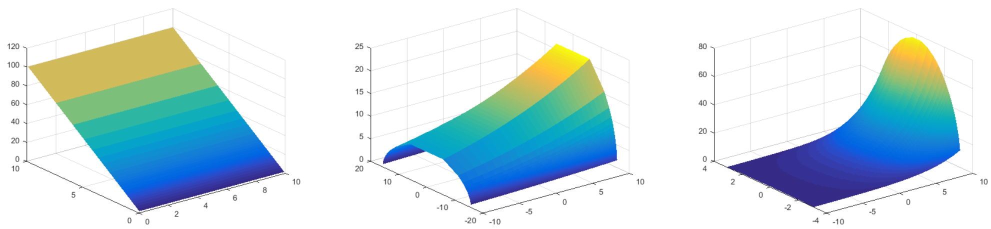
Figure 4.4: The approximation to u of the Theorem 4.1.2 i, depending on the parameter A.



(a) $A = -0.5$, min $= 0.0067$, max $= 663$

(b) $A = -0.25$, min $= 0.0821$, max $= 76$

(c) $A = -0.05$, min $= 0.0665$, max $= 21$



(d) $A = 0$, min $= 1$, max $= 100$

(e) $A = 0.05$, min $= 0.6065$, max $= 21$

(f) $A = 0.25$, min $= 0.0821$, max $= 76$

Chapter 5

Vectorial variational principles in L^∞ and their characterisation through PDE systems

5.1 Introduction

Let $n, N \in \mathbb{N}$ and $\mathrm{H} \in C^2\big(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}\big)$ with $\Omega \subseteq \mathbb{R}^n$ an open set. In this paper we consider the supremal functional

$$\mathrm{E}_\infty(u, \mathcal{O}) := \operatorname*{ess\,sup}_{\mathcal{O}} \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega, \tag{5.1.1}$$

defined on maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. In (5.1.1) and subsequently, we see the gradient as a matrix map $\mathrm{D}u = (\mathrm{D}_i u_\alpha)^{\alpha=1\dots N}_{i=1\dots n} : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N\times n}$. Variational problems for (5.1.1) have been pioneered by Aronsson in the 1960s in the scalar case $N = 1$ ([4]-[8]). Nowadays the study of such functionals (and of their associated PDEs describing critical points) form a fairly well-developed area of vivid interest, called Calculus of Variations in L^∞. For pedagogical general introductions to the theme we refer to [13, 28, 54].

One of the main difficulties in the study of (5.1.1) which prevents us from utilising the standard machinery of Calculus of Variations for conventional (integral) functionals as e.g. in [34] is that it is non-local, in the sense that a global minimisers u of $\mathrm{E}_\infty(\cdot, \Omega)$ in $W^{1,\infty}_g(\Omega; \mathbb{R}^N)$ for some fixed boundary data g may not minimise $\mathrm{E}_\infty(\cdot, \mathcal{O})$ in $W^{1,\infty}_u(\mathcal{O}; \mathbb{R}^N)$. Namely, global minimisers are not generally local minimisers, a property which is automatic for integral functionals. The remedy proposed by Aronsson (adapted) to the vector case is to build locality into the minimality notion:

5.1.1 Definition [Absolute Minimiser]

Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. We say that u is an *absolute minimiser* of (5.1.1) on Ω if

$$\left.\begin{array}{l} \forall\ \mathcal{O} \Subset \Omega, \\ \forall\ \phi \in W^{1,\infty}_0(\mathcal{O};\mathbb{R}^N) \end{array}\right\} \implies \mathrm{E}_\infty(u,\mathcal{O}) \,\leq\, \mathrm{E}_\infty(u+\phi,\mathcal{O}). \tag{5.1.2}$$

In the scalar case of $N=1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([3, 13, 15–18, 25–27, 29, 48, 68, 78]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$\mathrm{H}_P(\cdot,u,\mathrm{D}u)\cdot \mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) \,=\, 0. \tag{5.1.3}$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, $\mathrm{H}_P, \mathrm{H}_\eta, \mathrm{H}_x$ denotes the derivatives of $\mathrm{H}(x,\eta,P)$ with respect to the respective arguments and "$\cdot$" is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (5.1.1) in the general vectorial case of $N \geq 2$ through solvability of associated PDE systems which generalise the Aronsson equation (5.1.3). As the wording suggests and we explain below, when $N \geq 2$ Aronsson's notion of Definition 5.1.1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$\begin{aligned} &\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) \\ &+\, \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\mathrm{H}_P(\cdot,u,\mathrm{D}u)]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big) \,=\, 0. \end{aligned} \tag{5.1.4}$$

In the above, for any linear map $A:\mathbb{R}^n \longrightarrow \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\mathrm{Proj}_{\mathrm{R}(A)^\perp}$ on the orthogonal complement of its range $\mathrm{R}(A) \subseteq \mathbb{R}^N$. We will refer to the PDE system (5.1.4) as the "Aronsson system", in spite of the fact it was actually derived by N.Katzourakis in [49], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞. The Aronsson system was derived through the well-known method of L^p-approximations and is being studied quite systematically since its discovery, see e.g. [49]-[50], [57, 66]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [50, 53]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications, *in this paper we restrict our attention exclusively to* *regular minimisers and solutions.* In general, solutions to (5.1.4) are nonsmooth and the lack of divergence structure combined with its vectorial nature renders its study beyond the reach of viscosity solutions. To this end, the theory of $\mathcal{D}$-solutions introduced in [57] and subsequently utilised in several works (see e.g. [14, 31, 56, 57]) offers a viable alternative for the study of general locally Lipschitz solutions to (5.1.4), and in fact it works far beyond the realm of Calculus of Variations in L^∞. We therefore leave the generalisation of the results herein to a lower regularity setting for future work.

