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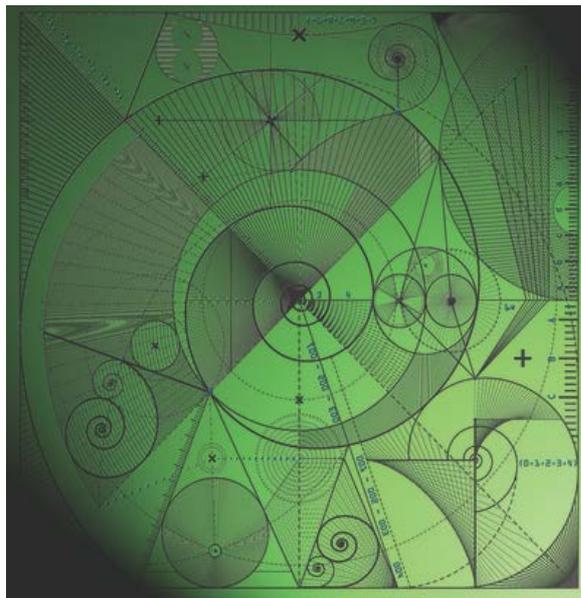
A shape calculus based method for a transmission problem with random interface

by

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A SHAPE CALCULUS BASED METHOD FOR A TRANSMISSION PROBLEM WITH RANDOM INTERFACE[*]

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Abstract. The present work is devoted to approximation of the statistical moments of the unknown solution of a class of elliptic transmission problems in $\mathbb{R}^3$ with uncertainly located transmission interfaces. Within this model, the diffusion coefficient has a jump discontinuity across the random transmission interface which models linear diffusion in two different media separated by an uncertain surface. We apply the shape calculus approach to approximate solution's perturbation by the so-called *shape derivative*, correspondingly statistical moments of the solution are approximated by the moments of the shape derivative. We characterize the shape derivative as a solution of a related homogeneous transmission problem with nonzero jump conditions which can be solved with the aid of boundary integral equations. We develop a rigorous theoretical framework for this method, particularly i) extending the method to the case of unbounded domains and ii) closing the gaps, clarifying and adapting results in the existing literature. The theoretical findings are supported by and illustrated in two particular examples.

Key words. Shape derivative, transmission problem, random domain, uncertainty quantification, statistical moments, pseudodifferential equations, asymptotic expansions

AMS subject classifications. 35R60, 65J15, 35C20, 65N30

1. Introduction. Elliptic transmission or interface problems arise in many fields in science and engineering, such as tomography, deformation of an elastic body with inclusions, stationary groundwater flow in heterogeneous medium, fluid-structure interaction, scattering of an elastic body and many others. Combined with the state-of-the-art hardware, advanced numerical schemes are capable of producing a highly accurate and efficient deterministic numerical simulation, provided that the problem data are known exactly. However, in real applications, a complete knowledge of the problem parameters is not realistic for many reasons. First, the simulation parameters are often estimated from measurements which can be *inexact* e.g. due to imperfect measurement devices. Second, the parameters are estimated based on a large but *finite* number of system samples (snapshots); this information can be incomplete or stochastic. Finally, parameters of the system originate from a mathematical model which is itself only an approximation of the actual process. Under such circumstances, *highly accurate results of a single deterministic simulation for one particular set of problem parameters are of limited use.* An important paradigm, becoming rapidly popular over the last years, see e.g. [2, 3, 6, 7, 8, 9, 10, 11, 14, 18, 19] and references therein, is to treat the lack of knowledge via modeling uncertain parameters as random fields. If the forward solution operator is continuous, the solution of the forward problem with random parameters becomes a well-defined random field. Efficient numerical approximation of the random (or stochastic) solution and its probabilistic characteristics, e.g. statistical moments, is a highly non-trivial task representing numerous new interdisciplinary challenges: from regularity analysis and numerical analysis to modeling and efficient parallel large scale computing.

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In this article we develop a deterministic method for numerical solution for a class of transmission problems with randomly perturbed interfaces. The equation to be solved is of the form

$$-\nabla \cdot (\alpha \nabla u) = f \quad \text{in } D_{\pm},$$

where D_- is a random bounded domain in $\mathbb{R}^3$ and $D_+ = \mathbb{R}^3 \setminus \overline{D_-}$ is its complement. The domains share a common random surface Γ, and the coefficient function α takes (in general) distinct constant values in D_- and D_+, respectively. The solution u is subject to jump conditions across Γ. A precise description of the model problem is deferred until Section 2.3, where a probabilistic perturbation model for the surface Γ (and thus $D_{\pm}$) will be rigorously introduced. Within this model, the transmission interface depends on the "random event" ω and the parameter $\epsilon \geq 0$ controlling the amplitude of the perturbation. Therefore, the solution u depends on ω and ϵ, and will be denoted by $u^\epsilon(\omega)$. The case $\varepsilon = 0$ corresponds to the *zero perturbation*. In the present paper we are aiming at estimating probabilistic properties of the solution perturbation $u^\epsilon(\omega) - u^0$ when the perturbation parameter is small, $\epsilon \ll 1$.

More precisely, we exploit the ideas from the recent publications [4, 6, 12, 13, 14] and propose to approximate the statistical moments of the solution perturbation by the moments of the linearized solution, i.e. for a fixed (small) value of the perturbation parameter ϵ the k-th order statistical moments of the solution perturbation are approximated by

$$(1.1) \qquad\qquad \mathcal{M}^k[u^\epsilon - u^0] \approx \epsilon^k \mathcal{M}^k[u']$$

and similarly

$$(1.2) \qquad\qquad \mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]] \approx \epsilon^k \mathcal{M}^k[u'].$$

Here u' is the *shape derivative* of u^ϵ *formally* understood as the linear order term in the asymptotic expansion

$$(1.3) \qquad\qquad u^\epsilon(\boldsymbol{x}, \omega) = u^0(\boldsymbol{x}) + \epsilon u'(\boldsymbol{x}, \omega) + \cdots, \qquad \epsilon \to 0,$$

for almost all random events $\omega \in \Omega$ at a certain *fixed* point $\boldsymbol{x}$ in the Euclidean space $\mathbb{R}^3$. The notion of the shape derivative has been introduced in the context of the shape optimization (see e.g. the monograph [20] and the references therein) and allows to quantify sensitivity of the solution of a PDE to small perturbation of the boundary.

Although very intuitive, (1.3) cannot be used as a rigorous definition of $u'(\boldsymbol{x}, \omega)$. In particular, convergence of the asymptotic expansion and herewith the existence of the shape derivative is unclear. In the first part of this article (Section 3) we develop a rigorous mathematical theory of existence of the shape derivative for the class of elliptic transmission problems under consideration. Similarly to [13, Lemma 1], we obtain a characterization of the shape derivative $u'(\boldsymbol{x}, \omega)$ as a solution of a deterministic transmission problem on a fixed interface. Our contribution in this section is twofold: *i)* we extend the notion of shape derivatives to the case of unbounded domains, and *ii)* we fill the gaps and unclarities in existing literature where no rigorous discussion on existence of shape derivatives is presented.

As mentioned above, for almost all $\omega \in \Omega$ the shape derivative $u'(\cdot, \omega)$ is a solution of a deterministic problem in $\mathbb{R}^3$ with (in general) nonhomogeneous jump conditions but with vanishing

volume source term. The second contribution of this article is the derivation and analysis of boundary integral equations [15, 17, 21] which are used to solve this transmission problem on deterministic domains with deterministic interface. A tensorization argument is then used to obtain the approximation (1.1) for the statistical moments.