Additionally to absolute minimisers, for reasons to be explained later, in the paper [52] a special case of the next L^∞ variational concept was introduced (therein for $\mathrm{H}(x,\eta,P) = |P|^2$):

5.1.2 Definition [∞-Minimal Map]

Let $u \in C^1(\Omega;\mathbb{R}^N)$. We say that u is an *∞-minimal map* for (5.1.1) on Ω if (i) and (ii) below hold true:

(i) u is a *rank-one absolute minimiser*, namely it minimises with respect to essentially scalar variations vanishing on the boundary along fixed unit directions:

$$\left.\begin{array}{l} \forall\ \mathcal{O} \Subset \Omega,\ \forall\ \xi \in \mathbb{R}^N \\ \forall\ \phi \in C^1_0(\overline{\mathcal{O}};\mathrm{span}[\xi]) \end{array}\right\} \implies \mathrm{E}_\infty(u,\mathcal{O}) \leq \mathrm{E}_\infty(u+\phi,\mathcal{O}). \tag{5.1.5}$$

(ii) u has *∞-minimal area*, namely it minimises with respect to variations which are normal to the range of the matrix field $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ and free on the boundary:

$$\left.\begin{array}{l} \forall\ \mathcal{O} \Subset \Omega,\quad \forall\ \phi \in C^1(\mathbb{R}^n;\mathbb{R}^N) \\ \text{with } \phi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u) = 0 \text{ on } \mathcal{O} \end{array}\right\} \implies \mathrm{E}_\infty(u,\mathcal{O}) \leq \mathrm{E}_\infty(u+\phi,\mathcal{O}). \tag{5.1.6}$$

In the above,

$$C^1_0(\overline{\mathcal{O}};\mathbb{R}^N) := \big\{\psi \in C^1(\mathbb{R}^n;\mathbb{R}^N)\ :\ \psi = 0 \text{ on } \partial\mathcal{O}\big\}.$$

Note also that *when $N = 1$ absolute minimisers and ∞-minimal maps coincide*, at least when $\{\mathrm{H}_P = 0\} \subseteq \{\mathrm{H} = 0\}$. Further, in the event that $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ has discontinuous rank on $\mathcal{O}$, the only continuous normal vector fields ϕ may be only those vanishing on the set of discontinuities.

In [52] it was proved that C^2 ∞-minimal maps of full rank (namely immersions or submersions) are ∞-Harmonic, that is solutions to the so-called ∞-Laplace system. The latter is a special case of (5.1.4), corresponding to the choice $\mathrm{H}(x,\eta,P) = |P|^2$:

$$\mathrm{D}u\,\mathrm{D}\big(|\mathrm{D}u|^2\big)\ +\ |\mathrm{D}u|^2\,[\mathrm{D}u]^\perp\Delta u\ =\ 0. \tag{5.1.7}$$

The fullness of rank was assumed because of the possible discontinuity of the coefficient $[\mathrm{D}u]^\perp$, which may well happen even for smooth solutions (for explicit examples see [50]). In this paper we bypass this difficulty by replacing the orthogonal projection $[\,\cdot\,]^\perp$ by the projection on the subspace of those normal vectors which have local normal C^1 extensions in a open neighbourhood:

5.1.3 Definition [Orthogonal Projection]

Let $V : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N\times n}$ be a matrix field and note that

$$\mathrm{R}(V(x))^\perp = \mathrm{N}(V(x)^\top),$$

where for any $x \in \Omega$, $\mathrm{N}(V(x)^\top)$ is the nullspace of the transpose $V(x)^\top \in \mathbb{R}^{n\times N}$. We define the orthogonal projection

$$[\![V(x)]\!]^\perp := \mathrm{Proj}_{\tilde{\mathrm{N}}(V(x)^\top)}, \quad [\![V(\cdot)]\!]^\perp : \ \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N\times N},$$

where $\tilde{\mathrm{N}}(V(x)^\top)$ is the reduced nullspace, given by

$$\begin{aligned}\tilde{\mathrm{N}}(V(x)^\top) := \Big\{\xi \in \mathrm{N}(V(x)^\top) \ \Big|\ &\exists\, \varepsilon > 0 \ \& \ \exists\, \bar{\xi} \in C^1(\mathbb{R}^n;\mathbb{R}^N) : \\ &\bar{\xi}(x) = \xi \ \& \ \bar{\xi}(y) \in \mathrm{N}(V(y)^\top), \ \forall\, y \in \mathbb{B}_\varepsilon(x)\Big\}.\end{aligned}$$

It is a triviality to check that $\tilde{\mathrm{N}}(V(x)^\top)$ is indeed a vector space and that

$$[\![V(x)]\!]^\perp [V(x)]^\perp = [\![V(x)]\!]^\perp,$$

where $[V(x)]^\perp = \mathrm{Proj}_{\mathrm{N}(V(x)^\top)}$. Note that the definition could be written in a more concise manner by using the algebraic language of *sheaves and germs*, but we refrained from doing so as there is no real benefit in this simple case.

The first main result in this paper is the next variational characterisation of the Aronsson system (5.1.4).

5.1.4 Theorem [Variational Structure of Aronsson's system]

Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a map in $C^2(\Omega;\mathbb{R}^N)$. Then:

(I) If u is a rank-one absolute minimiser for (5.1.1) on Ω (Definition 5.1.2(i)), then it solves

$$\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) \ = \ 0 \ \text{ on } \Omega. \tag{5.1.8}$$

The converse statement is true if in addition H does not depend on $\eta \in \mathbb{R}^N$ and $\mathrm{H}_P(\cdot,\mathrm{D}u)$ has full rank on Ω.

(II) If u has ∞-minimal area for (5.1.1) on Ω (Definition 5.1.2(ii)), then it solves

$$\mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^{\perp}\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\,=\,0\ \text{ on }\Omega. \tag{5.1.9}$$

The converse statement is true if in addition for any $x\in\Omega$, $\mathrm{H}(x,\cdot,\cdot)$ is convex on $\mathbb{R}^n\times\mathbb{R}^{N\times n}$.

(III) If u is ∞-minimal map for (5.1.1) on Ω, then it solves the (reduced) Aronsson system

$$\begin{aligned}\mathrm{A}_\infty u\,:=\,&\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\\ &+\,\mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^{\perp}\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\,=\,0.\end{aligned}$$

The converse statement is true if in addition H does not depend on $\eta\in\mathbb{R}^N$, $\mathrm{H}_P(\cdot,\mathrm{D}u)$ has full rank on Ω and for any $x\in\Omega$ $H(x,\cdot)$ is convex in $\mathbb{R}^{N\times n}$.

The emergence of two distinct sets of variations and a pair of separate PDE systems comprising (5.1.4) might seem at first glance mysterious. However, it is a manifestation of the fact that the (reduced) Aronsson system in fact consists of two linearly independent differential operators because of the perpendicularity between $[\![\mathrm{H}_P]\!]^{\perp}$ and H_P; in fact, one may split $\mathrm{A}_\infty u=0$ to

$$\left\{\begin{aligned}\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\,&=\,0,\\ \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^{\perp}\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\,&=\,0.\end{aligned}\right.$$

Theorem 5.1.4 makes clear that Aronsson's absolute minimisers do **not** characterise the Aronsson system when $N\geq 2$, at least when the additional natural assumptions hold true. This owes to the fact that, unlike the scalar case, *the Aronsson system admits arbitrarily smooth non-minimising solutions*, even in the model case of the ∞-Laplacian. For details we refer to [66].

Since Aronsson's absolute minimisers do not characterise the Aronsson system, the natural question arises as to what is their PDE counterpart. The next theorem which is our second main result answers this question:

5.1.5 Theorem [Divergence PDE characterisation of Absolute minimisers]

Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be a map in $C^1(\Omega;\mathbb{R}^N)$. Fix also $\mathcal{O}\Subset\Omega$ and consider the following statements:

(I) u is a vectorial minimiser of $\mathrm{E}_\infty(\cdot,\mathcal{O})$ in $C^1_u(\overline{\mathcal{O}};\mathbb{R}^N)$[1].

[1] We remind the reader that $u\in C^1_g(\overline{\mathcal{O}};\mathbb{R}^N)$ means $u-g\in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$.

(II) We have

$$\max_{\mathrm{Argmax}\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\}} \Big[\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big] \,\geq\, 0,$$

for any $\psi \in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$.

(III) For any $\psi \in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$, there exists a non-empty compact set

$$\mathrm{K}_\psi \equiv \mathrm{K} \,\subseteq\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\} \tag{5.1.10}$$

such that,

$$\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big)\Big|_{\mathrm{K}} \,=\, 0. \tag{5.1.11}$$

Then, (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III). If additionally $\mathrm{H}(x,\cdot,\cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$ for any fixed $x \in \Omega$, then (III) $\Longrightarrow$ (I) and all three statements are equivalent. Further, any of the statements above are deducible from the statement:

(IV) For any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ satisfying

$$\mathrm{supp}(\sigma) \,\subseteq\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\}, \tag{5.1.12}$$

we have

$$-\,\mathrm{div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\sigma\big) \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\sigma \,=\, 0, \tag{5.1.13}$$

in the dual space $(C^1_0(\overline{\mathcal{O}};\mathbb{R}^N))^*$.

Finally, all statement are equivalent if $\mathrm{K} = \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\}$ in (III) (this happens for instance when the argmax is a singleton set).

The result above provides an interesting characterisation of Aronsson's concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (5.1.13) is

$$\int_{\overline{\mathcal{O}}}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big)\mathrm{d}\sigma \,=\, 0$$

for all $\psi \in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$, where the ":" notation in the PDE symbolises the Euclidean (Frobenius) inner product in $\mathbb{R}^{N\times n}$.

The idea of Theorem 5.1.5 is inspired by the paper [40] of Evans and Yu, wherein a particular case of the divergence system is derived (in the special scalar case $N=1$ for the ∞-Laplacian and only for $\Omega = \mathcal{O}$), as well as by new developments on higher order Calculus of variations in L^∞ in [61, 64, 70].

Note that, it does not suffice to consider only $\Omega = \mathcal{O}$ as in [40] in order to describe absolute minimisers. For a subdomain $\mathcal{O} \subseteq \Omega$, it may well happen that the only measure σ "charging" the points of $\overline{\mathcal{O}}$ where the energy density $\mathrm{H}(\cdot,u,\mathrm{D}u)$ is maximised is the Dirac measure at a single point $x \in \partial\mathcal{O}$. This is for instance the case for the standard "Aronsson solution" of the ∞-Laplacian on $\mathbb{R}^2$, given by

$u(x,y) = |x|^{4/3} - |y|^{4/3}$, as well as for any other ∞-Harmonic function which is nowhere Eikonal (i.e. $|\mathrm{D}u|$ is non-constant on all open subsets).