Finally, we illustrate the accuracy of the linearization approach by considering two examples setting on the unit sphere $\Gamma := \{|\boldsymbol{x}| = 1\}$ with uniform radial perturbation. The first example involves a pre-determined solution with radial symmetry, so that the exact and the linearized solutions as well as their second moments are available explicitly. We observe that in this particular case the linearization error for the second order statistical moments is of the order $\mathcal{O}(\epsilon^4)$ rather than $o(\epsilon^2)$ as confirmed by the theory. The second example involves non-symmetric data so that the linearized solution is not available explicitly. To solve this problem numerically we use the sparse spectral tensor product BEM developed in [5]. This method exploits the underlying geometry of the formulation and uses the basis of spherical harmonics being the eigenfunctions of the integral operator governing the problem.

The paper is organized as follows. Section 2 contains the description of the random surface perturbation model and the rigorous formulation of the model transmission problem, preceded by the details on the function spaces involved in the analysis. Section 3 contains the generalization of the shape calculus to the case of unbounded domains, definition and characterization of the material and shape derivatives for the underlying model transmission problem and a rigorous proof and error bounds for the approximation (1.1). Section 4 contains the details of the boundary reduction for the linearized problem. Section 5 contains two examples, an analytic and a numerical, illustrating the accuracy of the method.

2. Model elliptic transmission problem on a random interface. We start with some preliminary definitions and notations in Section 2.1. Section 2.2 contains the description of a model for the random surface perturbation. We introduce the randomized model problem in the strong form in Section 2.3. The details on Sobolev spaces involved are summarized in Section 2.4.

2.1. Bochner spaces and statistical moments. Throughout this paper we denote by $(\Omega, \Sigma, \mathbb{P})$ a generic complete probability space and let X be a Banach space. For any $1 \leq k \leq \infty$, the Bochner space $L^k(\Omega, X)$ is defined as usual by

$$(2.1) \qquad L^k(\Omega, X) := \big\{ v : \Omega \to X, \text{ measurable} : \|v\|_{L^k(\Omega,X)} < \infty \big\}$$

with the norm

$$(2.2) \qquad \|v\|_{L^k(\Omega,X)} := \begin{cases} \left(\int_\Omega \|v(\omega)\|_X^k \, d\mathbb{P}(\omega) \right)^{1/k}, & 1 \leq k < \infty, \\ \operatorname*{esssup}_{\omega \in \Omega} \|v(\omega)\|_X, & k = \infty. \end{cases}$$

The elements of $L^k(\Omega, X)$ are called *random fields*. We remark that for a part of the subsequent analysis we may restrict to the special case when X is a Hilbert space. In particular, when X_1 and X_2 are two separable Hilbert spaces, their tensor product $X_1 \otimes X_2$ is a separable Hilbert space with the natural inner product extended by linearity from $\langle v \otimes a, w \otimes b \rangle_{X_1 \otimes X_2} = \langle v, w \rangle_{X_1} \langle a, b \rangle_{X_2}$, cf. e.g. [16, p. 20], [1, Definition 12.3.2, p.298]. In this paper we work with k-fold tensor products of Hilbert spaces

$$(2.3) \qquad X^{(k)} := X \otimes \cdots \otimes X.$$

with the natural inner product satisfying $\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k \rangle_{X^{(k)}} = \langle v_1, w_1 \rangle_X \ldots \langle v_k, w_k \rangle_X$.

DEFINITION 2.1. *For a random field $v \in L^k(\Omega, X)$, its k-order moment $\mathcal{M}^k[v]$ is an element of $X^{(k)}$ defined by*

$$(2.4) \qquad \mathcal{M}^k[v] := \int_\Omega \big(\underbrace{v(\omega) \otimes \cdots \otimes v(\omega)}_{k\text{-}times} \big) \, d\mathbb{P}(\omega).$$

In the case $k = 1$, the statistical moment $\mathcal{M}^1[v]$ coincides with the *mean value* of v and is denoted by $\mathbb{E}[v]$. If $k \geq 2$, the statistical moment $\mathcal{M}^k[v]$ is the *k-point autocorrelation function* of v. The quantity $\mathcal{M}^k[v - \mathbb{E}[v]]$ is termed the k-th central moment of v. We distinguish in particular second order moments: the *correlation* and *covariance* defined by

$$(2.5) \qquad \mathrm{Cor}[v] := \mathcal{M}^2[v], \qquad \mathrm{Cov}[v] := \mathcal{M}^2[v - \mathbb{E}[v]].$$

In this paper we work with X being Sobolev spaces of real-valued functions defined on a domain $U \subset \mathbb{R}^3$ yielding, in particular, the representation

$$(2.6) \qquad \mathrm{Cor}[v](\boldsymbol{x}, \boldsymbol{y}) := \int_\Omega v(\boldsymbol{x}, \omega) v(\boldsymbol{y}, \omega) \, d\mathbb{P}(\omega), \quad \boldsymbol{x}, \boldsymbol{y} \in U.$$

We observe that $\mathrm{Cor}[v]$ is defined on the Cartesian product $U \times U$. Similarly, $\mathcal{M}^k[v]$ is defined on the k-fold Cartesian product $U \times \cdots \times U$. Here, the dimension of the underlying domain grows rapidly with increasing moment order k.

2.2. Random interfaces. Consider a fixed bounded domain $D_-^0 \subset \mathbb{R}^3$ and let $D_+^0 := \mathbb{R}^3 \setminus \overline{D_-^0}$ be its complement. Then the interface $\Gamma^0 = \overline{D_-^0} \cap \overline{D_+^0}$ is a closed manifold in $\mathbb{R}^3$. For the subsequent analysis we assume that Γ^0 is at least of the class $C^{1,1}$. This implies that the outward normal vector $\boldsymbol{n}^0$ to Γ^0 is Lipschitz continuous: $\boldsymbol{n}^0 \in C^{0,1}(\Gamma^0)$. The partition $\mathbb{R}^3 = \overline{D_+^0 \cup D_-^0}$ and the interface Γ^0 will be fixed throughout the paper and will be called the *nominal partition* and *nominal interface*, respectively.

In the present paper we utilize the domain perturbation model based on the *speed method* (see e.g. the monograph [20] and references therein) and random domain perturbation model from [4, 6, 12, 13, 14]. Suppose $\kappa \in L^k(\Omega, C^{0,1}(\Gamma^0))$ is a random field, i.e. for almost any realization $\omega \in \Omega$, we have $\kappa(\cdot, \omega) \in C^{0,1}(\Gamma^0)$. For some sufficiently small, nonnegative ϵ we consider a family of random interfaces of the form

$$(2.7) \qquad \Gamma^\epsilon(\omega) = \{\boldsymbol{x} + \epsilon \kappa(\boldsymbol{x}, \omega) \boldsymbol{n}^0(\boldsymbol{x}) : \boldsymbol{x} \in \Gamma^0\}, \quad \omega \in \Omega.$$

Here, the uncertainty of the surfaces $\Gamma^\epsilon(\omega)$ is represented by the uncertainty in $\kappa(\cdot, \omega)$. Notice that the interface $\Gamma^\epsilon(\omega)|_{\epsilon=0}$ is identical with Γ^0 and therefore is a deterministic closed manifold. Moreover, the limit $\Gamma^\epsilon(\omega) \to \Gamma^0$ as $\epsilon \to 0$ is well defined in $L^k(\Omega, C^{0,1})$. If we identify Γ^ϵ and Γ^0 with their graphs, then

$$(2.8) \quad \|\Gamma^\epsilon - \Gamma^0\|_{L^k(\Omega, C^{0,1})} = \epsilon \left(\int_\Omega \|\kappa(\cdot, \omega) \boldsymbol{n}^0\|_{C^{0,1}(\Gamma^0)}^k \, d\mathbb{P}(\omega) \right)^{\frac{1}{k}} \leq 2\epsilon \|\kappa\|_{L^k(\Omega, C^{0,1}(\Gamma^0))} \|\boldsymbol{n}^0\|_{C^{0,1}(\Gamma^0)}.$$

This implies that for almost all $\omega \in \Omega$ and a sufficiently small $\epsilon \geq 0$ the surface $\Gamma^\epsilon(\omega)$ is a Lipschitz continuous closed manifold separating the interior domain $D_-^\epsilon(\omega)$ and its complement

$D_+^\epsilon(\omega) := \mathbb{R}^3 \setminus \overline{D_-^\epsilon}$. The *shape calculus* in Section 3 requires a somewhat stronger smoothness assumption on κ, namely that the realizations of κ belong to $C^1(\Gamma^0)$. From (2.7) we observe that the *mean random interface* is represented by

$$\mathbb{E}[\Gamma^\epsilon] = \{\boldsymbol{x} + \epsilon\mathbb{E}[\kappa(\boldsymbol{x},\cdot)]\boldsymbol{n}^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Gamma^0\}.$$

Without loss of generality, we may assume that the random perturbation amplitude $\kappa(\boldsymbol{x},\omega)$ is centered, i.e.,

$$(2.9) \qquad\qquad\qquad \mathbb{E}[\kappa(\boldsymbol{x},\cdot)] = 0 \qquad \forall \boldsymbol{x} \in \Gamma^0.$$

In this case

$$\mathbb{E}[\Gamma^\epsilon] = \Gamma^0 \qquad \text{and} \qquad \operatorname{Cov}[\kappa](\boldsymbol{x},\boldsymbol{y}) = \operatorname{Cor}[\kappa](\boldsymbol{x},\boldsymbol{y}).$$

2.3. The model problem. As shown above, for a sufficiently small value $\epsilon \geq 0$ the surface perturbation model (2.7) generates a well defined partition of $\mathbb{R}^3$ into a bounded Lipschitz domain $D_-^\epsilon(\omega)$ and its complement $D_+^\epsilon(\omega) = \mathbb{R}^3 \setminus \overline{D_-^\epsilon}$ separated by the closed Lipschitz manifold $\Gamma^\epsilon(\omega) = \overline{D_-^\epsilon(\omega)} \cap \overline{D_+^\epsilon(\omega)}$. We consider a piecewise constant diffusion function subjected to this partition:

$$(2.10) \qquad\qquad\qquad \alpha^\epsilon(\boldsymbol{x},\omega) = \begin{cases} \alpha_-, & \boldsymbol{x} \in D_-^\epsilon(\omega), \\ \alpha_+, & \boldsymbol{x} \in D_+^\epsilon(\omega), \end{cases}$$

where α_- and α_+ are two positive constants independent of $\boldsymbol{x}$, ϵ, and ω. Having this we introduce the model elliptic transmission problem as a problem of finding u^ϵ satisfying

$$(2.11\text{a}) \qquad\qquad -\nabla \cdot \left(\alpha^\epsilon(\boldsymbol{x},\omega)\nabla u^\epsilon(\boldsymbol{x},\omega)\right) = f(\boldsymbol{x}) \quad \text{in } D_\pm^\epsilon(\omega),$$

$$(2.11\text{b}) \qquad\qquad\qquad [u^\epsilon(\boldsymbol{x},\omega)] = 0 \quad \text{on } \Gamma^\epsilon(\omega),$$

$$(2.11\text{c}) \qquad\qquad \left[\alpha^\epsilon(\boldsymbol{x},\omega)\frac{\partial u^\epsilon}{\partial \boldsymbol{n}}(\boldsymbol{x},\omega)\right] = 0 \quad \text{on } \Gamma^\epsilon(\omega),$$

$$(2.11\text{d}) \qquad\qquad u^\epsilon(\boldsymbol{x},\omega) = O(|\boldsymbol{x}|^{-1}) \quad \text{as } |\boldsymbol{x}| \to +\infty.$$

Here, $\partial/\partial\boldsymbol{n}$ denotes the normal derivative on $\Gamma^\epsilon(\omega)$, i.e. $\partial/\partial\boldsymbol{n} = \boldsymbol{n}^\epsilon(\boldsymbol{x},\omega) \cdot \nabla$, where $\boldsymbol{n}^\epsilon(\boldsymbol{x},\omega)$ is the unit normal vector to the interface $\Gamma^\epsilon(\omega)$ pointing into the interior of $D_+^\epsilon(\omega)$. Let $u_-^\epsilon(\omega)$ and $u_+^\epsilon(\omega)$ be the restrictions of $u^\epsilon(\omega)$ on $D_-^\epsilon(\omega)$ and $D_+^\epsilon(\omega)$, respectively. Then the jump $[u^\epsilon(\omega)]$ is understood to be $u_-^\epsilon(\omega) - u_+^\epsilon(\omega)$ on $\Gamma^\epsilon(\omega)$ in the sense of trace for each sample ω. Similarly

$$\left[\alpha^\epsilon(\boldsymbol{x},\omega)\frac{\partial u^\epsilon}{\partial \boldsymbol{n}}(\boldsymbol{x},\omega)\right] = \alpha_-^\epsilon \frac{\partial u_-^\epsilon}{\partial \boldsymbol{n}}(\boldsymbol{x},\omega) - \alpha_+^\epsilon \frac{\partial u_+^\epsilon}{\partial \boldsymbol{n}}(\boldsymbol{x},\omega), \quad \boldsymbol{x} \in \Gamma^\epsilon(\omega).$$

The function $f \in H^1(\mathbb{R}^3)$ is assumed to be independent of ω and thereby it represents a deterministic source function in $\mathbb{R}^3$.

The model problem (2.11a)–(2.11d) represents a stationary diffusion in $\mathbb{R}^3$ with piecewise constant diffusivity in the interior and exterior domain. The uncertainty in the random solution $u^\epsilon(\boldsymbol{x},\omega)$ is implied by the uncertain location of the transmission interface $\Gamma^\epsilon(\omega)$. The solution depends nonlinearly on the interface and a linearization process will first be used to linearize the initial problem. The tool in this process is shape calculus which will be presented in Section 3. In what follows we address the problem of approximation of the statistical moments

$$(2.12) \qquad\qquad \mathbb{E}[u^\epsilon], \quad \mathcal{M}^k[u^\epsilon - u^0], \quad \text{and} \quad \mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]], \quad k \geq 2,$$

with this strategy as well as the rigorous control of the approximation error.

2.4. Sobolev spaces. In this section we introduce function spaces needed for the forthcoming analysis. These spaces will allow to identify the unique weak solution of the model problem (2.11a)–(2.11d) and characterize the moments (2.12).