We conclude this introduction by noting that the two vectorial variational concepts we are considering herein (Definitions 5.1.1-5.1.2) do not exhaust the plethora variational concepts in L^∞. In particular, in the paper [76] the concept of *tight maps* was introduced in the case of $\mathrm{H}(x,\eta,P) = \|P\|$ where $\|\cdot\|$ is the operator norm on $\mathbb{R}^{N\times n}$. Additionally, in the papers [14, 56] a concept of special affine variations was considered which also characterises the Aronsson system, in fact in the generality of merely locally Lipschitz $\mathcal{D}$-solutions. Finally, in the paper [12] new concepts of absolute minimisers for constrained minimisation problems have been proposed, whilst results relevant to variational principles in L^∞ and applications appear in [21, 22, 26, 45, 73, 74].

5.2 Proofs and a maximum-minimum principle for $\mathbf{H}(\cdot, u, \mathrm{D}u)$

In this section we prove our main results Theorems 5.1.4-5.1.5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [52].

5.2.1 Proposition [Maximum-Minimum Principles]

Suppose Let $u \in C^2(\Omega;\mathbb{R}^N)$ be a solution to (5.1.8), such that H satisfies

(a) $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ has full rank on Ω,

(b) there exists $c>0$ such that

$$\big(\xi^\top \mathrm{H}_P(x,\eta,P)\big)\cdot\big(\xi^\top P\big) \geq c\big|\xi^\top \mathrm{H}_P(x,\eta,P)\big|^2,$$

for all $\xi \in \mathbb{R}^N$ and all $(x,\eta,P) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$.

Then, for any $\mathcal{O} \Subset \Omega$ we have:

$$\sup_{\mathcal{O}} \mathrm{H}(\cdot,u,\mathrm{D}u) \;=\; \max_{\partial\mathcal{O}} \mathrm{H}(\cdot,u,\mathrm{D}u), \tag{5.2.1}$$

$$\inf_{\mathcal{O}} \mathrm{H}(\cdot,u,\mathrm{D}u) \;=\; \min_{\partial\mathcal{O}} \mathrm{H}(\cdot,u,\mathrm{D}u). \tag{5.2.2}$$

The proof is based on the usage of the following flow with parameters:

5.2.2 Lemma

Let $u \in C^2(\Omega;\mathbb{R}^N)$. Consider the parametric ODE system

$$\begin{cases} \dot{\gamma}(t) \ = \ \xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}, \quad t \neq 0, \\ \gamma(0) \ = \ x, \end{cases} \tag{5.2.3}$$

for given $x \in \Omega$ and $\xi \in \mathbb{R}^N$. Then, we have

$$\frac{d}{dt}\Big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big) \ = \ \xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{\gamma(t)}, \tag{5.2.4}$$

$$\frac{d}{dt}\xi^\top u\big(\gamma(t)\big) \ \geq \ c\Big|\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big|^2. \tag{5.2.5}$$

5.2.3 Proof of Lemma 5.2.2

The identity (5.2.4) follows by a direct computation and (5.2.3). For the inequality (5.2.5), we have

$$\begin{aligned} \frac{d}{dt}\xi^\top u\big(\gamma(t)\big) \ &= \ \Big(\xi^\top \mathrm{D}u\big(\gamma(t)\big)\Big)\cdot\dot{\gamma}(t) \\ &= \ \Big(\xi^\top \mathrm{D}u\big(\gamma(t)\big)\Big)\cdot\Big(\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big) \\ &\geq \ c\Big|\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big|^2. \end{aligned}$$

The lemma ensues. $\square$

5.2.4 Proof of Proposition 5.2.1

Fix $\mathcal{O} \Subset \Omega$. Without loss of generality, we may suppose $\mathcal{O}$ is connected. Consider first the case where $\mathrm{rk}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big) \equiv n \leq N$. Then, the matrix-valued map $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ is pointwise left invertible. Therefore, by (5.1.8),

$$\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)^{-1}\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) \ = \ 0$$

which, by the connectivity of $\mathcal{O}$, gives $\mathrm{H}(\cdot,u,\mathrm{D}u) \equiv \mathrm{const}$ on $\mathcal{O}$. The latter equality readily implies the desired conclusion. Consider now the case where $\mathrm{rk}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big) \equiv N \leq n$. Fix $x \in \mathcal{O}$ and a unit vector $\xi \in \mathbb{R}^n$ and consider the parametric ODE system (5.2.3) of Lemma 5.2.2. By the fullness of the rank of $\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)$, we have that

$$\big|\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)\big| \ \geq \ c_1 > 0 \quad \text{on } \mathcal{O}.$$