Let $\mathcal{G}$ be a sphere-like surface, i.e., there exists a diffeomorphism $\rho : \mathbb{S} \to \mathcal{G}$ such that

$$\mathcal{G} = \{\rho(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{S}\}.$$

Here, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$. The surface $\mathcal{G}$ divides $\mathbb{R}^3$ into two subdomains, a bounded domain D_- and an unbounded domain D_+. For any distribution v defined on $\mathcal{G}$, and for any point $\rho(\boldsymbol{x})$ on $\mathcal{G}$, we can write

$$(v \circ \rho)(\boldsymbol{x}) = v(\rho(\boldsymbol{x})) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{v}_{\ell,m} Y_{\ell,m}(\boldsymbol{x}),$$

where

$$(2.13) \qquad \widehat{v}_{\ell,m} = \int_{\mathbb{S}} (v \circ \rho)(\boldsymbol{x}) \, Y_{\ell,m}(\boldsymbol{x}) \, d\sigma_{\boldsymbol{x}}$$

are the Fourier coefficients of v. Here $Y_{\ell,m}$ are spherical harmonics, which are the restrictions on the unit sphere $\mathbb{S}$ of homogeneous harmonics polynomials in $\mathbb{R}^3$. The Sobolev space $H^s(\mathcal{G})$, for $s \in \mathbb{R}$, is defined by

$$(2.14) \qquad H^s(\mathcal{G}) = \left\{ v \in \mathcal{D}'(\mathcal{G}) : \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1+\ell)^{2s} \, |\widehat{v}_{\ell,m}|^2 < +\infty \right\},$$

where $\mathcal{D}'(\mathcal{G})$ is the set of distributions on $\mathcal{G}$. The corresponding inner product and the norm are given by

$$(2.15) \qquad \langle v, w \rangle_{H^s(\mathcal{G})} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1+\ell)^{2s} \widehat{v}_{\ell,m} \widehat{w}_{\ell,m}, \quad v, w \in H^s(\mathcal{G}),$$

and

$$(2.16) \qquad \|v\|_{H^s(\mathcal{G})} = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1+\ell)^{2s} \, |\widehat{v}_{\ell,m}|^2 \right)^{1/2}, \quad v \in H^s(\mathcal{G}).$$

We note here that the inner product (2.15) and the norm (2.16) satisfy

$$(2.17) \qquad \langle v, w \rangle_{H^s(\mathcal{G})} = \langle v \circ \rho, w \circ \rho \rangle_{H^s(\mathbb{S})} \quad \text{and} \quad \|v\|_{H^s(\mathcal{G})} = \|v \circ \rho\|_{H^s(\mathbb{S})}$$

for any $v, w \in H^s(\mathcal{G})$. The set $\{Y_{\ell,m} \circ \rho^{-1} : \ell \in \mathbb{N}, \ m = -\ell, \dots, \ell\}$ is an orthogonal basis for $H^s(\mathcal{G})$. We also note that the space $H^0(\mathcal{G})$ can be understood as a weighted L_2-space on the interface $\mathcal{G}$.

We now introduce the tensor product of Sobolev spaces on the k-fold Cartesian product domains $\mathcal{G}^k = \mathcal{G} \times \cdots \times \mathcal{G}$. These spaces will be used later on for characterization of statistical moments. By boldface symbols we denote multiindices with k integer components, e.g. $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$. Given

$s \in \mathbb{R}$, the Sobolev space $H^s_{\mathrm{mix}}(\mathcal{G}^k)$ is defined to be the space of all distributions $v(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$ with $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k \in \mathcal{G}$ satisfying

$$
\|v\|_{H^s_{\mathrm{mix}}(\mathcal{G}^k)} := \langle v, v \rangle^{1/2}_{H^s_{\mathrm{mix}}(\mathcal{G}^k)} < \infty,
$$

(2.18)

$$
\langle v, w \rangle_{H^s_{\mathrm{mix}}(\mathcal{G}^k)} := \sum_{\boldsymbol{\ell}=0}^{\infty} \sum_{\boldsymbol{m}=-\boldsymbol{\ell}}^{\boldsymbol{\ell}} \left(\prod_{i=1}^{k} (1 + \ell_i)^{2s} \right) \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} \widehat{w}_{\boldsymbol{\ell},\boldsymbol{m}}
$$

with the Fourier coefficients

(2.19) $\qquad \widehat{v}_{\boldsymbol{\ell},\boldsymbol{m}} := \displaystyle\int_{\boldsymbol{x}_1 \in \mathbb{S}} \cdots \int_{\boldsymbol{x}_k \in \mathbb{S}} v(\rho(\boldsymbol{x}_1), \ldots, \rho(\boldsymbol{x}_k)) \left(\prod_{i=1}^{k} Y_{\ell_i, m_i}(\boldsymbol{x}_i) \right) d\sigma_{\boldsymbol{x}_1} \ldots d\sigma_{\boldsymbol{x}_k}$

Recalling definition (2.3) we observe that $H^s_{\mathrm{mix}}(\mathcal{G}^k)$ is isometrically isomorphic to the tensor product space $H^s(\mathcal{G})^{(k)}$. These spaces will be identified in what follows. We also use the notation $H^s_{\mathrm{mix}}(K^k)$ for the tensor product $H^s(K)^{(k)}$ where K is a compact subset of $\mathbb{R}^3$.

Sobolev spaces on bounded domains in $\mathbb{R}^3$ are defined, as usual, as spaces of all distributions whose partial derivatives are square integrable. Proper treatment of the transmission problem (2.11a)–(2.11d) in unbounded domains in $\mathbb{R}^3$ requires a special care. Following [17], for an unbounded domain $U \subset \mathbb{R}^3$ we introduce the space

(2.20) $\qquad H^1_w(U) := \left\{ v \in \mathcal{D}'(U) : \|v\|_{H^1_w(U)} = \left(\displaystyle\int_U \left(|\nabla v|^2 + \frac{|v(\boldsymbol{x})|^2}{1 + |\boldsymbol{x}|^2} \right) d\boldsymbol{x} \right)^{1/2} < +\infty \right\}.$

Specifically, for a given partition $\mathbb{R}^3 = \overline{D^\epsilon_-} \cup \overline{D^\epsilon_+}$ we define the space

(2.21) $\qquad W_\epsilon := \left\{ v = (v_-, v_+) \in H^1(D^\epsilon_-) \times H^1_w(D^\epsilon_+) : [v]_{\Gamma^\epsilon} = 0 \right\}$

which is a weighted Sobolev space on $D^\epsilon_- \cup D^\epsilon_+$ with corresponding norm and seminorm
(2.22)

$$
\|v\|_{W_\epsilon} := \left(\|v_-\|^2_{H^1(D^\epsilon_-)} + \|v_+\|^2_{H^1_w(D^\epsilon_+)} \right)^{1/2}, \quad |v|_{W_\epsilon} := \left(\int_{D^\epsilon_-} |\nabla v_-|^2 \, d\boldsymbol{x} + \int_{D^\epsilon_+} |\nabla v_+|^2 \, d\boldsymbol{x} \right)^{1/2}.
$$

The following lemma which will be frequently used in the rest of the paper states the equivalence between the norm $\|\cdot\|_{W_\epsilon}$ and seminorm $|\cdot|_{W_\epsilon}$. The proof of this result follows by the Friedrichs inequality and the technique in the proof of [17, Theorem 2.10.10].

LEMMA 2.2. *The seminorm $|\cdot|_{W_\epsilon}$ is also a norm in W_ϵ which is equivalent to $\|\cdot\|_{W_\epsilon}$.*

3. Shape calculus. The aim of the present section is the systematic development of the linearization theory for the solution u^ϵ of the model problem (2.11a)–(2.11d) with respect to the shape of the perturbed interface Γ^ϵ. This techniques is also known as *shape calculus* and originates from shape optimization; see [20] and references therein. For this purpose, in the first three subsections that follow, we temporarily stay away from randomness and consider only deterministic perturbed interfaces.

3.1. Perturbation of deterministic interfaces. In this subsection we collect several properties of perturbed interfaces which are important for the subsequent analysis. Assume that the perturbation function κ is a fixed deterministic function in $W^{1,\infty}(\Gamma^0)$, in particular κ is independent of ω. Then Γ^ϵ is defined by

$$(3.1) \qquad \Gamma^\epsilon := \{\boldsymbol{x} + \epsilon\kappa(\boldsymbol{x})\boldsymbol{n}^0(\boldsymbol{x}) : \boldsymbol{x} \in \Gamma^0\}, \quad \epsilon > 0.$$

As already noticed in Section 2.2, Γ^ϵ is a closed Lipschitz manifold in $\mathbb{R}^3$ provided $0 \le \epsilon \le \epsilon_0$ and ϵ_0 is sufficiently small. In this case Γ^ϵ introduces a decomposition of $\mathbb{R}^3$ into the interior and exterior subdomains D_-^ϵ and D_+^ϵ, respectively.

Following [20], we define a mapping $T^\epsilon : \mathbb{R}^3 \to \mathbb{R}^3$ which transforms Γ^0 into Γ^ϵ and $D_\pm^0$ into $D_\pm^\epsilon$, respectively, by

$$(3.2) \qquad T^\epsilon(\boldsymbol{x}) := \boldsymbol{x} + \epsilon\tilde{\kappa}(\boldsymbol{x})\tilde{\boldsymbol{n}}^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3,$$

where $\tilde{\kappa}$ and $\tilde{\boldsymbol{n}}^0$ are any smoothness-preserving extensions of κ and $\boldsymbol{n}^0$ into $\mathbb{R}^3$. We require in particular that $\tilde{\kappa} \in W^{1,\infty}(\mathbb{R}^3)$. Without loss of generality we assume that the extension $\tilde{\kappa}$ vanishes outside a sufficiently large ball $B_R := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < R\}$ containing Γ^ϵ for any $0 \le \epsilon \le \epsilon_0$. This implies that the perturbation mapping $T^\epsilon(\boldsymbol{x})$ is an identity in the complement $B_R^c := \mathbb{R}^3 \setminus \overline{B_R}$, i.e.

$$(3.3) \qquad T^\epsilon(\boldsymbol{x}) = \boldsymbol{x} \qquad \forall \boldsymbol{x} \in B_R^c.$$

For the ease of notation we abbreviate

$$(3.4) \qquad V(\boldsymbol{x}) := \tilde{\kappa}(\boldsymbol{x})\tilde{\boldsymbol{n}}^0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

In [20], V is called *the velocity field* of the mapping T^ϵ. The following result is straightforward.

LEMMA 3.1. *Assuming $\tilde{\kappa} \in W^{1,\infty}(\mathbb{R}^3)$ and $\tilde{\kappa}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in B_R^c$, there hold $V \in \left(H^1(\mathbb{R}^3)\right)^3$ and*

$$\frac{\partial^m V(\boldsymbol{x})}{\partial x_l^m} = \boldsymbol{0} \quad \forall \boldsymbol{x} \in B_R^c, \quad l = 1, 2, 3, \quad m = 0, 1.$$

Recall the definition (2.21) of the weighted space W_ϵ associated to the splitting $\mathbb{R}^3 = \overline{D_-^\epsilon} \cup \overline{D_+^\epsilon}$. It can be proved that a function v belongs to W_ϵ if and only if the composition $v \circ T^\epsilon$ belongs to W_0, and there hold

$$(3.5) \qquad \begin{aligned} \|(v^\epsilon)_-\|_{H^1(D_-^\epsilon)} &\simeq \|(v^\epsilon \circ T^\epsilon)_-\|_{H^1(D_-^0)} \\ \|(v^\epsilon)_+\|_{H_w^1(D_+^\epsilon)} &\simeq \|(v^\epsilon \circ T^\epsilon)_+\|_{H_w^1(D_+^0)} \\ \|v^\epsilon\|_{W_\epsilon} &\simeq \|v^\epsilon \circ T^\epsilon\|_{W_0}. \end{aligned}$$

In the subsequent analysis, for any 3 by 3 matrix $A(\boldsymbol{x})$ whose entries are functionals of $\boldsymbol{x} \in U \subset \mathbb{R}^3$, we denote

$$\|A(\cdot)\|_{L^p(U)} := \max_{i,j=1,2,3}\{\|A_{i,j}(\cdot)\|_{L^p(U)}\}, \quad 1 \le p \le \infty,$$

where A_{ij} are components of A.

The following three lemmas state some important properties of the mapping T^ϵ which will be used later in this section. Until the end of this section we assume that T^ϵ is defined by (3.2) and (3.3) with $\tilde\kappa \in C^1(\mathbb{R}^3)$, and denote its Jacobian matrix and Jacobian determinant by J_{T^ϵ} and $\gamma(\epsilon, \cdot)$, respectively.

LEMMA 3.2. *Consider* $A(\epsilon, \cdot) := \gamma(\epsilon, \cdot) J_{T^\epsilon}^{-1} J_{T^\epsilon}^{-\top}$, *where* $J_{T^\epsilon}^\top$ *is the transpose of* J_{T^ϵ}. *Then there hold*

$$(3.6) \qquad \lim_{\epsilon \to 0} \|A(\epsilon, \cdot) - I\|_{L^\infty(\mathbb{R}^3)} = 0$$

and

$$(3.7) \qquad \lim_{\epsilon \to 0} \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} - A'(0, \cdot) \right\|_{L^2(\mathbb{R}^3)} = 0.$$

Here, $A'(0, \cdot)$ *is the Gâteaux derivative of* A *(determined by* T^ϵ*) at* $\epsilon = 0$*, namely*

$$A'(0, \boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{A(\epsilon, \boldsymbol{x}) - I(\boldsymbol{x})}{\epsilon}, \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Proof. Denoting $V(\boldsymbol{x}) := (V_1(\boldsymbol{x}), V_2(\boldsymbol{x}), V_3(\boldsymbol{x}))^\top$, the Jacobian matrix and the Jacobian of T^ϵ are given by

$$(3.8) \qquad J_{T^\epsilon}(\boldsymbol{x}) = \begin{bmatrix} 1 + \epsilon \dfrac{\partial V_1(\boldsymbol{x})}{\partial x_1} & \epsilon \dfrac{\partial V_1(\boldsymbol{x})}{\partial x_2} & \epsilon \dfrac{\partial V_1(\boldsymbol{x})}{\partial x_3} \\ \epsilon \dfrac{\partial V_2(\boldsymbol{x})}{\partial x_1} & 1 + \epsilon \dfrac{\partial V_2(\boldsymbol{x})}{\partial x_2} & \epsilon \dfrac{\partial V_2(\boldsymbol{x})}{\partial x_3} \\ \epsilon \dfrac{\partial V_3(\boldsymbol{x})}{\partial x_1} & \epsilon \dfrac{\partial V_3(\boldsymbol{x})}{\partial x_2} & 1 + \epsilon \dfrac{\partial V_3(\boldsymbol{x})}{\partial x_3} \end{bmatrix}$$