We will now show that the trajectory $\gamma(t)$ reaches $\partial\mathcal{O}$ in finite time. To this end, we estimate

$$\|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\mathrm{diam}(\mathcal{O}) \,\geq\, \|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\big|\gamma(t)\,-\,\gamma(0)\big| \,\geq\, \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\hat{t}}\xi^\top u(\gamma(t))\right| t,$$

for some $\hat{t}\in(0,t)$, by the mean value theorem. Hence,

$$\begin{aligned}
\|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\mathrm{diam}(\mathcal{O}) &\geq \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\hat{t}}\xi^\top u(\gamma(t))\right| t\\
&= \left|\xi^\top \mathrm{D}u(\gamma(\hat{t}))\cdot\dot{\gamma}(\hat{t})\right| t\\
&= \left|\xi^\top \mathrm{D}u(\gamma(\hat{t}))\cdot\Big(\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(\hat{t})}\Big)\right| t\\
&\geq c_0\left|\xi^\top \mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(\hat{t})}\right|^2 t\\
&\geq \big(c_0c_1^2\big)\, t.
\end{aligned}$$

This proves the desired claim. Further, since u solves (5.1.8), by (5.2.4) of Lemma 5.2.2 it follows that $\mathrm{H}(\cdot,u,\mathrm{D}u)$ is constant along the trajectory. Thus, if $x\in\mathcal{O}$ is chosen as a point realising either the maximum or the minimum in $\overline{\mathcal{O}}$, then by moving along the trajectory, we reach a point $y\in\partial\mathcal{O}$ such that $\mathrm{H}(\cdot,u,\mathrm{D}u)\big|_x = \mathrm{H}(\cdot,u,\mathrm{D}u)\big|_y$. This establishes both the maximum and minimum principle. The proposition ensues. $\square$

5.2.5 Remark [Danskin's theorem]

The central ingredient in the proofs of Theorems 5.1.4-5.1.5 is the next consequence of Danskin's theorem: for any $\mathcal{O}\Subset\Omega$ and any $u,\phi\in C^1(\Omega;\mathbb{R}^N)$, we have the identities

$$\left\{\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}\mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,&=\, \max_{\mathcal{O}(u)}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi\,+\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big),\\
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^-}\mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,&=\, \min_{\mathcal{O}(u)}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi\,+\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big),
\end{aligned}\right. \tag{5.2.6}$$

where

$$\mathcal{O}(u) \,:=\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\}.$$

Indeed, by [34, Theorem 1, page 643] and the chain rule we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}\mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,&=\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}\Big(\max_{\overline{\mathcal{O}}}\mathrm{H}\big(\cdot,u+t\phi,\mathrm{D}u+t\mathrm{D}\phi\big)\Big)\\
&=\, \max_{\mathcal{O}(u)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}\mathrm{H}\big(\cdot,u+t\phi,\mathrm{D}u+t\mathrm{D}\phi\big)\right)\\
&=\, \max_{\mathcal{O}(u)}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi\,+\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big).
\end{aligned}$$

This establishes the first identity of (5.2.6). The second one follows through the substitutions $\phi \rightsquigarrow -\phi$, $t \rightsquigarrow -t$.

Now we may establish Theorem 5.1.4.

5.2.6 Proof of Theorem 5.1.4

(I) Suppose first that u is a rank-one absolute minimiser on Ω. The aim is to show that (5.1.8) is satisfied on Ω. This conclusion in fact follows by the results in [49], but below we provide a new shorter proof. To this end, fix $x \in \Omega$ and $\rho \in (0, \mathrm{dist}(x, \partial\Omega))$ and let $\mathcal{O} := \mathbb{B}_\rho(x)$. We fix also $\xi \in \mathbb{R}^N$ and choose

$$\phi(y) \,:=\, \xi\big(|y-x|^2 - \rho^2\big).$$

Then, $\phi \in C^1_0\big(\bar{\mathbb{B}}_\rho(x); \mathrm{span}[\xi]\big)$. By Remark 5.2.5 and our minimality assumption, the definition of one-sided derivatives yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^-} \mathrm{E}_\infty(u+t\phi, \mathcal{O}) \,\leq\, 0 \,\leq\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+} \mathrm{E}_\infty(u+t\phi, \mathcal{O}). \tag{5.2.7}$$

Hence, by (5.2.7), (5.2.6) and continuity there exists a point x_ρ with $|x_\rho - x| \leq \rho$ which lies in the argmax set

$$(\mathbb{B}_\rho(x))(u) \,=\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \,:\, \bar{\mathbb{B}}_\rho(x)\big\}$$

such that

$$\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\cdot \phi\Big)\Big|_{x_\rho} \,=\, 0. \tag{5.2.8}$$

Therefore,

$$\xi^\top\Big(2\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big|_{x_\rho}(x_\rho - x) \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\big|_{x_\rho}\big(|x_\rho - x|^2 - \rho^2\big)\Big) \,=\, 0. \tag{5.2.9}$$

If x_ρ lies in the interior of $\mathbb{B}_\rho(x)$, then it is an interior maximum and therefore

$$\mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big)\big|_{x_\rho} \,=\, 0.$$

This means that (5.1.8) is satisfied at x_ρ. If x_ρ lies on the boundary of $\mathbb{B}_\rho(x)$, then this means that

$$\forall\, y \in \bar{\mathbb{B}}_\rho(x),\ \text{ we have } \mathrm{H}(\cdot, u, \mathrm{D}u)\big|_y \leq\, \mathrm{H}(\cdot, u, \mathrm{D}u)\big|_{x_\rho}.$$