and

$$\gamma(\epsilon, \boldsymbol{x}) = \left| 1 + \epsilon \left(\sum_{k=1}^3 \frac{\partial V_k(\boldsymbol{x})}{\partial x_k} \right) + \epsilon^2 \left(\sum_{\substack{k,l=1 \\ k \neq l}}^3 \frac{\partial V_k(\boldsymbol{x})}{\partial x_k} \frac{\partial V_l(\boldsymbol{x})}{\partial x_l} - \frac{\partial V_l(\boldsymbol{x})}{\partial x_k} \frac{\partial V_k(\boldsymbol{x})}{\partial x_l} \right) \right.$$

$$\left. + \epsilon^3 \left(\sum_{i,j,k=1}^3 \mathrm{sign}(i,j,k) \frac{\partial V_i(\boldsymbol{x})}{\partial x_1} \frac{\partial V_j(\boldsymbol{x})}{\partial x_2} \frac{\partial V_k(\boldsymbol{x})}{\partial x_3} \right) \right|$$

$$(3.9) \qquad =: \left| 1 + \epsilon \gamma_1(\boldsymbol{x}) + \epsilon^2 \gamma_2(\boldsymbol{x}) + \epsilon^3 \gamma_3(\boldsymbol{x}) \right|.$$

Here $\mathrm{sign}(i,j,k)$ denotes the sign of the permutation (i,j,k). The entries $A_{ij}(\epsilon, \boldsymbol{x})$, $i,j = 1,2,3$, of the matrix $A(\epsilon, \boldsymbol{x})$ are given by

$$(3.10) \qquad A_{ij}(\epsilon, \boldsymbol{x}) = \gamma(\epsilon, \boldsymbol{x})^{-1} \left(\delta_{ij} + \sum_{n=1}^4 \epsilon^n h_{ijn}(\boldsymbol{x}) \right),$$

where h_{ijn} is a polynomial of partial derivatives of V and δ_{ij} is the Kronecker delta. Using Lemma 3.1, we deduce

$$(3.11) \qquad \begin{aligned} &\gamma_n, \ h_{ijn} \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \quad i,j = 1,2,3 \text{ and } n = 1, \dots, 4, \\ &\lim_{\epsilon \to 0} \|\gamma(\epsilon, \cdot)\|_{L^\infty(\mathbb{R}^3)} > 0, \end{aligned}$$

where γ_1, γ_2, γ_3 are defined by (3.9) and $\gamma_4 := 0$ for notational convenience later. In particular, for sufficiently small $\epsilon > 0$, there holds

(3.12) $$\gamma(\epsilon, \boldsymbol{x}) = 1 + \epsilon\gamma_1(\boldsymbol{x}) + \epsilon^2\gamma_2(\boldsymbol{x}) + \epsilon^3\gamma_3(\boldsymbol{x}) \geq c > 0 \qquad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

Consider from now on sufficiently small $\epsilon > 0$. It follows from (3.10) and (3.12) that the ij-entry of the matrix $A(\epsilon, \boldsymbol{x}) - I$ is

(3.13) $$A_{ij}(\epsilon, \cdot) - \delta_{ij} = \epsilon\,\gamma(\epsilon, \cdot)^{-1} \sum_{n=1}^{4} \epsilon^{n-1}\big(h_{ijn} - \delta_{ij}\gamma_n\big).$$

Hence, (3.11) yields

$$\|A_{ij}(\epsilon, \cdot) - \delta_{ij}\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \text{as} \quad \epsilon \to 0,$$

proving (3.6).

From (3.13), we have

(3.14) $$\frac{A_{ij}(\epsilon, \cdot) - \delta_{ij}}{\epsilon} = \gamma(\epsilon, \cdot)^{-1} \sum_{n=1}^{4} \epsilon^{n-1}\big(h_{ijn} - \delta_{ij}\gamma_n\big).$$

Taking the limit when ϵ goes to 0, noting that $\gamma(\epsilon, \cdot) \to 1$, we obtain

(3.15) $$A'_{ij}(0, \cdot) = h_{ij1} - \delta_{ij}\gamma_1, \quad i, j = 1, 2, 3.$$

Subtracting (3.15) from (3.14) side by side, we obtain

(3.16) $$\frac{A_{ij}(\epsilon, \cdot) - \delta_{ij}}{\epsilon} - A'_{ij}(0, \cdot) = \gamma(\epsilon, \cdot)^{-1} \Big(\sum_{n=2}^{4} \epsilon^{n-1}(h_{ijn} - \delta_{ij}\gamma_n) - (h_{ij1} - \delta_{ij}\gamma_1)(\gamma(\epsilon, \cdot) - 1) \Big).$$

Noting (3.11), we infer

$$\lim_{\epsilon \to 0} \left\| \frac{A_{ij}(\epsilon, \cdot) - \delta_{ij}}{\epsilon} - A'_{ij}(0, \cdot) \right\|_{L^2(\mathbb{R}^3)} = 0,$$

proving (3.7). $\square$

LEMMA 3.3. *For any function $v \in L^2(\mathbb{R}^3)$, there holds*

$$\lim_{\epsilon \to 0} \left\| \sqrt{1 + |\cdot|^2}\big(\gamma(\epsilon, \cdot)\,v \circ T^\epsilon - v\big) \right\|_{L^2(\mathbb{R}^3)} = 0.$$

Proof. Since $T^\epsilon(\boldsymbol{x}) = \boldsymbol{x}$ for any $\boldsymbol{x} \in B_R^c$, see (3.3), the Jacobian satisfies

(3.17) $$\gamma(\epsilon, \boldsymbol{x}) = 1 \quad \text{for any} \quad \boldsymbol{x} \in B_R^c.$$

Therefore,

$$\left\| \sqrt{1 + |\cdot|^2}\big(\gamma(\epsilon, \cdot) - 1\big)(v \circ T^\epsilon) \right\|_{L^2(\mathbb{R}^3)} = \left\| \sqrt{1 + |\cdot|^2}\big(\gamma(\epsilon, \cdot) - 1\big)(v \circ T^\epsilon) \right\|_{L^2(B_R)}$$
$$\leq \sqrt{1 + R^2}\, \|\gamma(\epsilon, \cdot) - 1\|_{L^\infty(\mathbb{R}^3)}\, \|v \circ T^\epsilon\|_{L^2(\mathbb{R}^3)}$$
$$\leq C\epsilon\, \|v \circ T^\epsilon\|_{L^2(\mathbb{R}^3)}.$$

Using the change of variables $\boldsymbol{y} = T^\epsilon(\boldsymbol{x})$ and noting (3.11), we have

$$\|v \circ T^\epsilon\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |v(\boldsymbol{y})|^2 \left(\gamma(\epsilon, (T^\epsilon)^{-1}(\boldsymbol{y}))\right)^{-1} d\boldsymbol{y} \leq C \|v\|_{L^2(\mathbb{R}^3)}.$$

Therefore,

(3.18) $$\lim_{\epsilon \to 0} \left\| \sqrt{1 + |\cdot|^2} (\gamma(\epsilon, \boldsymbol{x}) - 1)(v \circ T^\epsilon) \right\|_{L^2(\mathbb{R}^3)} = 0.$$

Furthermore, (3.3) also gives

$$\left\| \sqrt{1 + |\cdot|^2} (v \circ T^\epsilon - v) \right\|_{L^2(\mathbb{R}^3)} = \left\| \sqrt{1 + |\cdot|^2} (v \circ T^\epsilon - v) \right\|_{L^2(B_R)} \leq \sqrt{1 + R^2} \|v \circ T^\epsilon - v\|_{L^2(B_R)}.$$

Note that $\lim_{\epsilon \to 0} \|v \circ T^\epsilon - v\|_{L^2(B_R)} = 0$ if v is continuous. By using a density argument we deduce that $\lim_{\epsilon \to 0} \|v \circ T^\epsilon - v\|_{L^2(B_R)} = 0$ for $v \in L^2(B_R)$. Hence,

$$\lim_{\epsilon \to 0} \left\| \sqrt{1 + |\cdot|^2} (v \circ T^\epsilon - v) \right\|_{L^2(\mathbb{R}^3)} = 0.$$

The above identity and (3.18) together with the triangle inequality give the required result. $\square$

LEMMA 3.4. *For any function $v \in H^1(\mathbb{R}^3)$, there holds*

$$\lim_{\epsilon \to 0} \left\| \sqrt{1 + |\cdot|^2} \left(\frac{\gamma(\epsilon, \cdot)(v \circ T^\epsilon) - v}{\epsilon} - \operatorname{div}(vV) \right) \right\|_{L^2(\mathbb{R}^3)} = 0.$$

Proof. Noting (3.3), Lemma 3.1, (3.17) and the triangle inequality, we obtain

$$\left\| \sqrt{1 + |\cdot|^2} \left(\frac{\gamma(\epsilon, \cdot)(v \circ T^\epsilon) - v}{\epsilon} - \operatorname{div}(vV) \right) \right\|_{L^2(\mathbb{R}^3)}$$

$$= \left\| \sqrt{1 + |\cdot|^2} \left(\frac{\gamma(\epsilon, \cdot)(v \circ T^\epsilon) - v}{\epsilon} - \operatorname{div}(vV) \right) \right\|_{L^2(B_R)}$$

$$\lesssim \left\| \frac{\gamma(\epsilon, \cdot)(v \circ T^\epsilon) - v}{\epsilon} - \operatorname{div}(vV) \right\|_{L^2(B_R)}$$

(3.19) $$\leq \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} (v \circ T^\epsilon) - v \operatorname{div} V \right\|_{L^2(B_R)} + \left\| \frac{v \circ T^\epsilon - v}{\epsilon} - V \cdot \nabla v \right\|_{L^2(B_R)}.$$

Recall from (3.9) that $\gamma_1 = \operatorname{div} V$. It follows from (3.12) that

$$\frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} (v \circ T^\epsilon) - v \operatorname{div} V = \gamma_1(v \circ T^\epsilon - v) + \epsilon(\gamma_2 + \epsilon\gamma_3)(v \circ T^\epsilon).$$

Employing the density argument as in proof of Lemma 3.3, we obtain

$$\lim_{\epsilon \to 0} \|\gamma_1(v \circ T^\epsilon - v)\|_{L^2(B_R)} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \|\epsilon(\gamma_2 + \epsilon\gamma_3)(v \circ T^\epsilon)\|_{L^2(B_R)} = 0,$$

so that

$$\lim_{\epsilon \to 0} \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} (v \circ T^\epsilon) - v \operatorname{div} V \right\|_{L^2(B_R)} = 0.$$

The second term on the right hand side of (3.19) also tends to zero by a density argument, noting that $V = \partial T^\epsilon / \partial \epsilon$ at $\epsilon = 0$. This completes the proof of the lemma. $\square$

3.2. Material and shape derivatives. In this subsection, for notational convenience we use the notation D^ϵ for D^ϵ_- or D^ϵ_+, and $\mathcal{H}^1(D^\epsilon)$ for $H^1(D^\epsilon_-)$ or $H^1_w(D^\epsilon_+)$.

DEFINITION 3.5. *For any sufficiently small ϵ, let v^ϵ be an element in $\mathcal{H}^1(D^\epsilon)$ or $H^{1/2}(\Gamma^\epsilon)$. The material derivative of v^ϵ, denoted by $\dot{v}$, is defined by*

$$(3.20) \qquad \dot{v} := \lim_{\epsilon \to 0} \frac{v^\epsilon \circ T^\epsilon - v^0}{\epsilon},$$

if the limit exists in the corresponding space $\mathcal{H}^1(D^0)$ or $H^{1/2}(\Gamma^0)$. The shape derivative of v^ϵ is defined by

$$(3.21) \qquad v' = \begin{cases} \dot{v} - \nabla v^0 \cdot V & \text{if } v^\epsilon \in \mathcal{H}^1(D^\epsilon), \\ \dot{v} - \nabla_{\Gamma^0} v^0 \cdot V & \text{if } v^\epsilon \in H^{1/2}(\Gamma^\epsilon), \end{cases}$$

where ∇_{Γ^0} denotes the surface gradient.

LEMMA 3.6. *If v' is a shape derivative of $v^\epsilon \in \mathcal{H}^1(D^\epsilon)$, then for any compact set $K \subset\subset D^0$ we have*

$$(3.22) \qquad v' = \lim_{\epsilon \to 0} \frac{v^\epsilon - v^0}{\epsilon} \quad \text{in} \quad H^1(K).$$

Proof. Given $K \subset\subset D^0$, there exists an $\epsilon_0 > 0$ such that $K \subset\subset D^\epsilon$ for all $0 \leq \epsilon \leq \epsilon_0$. We denote by $\mathcal{T} : [0, \epsilon_0] \times \mathbb{R}^3 \to \mathbb{R}^3$ the mapping given by

$$\mathcal{T}(\epsilon, \boldsymbol{x}) := T^\epsilon(\boldsymbol{x}), \quad \forall (\epsilon, \boldsymbol{x}) \in [0, \epsilon_0] \times \mathbb{R}^3.$$

We also denote by $\tilde{v}(\epsilon, \boldsymbol{x}) := v^\epsilon(\boldsymbol{x})$ for any $0 \leq \epsilon \leq \epsilon_0$ and $\boldsymbol{x} \in D^\epsilon$. By the definition of material derivative, we have

$$\dot{v} = \frac{\partial}{\partial \epsilon} \tilde{v}(\epsilon, \mathcal{T}(\epsilon, \cdot)) \Big|_{\epsilon=0}, \quad \text{in} \quad H^1(K).$$

Applying the chain rule, we obtain

$$\dot{v} = \frac{\partial \tilde{v}}{\partial \epsilon}(0, \mathcal{T}(0, \cdot)) + \nabla \tilde{v}(0, \mathcal{T}(0, \cdot)) \cdot \frac{\partial \mathcal{T}(0, \cdot)}{\partial \epsilon}$$
$$= \frac{\partial \tilde{v}(0, \cdot)}{\partial \epsilon} + \nabla v^0 \cdot V, \quad \text{in} \quad H^1(K).$$

This implies

$$v' = \frac{\partial \tilde{v}(0, \cdot)}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{v^\epsilon - v^0}{\epsilon} \quad \text{in} \quad H^1(K).$$

$\square$

REMARK 3.7. *The limit in the above lemma does not hold in $\mathcal{H}^1(D^0)$ since in general, v^ϵ does not belong to $\mathcal{H}^1(D^0)$.*

Similar definitions can be introduced for vector functions $\boldsymbol{v}$. The following lemmas state some useful properties of material and shape derivatives which will be used frequently in the remainder of the paper.