The above can be rewritten as

$$\bar{\mathbb{B}}_\rho(x) \,\subseteq\, \mathcal{H}(x_\rho) \,:=\, \Big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \leq\, \mathrm{H}(\cdot, u, \mathrm{D}u)\big|_{x_\rho}\Big\},$$

and note also that $x_\rho \in \partial\mathbb{B}_\rho(x) \cap \partial\mathcal{H}(x_\rho)$. Hence, the sublevel set $\mathcal{H}(x_\rho)$ satisfied an interior sphere condition at x_ρ. If $\mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big)\big|_{x_\rho} = 0$ then (5.1.8) is again satisfied at x_ρ. If on the other hand

$$\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho} \neq 0$$

then $\partial\mathcal{H}(x_\rho)$ is a C^1 manifold near x_ρ and the gradient above is the normal vector at the point x_ρ. Due to the interior sphere condition, this implies that this is also the normal vector to the sphere $\partial\mathbb{B}_\rho(x)$ at x_ρ. Thus, there exists $\lambda \neq 0$ such that

$$x_\rho - x \,=\, \lambda \mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho}. \tag{5.2.10}$$

By inserting (5.2.10) into (5.2.9) and noting that $|x_\rho - x| = \rho$, we infer that

$$2\lambda\, \xi^\top\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\Big)\big|_{x_\rho} \,=\, 0.$$

By dividing by 2λ and letting $\rho \to 0$, we deduce that (5.1.8) is satisfied at the arbitrary $x \in \Omega$.

Conversely, suppose that u satisfies (5.1.8) on Ω, together with the additional assumptions of the statement. Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1_0(\overline{\mathcal{O}};\mathrm{span}[\xi])$. Without loss of generality, we may suppose $\mathcal{O}$ is connected. Since $\phi = (\xi^\top\phi)\xi$, for convenience we set $g := \xi^\top\phi$ and then we may write $\phi = g\xi$ with $g \in C^1_0(\overline{\mathcal{O}})$. Then, the matrix-valued map $\mathrm{H}_P(\cdot,\mathrm{D}u)$ is pointwise left invertible. Therefore, by (5.1.8)

$$\big(\mathrm{H}_P(\cdot,\mathrm{D}u)\big)^{-1}\mathrm{H}_P(\cdot,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,\mathrm{D}u)\big) \,=\, 0 \ \text{ on } \mathcal{O},$$

which, by the connectivity of $\mathcal{O}$, gives

$$\mathrm{H}(\cdot,\mathrm{D}u) \equiv \text{const} \ \text{ on } \mathcal{O}.$$

Since $g \in C^1(\mathbb{R}^n)$ with $g = 0$ on $\partial\mathcal{O}$, there exists at least one interior critical point $\bar{x} \in \mathcal{O}$ such that $\mathrm{D}g(\bar{x}) = 0$. By the previous, we have

$$\begin{aligned}
\mathrm{E}_\infty(u,\mathcal{O}) \,&=\, \mathrm{H}\big(\bar{x},\mathrm{D}u(\bar{x})\big)\\
&=\, \mathrm{H}\Big(\bar{x},\mathrm{D}u(\bar{x}) + \xi\otimes\mathrm{D}g(\bar{x})\Big)\\
&=\, \mathrm{H}\Big(\bar{x},\mathrm{D}u(\bar{x}) + \mathrm{D}\phi(\bar{x})\Big)\\
&\leq\, \sup_{x\in\mathcal{O}}\mathrm{H}\Big(x,\mathrm{D}u(x) + \mathrm{D}\phi(x)\Big)\\
&=\, \mathrm{E}_\infty(u+\phi,\mathcal{O}).
\end{aligned}$$

The conclusion ensues.

(II) Suppose that u has ∞-minimal area. Fix $x \in \Omega$ and $\rho \in (0,\mathrm{dist}(x,\partial\Omega))$. Fix

$$\xi \,\in\, \tilde{\mathrm{N}}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)^\top\big|_x\Big),$$

noting also that by Definition 5.1.3 the above set is the reduced nullspace of

$\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top$ at x. This implies that there exists a C^1 extension $\bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\bar{\xi}(x) = \xi$ and $(\bar{\xi})^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ on the closed ball $\bar{\mathbb{B}}_\varepsilon(x)$ for some $\varepsilon \in (0, \rho)$. By differentiating the relation $(\bar{\xi})^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ and taking its trace, we obtain

$$\bar{\xi} \cdot \mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \, + \, \mathrm{D}\bar{\xi} : \mathrm{H}_P(\cdot, u, \mathrm{D}u) \, = \, 0, \tag{5.2.11}$$

on $\bar{\mathbb{B}}_\varepsilon(x)$. Since u has ∞-minimal area and $\bar{\xi}$ is an admissible normal variation, by using Remark 5.2.5 and arguing as in the beginning of part (I), it follows that

$$\Big(\bar{\xi} \cdot \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \, + \, \mathrm{D}\bar{\xi} : \mathrm{H}_P(\cdot, u, \mathrm{D}u)\Big)\Big|_{x_\varepsilon} \, = \, 0 \tag{5.2.12}$$

for some $x_\varepsilon \in (\mathbb{B}_\varepsilon(x))(u)$, where

$$(\mathbb{B}_\varepsilon(x))(u) \, = \, \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) \, : \, \bar{\mathbb{B}}_\varepsilon(x)\big\}.$$

By (5.2.11)-(5.2.12), we infer that

$$\bar{\xi}(x_\varepsilon) \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \, - \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big)\Big|_{x_\varepsilon} \, = \, 0$$

and by letting $\varepsilon \to 0$, we deduce that

$$\xi \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \, - \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big)\Big|_{x} \, = \, 0,$$

for any $\xi \in \tilde{\mathrm{N}}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top\big|_x\big)$. Hence, u satisfies (5.1.9) at the arbitrary $x \in \Omega$.

Conversely, suppose that u solves (5.1.9) on Ω. Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\phi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ on $\mathcal{O}$. Note further that by the continuity up to the boundary of all functions involved, the latter identity in fact holds on $\overline{\mathcal{O}}$. By the satisfaction of (5.1.9) and Definition 5.1.3, it follows that

$$\phi \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \, - \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \, = \, 0,$$

on $\overline{\mathcal{O}} \subseteq \Omega$. By differentiating $\phi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$, we obtain

$$\phi \cdot \mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \, + \, \mathrm{D}\phi : \mathrm{H}_P(\cdot, u, \mathrm{D}u) \, = \, 0,$$

on $\overline{\mathcal{O}}$. By the above two identities, we deduce

$$\phi \cdot \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \, + \, \mathrm{D}\phi : \mathrm{H}_P(\cdot, u, \mathrm{D}u) \, = \, 0,$$

on $\overline{\mathcal{O}}$. Since $\mathcal{O}(u) \subseteq \overline{\mathcal{O}}$, Remark 5.2.5 yields that u is a critical point since the left and right derivative of $\mathrm{E}_\infty(u + t\phi, \mathcal{O})$ at $t = 0$ coincide and vanish. Since by assumption $\mathrm{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$, it follows that $\mathrm{E}_\infty(\cdot, \mathcal{O})$ is convex on $C^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point for this class of variations. This establishes our claim.

(III) This is an immediate corollary of items (I) and (II). $\square$

Now we conclude by establishing Theorem 5.1.5.

5.2.7 Proof of Theorem 5.1.5

Fix $\mathcal{O} \Subset \Omega$ and $u, \phi \in C^1(\Omega; \mathbb{R}^N)$. We show that (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III) and that (III) $\Longrightarrow$ (I) under the additional convexity assumption. By recalling Remark 5.2.5, note that if

$$\mathrm{E}_\infty(u + t\phi, \mathcal{O}) \geq \mathrm{E}_\infty(u, \mathcal{O}), \quad \text{for all } t \in \mathbb{R}, \tag{5.2.13}$$

then directly by (5.2.13) and the definition of one-sided derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^-} \mathrm{E}_\infty(u + t\phi, \mathcal{O}) \leq 0 \leq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+} \mathrm{E}_\infty(u + t\phi, \mathcal{O}). \tag{5.2.14}$$

This shows (I) $\Longrightarrow$ (II). If (II) holds, note that one also has that

$$\min_{\mathrm{Argmax}\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\}} \Big[\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \, + \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi \Big] \leq 0,$$

for any $\phi \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$. By (5.2.6) we see that (5.2.14) is satisfied and by continuity we obtain the existence of a non-empty compact set $\mathrm{K} = \mathrm{K}_\phi \subseteq \mathcal{O}(u)$ such that

$$\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \, + \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi \Big)\Big|_{\mathrm{K}} = 0. \tag{5.2.15}$$

Hence, (III) ensues. If now (5.2.15) holds true for some non-empty compact set $\mathrm{K} \subseteq \mathcal{O}(u)$, then by (5.2.6) we have that (5.2.14) is true. If further $\mathrm{H}(x, \cdot, \cdot)$ is convex for all $x \in \Omega$, then by Lemma 5.2.8 given right after the proof, $t \mapsto \mathrm{E}_\infty(u + t\phi, \mathcal{O})$ is minimised at $t = 0$ and (5.2.13) holds true.

(IV) $\Longrightarrow$ (III): Let $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ be any Radon probability measure satisfying (5.1.12). Then, by assumption

$$\int_{\overline{\mathcal{O}}} \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \, + \, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi \Big) \mathrm{d}\sigma = 0$$

for all $\phi \in C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N)$. Fix any point $\bar{x} \in \mathcal{O}(u)$. By choosing the Dirac measure $\bar{\sigma} \in \mathcal{P}(\overline{\mathcal{O}})$ given by

$$\bar{\sigma} := \delta_{\bar{x}}$$

which evidently satisfies $\mathrm{supp}(\bar\sigma) = \{\bar x\} \subseteq \mathcal{O}(u)$, we obtain

$$\begin{aligned}
\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big)\Big|_{\bar x}& \\
&= \int_{\overline{\mathcal{O}}}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big)\mathrm{d}\bar\sigma \\
&= 0,
\end{aligned}$$

for any $\bar x \in \mathcal{O}(u)$. The conclusion ensues with $\mathrm{K} = \mathcal{O}(u)$.

(III) $\Longrightarrow$ (IV): If we have $\mathrm{K} = \mathcal{O}(u)$ and

$$\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big)\Big|_{\mathrm{K}} = 0,$$

then for any Radon probability measure $\sigma \in \mathcal{P}(\overline{\mathcal{O}})$ with $\mathrm{supp}(\sigma) \subseteq \mathrm{K}$, we have

$$\int_{\overline{\mathcal{O}}}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big)\mathrm{d}\sigma = 0$$

for all $\phi \in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$. Hence, we have shown that

$$-\mathrm{div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\sigma\big) \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\sigma = 0,$$

in the dual space $(C^1_0(\overline{\mathcal{O}};\mathbb{R}^N))^*$. □

The next result which was utilised in the proof of Theorem 5.1.5 completes our arguments.

5.2.8 Lemma

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. If the one-sided derivatives $f'(0^\pm)$ exist and $f'(0^-) \leq 0 \leq f'(0^+)$, then $f(0)$ is the global minimum of f on $\mathbb{R}$.

5.2.9 Proof of Lemma 5.2.8

By the convexity of f on $\mathbb{R}$, for any fixed $s \in \mathbb{R}$ there exists a sub-differential $p_s \in \mathbb{R}$ such that

$$f(t) - f(s) \,\geq\, p_s(t-s), \quad \text{for all } t \in \mathbb{R}. \tag{5.2.16}$$

For the choice $t = 0$ and $s > 0$, we have

$$\frac{f(s) - f(0)}{s} \,\leq\, p_s$$

and note also that since convex functions are locally Lipschitz, the set $(p_s)_{0<s<1}$ is bounded. Thus, since $f'(0^+)$ exists and is non-negative, the above inequality yields

$$0 \leq f'(0^+) \leq \liminf_{s\to 0^+} p_s < \infty.$$

Hence, by passing to the limit as $s \to 0^+$ in the inequality (5.2.16) for $t > 0$ fixed, we obtain $f(t) - f(0) \geq 0$. The case of $t < 0$ follows by arguing similarly. $\square$

Chapter 6

Conclusions and future work

6.1 Conclusions

We would like to mention that this thesis is a collection of published papers presented as chapters consist of original results. This work includes new results in the field of Calculus of Variations in L^∞. The new results are improved previous theorems by generalising and relaxing some of the conditions. Chapter 2 and Chapter 5 are joint papers with my supervisor Dr. N. Katzourakis. Chapter 3 is a joint paper with Dr. N. Katzourakis and Dr. H. Abugirda. While chapter 4 is single author paper.

The main result of Chapter 2 is that we characterise local minimiser of the following functional

$$\mathrm{E}_\infty(u, \mathcal{O}) := \operatorname*{ess\,sup}_{\mathcal{O}} \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega.$$

for appropriate classes of affine variations of the energy as generalised solutions of associated PDE system which plays the role of Euler-Lagrange equation. Similar result was proven for $\mathrm{H}(x, \eta, P) = |P|^2$ in [53]. That makes our result a generalisation of result in [53] since the Hamiltonian function H depends not only on gradient of the function but also on the function itself and the domain.

Chapter 3 is the joint paper with Dr. N. Katzourakis and Dr. H. Abugirda. The author of this thesis gave an idea which partly impacted on the proof of the main result "Rigidity and flatness of maps with tangential Laplacian in separated form", which states let $\Omega \subseteq \mathbb{R}^2$ be an open set and let also $N \geq 2$. Let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a classical solution to the nonlinear system

$$[\![\mathrm{D}u]\!]^\perp \Delta u = 0 \quad \text{in } \Omega,$$

having the separated form $u(x, y) = f(x) - f(y)$, for some curve $f \in (W^{3,p} \cap C^2)(\mathbb{R}, \mathbb{R}^N)$ and some $p > 1$. Then, the image $u(\Omega)$ of the solution is contained in an at most countable union of affine planes in $\mathbb{R}^N$.

Chapter 4 investigated with authors own initiative and he found a new classical ∞-harmonic functions in high dimensions, particularly when domains are two dimensional in polar coordinates and at least two dimensional in Cartesian coordinates. The challenges were the technical computations and the regularity of the solutions on uncertain domains which was assumed to be well defined.

We have two main outcomes of Chapter 5 which is a joint paper with Dr. N. Katzourakis. First result "Variational Structure of Aronsson's system" coincides with the result in [49] when the Hamiltonian function depends only on the gradient function, namely $\mathrm{H}(x, \eta, P) = |P|^2$. The result characterises C^2 ∞–minimal maps as solution of the Aronsson system and vice versa. One of the difficulty of proving this theorem was that we can not differentiate

$$(\bar{\xi})^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0.$$

if $\bar{\xi} \notin C^1(\mathbb{R}^n; \mathbb{R}^N)$. We avoided that using reduced nullspace in the definition 5.1.3. The second result "Divergence PDE characterisation of Absolute minimisers" is a completely new original result. Lets highlight the main differences of this theorem with previous results. Firstly it has been proved for C^1 maps and for C^1 variations vanishing on compactly contained boundaries. Secondly we did not use approximation techniques of the L^p space but rather techniques of the L^∞ space, namely Danskin's theorem.

6.2 Future work

We believe that the work in this field is interesting and there are still many open problems one can work on, for example:

1. It is common that a solutions to a PDEs might have less regularity than we require. So it is natural to work on extending the result of the theorem 5.1.5 from C^1 to Sobolev spaces or $\mathcal{D}$-solutions.
2. Theorem 5.1.5 gives us characterization of Absolute minimiser only on Argmax set. One of the methods to fill the gap is to study vectorial L^∞-absolute minimisers on Ω using vectorial L^p-absolute minimisers on Ω, i.e. for every $\mathcal{O} \Subset \Omega$ if u_p is minimiser of $\mathrm{E}_p(u, \mathcal{O}) := \|\mathrm{H}(\cdot, u, \mathrm{D}u)\|_{L^p(\mathcal{O})}$, then one can study "convergence" of u_p to u_∞ as $p \to \infty$, where u_∞ is minimiser of $\mathrm{E}_\infty(u, \mathcal{O}) := \|\mathrm{H}(\cdot, u, \mathrm{D}u)\|_{L^\infty(\mathcal{O})}$.
3. All known explicit solutions have at least $C^{1,\alpha}$ regularity. So it is challenging to find explicit solutions of the theorem 5.1.5 and find out how and/or why other results have to fail.

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