LEMMA 3.8. *Let $\dot{v}$, $\dot{w}$ be material derivatives, and v', w' be shape derivatives of v^ϵ, w^ϵ in $\mathcal{H}^1(D^\epsilon)$, $\epsilon \geq 0$, respectively. Then the following statements are true.*

(i) *The material and shape derivatives of the product $v^\epsilon w^\epsilon$ are $\dot{v}w^0 + v^0\dot{w}$ and $v'w^0 + v^0w'$,*
respectively.

(ii) *The material and shape derivatives of the quotient v^ϵ/w^ϵ are $(\dot{v}w^0 - v^0\dot{w})/(w^0)^2$ and $(v'w^0 -*
v^0w')/(w^0)^2$, respectively, provided that all the fractions are well-defined.

(iii) *If $v^\epsilon = v$ for all $\epsilon \geq 0$, then $\dot{v} = \nabla v^0 \cdot V = \nabla v \cdot V$ and $v' = 0$.*

(iv) *If*

$$\mathcal{J}_1(D^\epsilon) := \int_{D^\epsilon} v^\epsilon \, d\boldsymbol{x}, \ \ \mathcal{J}_2(D^\epsilon) := \int_{\Gamma^\epsilon} v^\epsilon \, d\sigma, \ \ and \ dJ_i(D^\epsilon)|_{\epsilon=0} := \lim_{\epsilon \to 0} \frac{J_i(D^\epsilon) - J_i(D^0)}{\epsilon}, \ \ i = 1, 2,$$

then

$$d\mathcal{J}_1(D^\epsilon)|_{\epsilon=0} = \int_{D^0} v' \, d\boldsymbol{x} + \int_{\Gamma^0} v^0 \langle V, \boldsymbol{n}^0 \rangle \, d\sigma$$

and

$$d\mathcal{J}_2(D^\epsilon)|_{\epsilon=0} = \int_{\Gamma^0} v' \, d\sigma + \int_{\Gamma^0} \left(\frac{\partial v^0}{\partial n} + \operatorname{div}_{\Gamma^0}(\boldsymbol{n}^0) \, v^0 \right) \langle V, \boldsymbol{n}^0 \rangle \, d\sigma.$$

Proof. Statements (i)–(iii) can be obtained by using elementary calculations. Statement (iv) is proved in [20, pages 113, 116]. $\square$

LEMMA 3.9. *The material and shape derivatives of the normal field $\boldsymbol{n}^\epsilon$ are given by*

$$\dot{\boldsymbol{n}} = \boldsymbol{n}' = -\nabla_{\Gamma^0}\kappa.$$

Proof. We start by recalling that the material and the shape derivative of surface fields are identical in the case of normal surface perturbation (3.4). Particularly, from (3.4) and (3.21) we find

$$\dot{\boldsymbol{n}} - \boldsymbol{n}' = \nabla_{\Gamma^0}\boldsymbol{n}^0 \cdot \kappa\boldsymbol{n}^0 = 0.$$

Recall that the unit normal vector field $\boldsymbol{n}^\epsilon$ of the perturbed interface Γ^ϵ is related to that of the reference interface Γ^0 by

$$\boldsymbol{n}^\epsilon \circ T^\epsilon(\boldsymbol{x}) = \frac{J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x})) \, \boldsymbol{n}^0(\boldsymbol{x})}{\left| J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x})) \, \boldsymbol{n}^0(\boldsymbol{x}) \right|}.$$

Therefore,

$$\dot{\boldsymbol{n}} = \lim_{\epsilon \to 0} \frac{\boldsymbol{n}^\epsilon \circ T^\epsilon(\boldsymbol{x}) - \boldsymbol{n}^0(\boldsymbol{x})}{\epsilon}$$

$$= \left(\lim_{\epsilon \to 0} \frac{J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x})) - I}{\epsilon} - \lim_{\epsilon \to 0} \frac{\left| J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x}))\boldsymbol{n}^0(\boldsymbol{x}) \right| - 1}{\epsilon} \right) \lim_{\epsilon \to 0} \frac{\boldsymbol{n}^0(\boldsymbol{x})}{\left| J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x})) \, \boldsymbol{n}^0(\boldsymbol{x}) \right|}$$

$$(3.23) \qquad = \left(\frac{dJ_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x}))}{d\epsilon} \bigg|_{\epsilon=0} - \frac{d \left| J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x}))\boldsymbol{n}^0(\boldsymbol{x}) \right|}{d\epsilon} \bigg|_{\epsilon=0} \right) \boldsymbol{n}^0(\boldsymbol{x}),$$

noting from (3.8) that

$$\lim_{\epsilon \to 0} J_{T^\epsilon}^{-\top} = \lim_{\epsilon \to 0} J_{T^\epsilon} = I.$$

Since $I = J_{T^\epsilon}^{-1}(T^\epsilon(\boldsymbol{x})) \, J_{T^\epsilon}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$, we have $\boldsymbol{0} = \frac{d}{d\epsilon}\big(J_{T^\epsilon}^{-1} J_{T^\epsilon}\big)|_{\epsilon=0}$, which together with the product rule and (3.8) yields

$$(3.24) \qquad \frac{d}{d\epsilon}\big(J_{T^\epsilon}^{-\top}(T^\epsilon(\boldsymbol{x}))\big)\Big|_{\epsilon=0} = -(J_{T^0})^{-\top}\Big(\frac{d}{d\epsilon}(J_{T^\epsilon}^\top)\Big|_{\epsilon=0}\Big)(J_{T^0})^{-1} = -\frac{d}{d\epsilon}(J_{T^\epsilon})\Big|_{\epsilon=0} = -J_V^\top,$$

We also have, using the fact that $\big|J_{T^0}^{-\top} \boldsymbol{n}^0\big| = 1$,

$$\frac{d}{d\epsilon}\big|J_{T^\epsilon}^{-\top}\boldsymbol{n}^0\big|\Big|_{\epsilon=0} = \big|J_{T^0}^{-\top}\boldsymbol{n}^0\big|\frac{d}{d\epsilon}\big|J_{T^\epsilon}^{-\top}\boldsymbol{n}^0\big|\Big|_{\epsilon=0} = \frac{1}{2}\frac{d}{d\epsilon}\Big(\big|J_{T^\epsilon}^{-\top}\boldsymbol{n}^0\big|^2\Big)\Big|_{\epsilon=0}$$

$$(3.25) \qquad\qquad = \frac{1}{2}\Big\langle \frac{d}{d\epsilon}\big(J_{T^\epsilon}^{-1} J_{T^\epsilon}^{-\top}\big)\boldsymbol{n}^0, \boldsymbol{n}^0 \Big\rangle = -\frac{1}{2}\big\langle(J_V^\top + J_V)\boldsymbol{n}^0, \boldsymbol{n}^0\big\rangle.$$

Simple calculation reveals that

$$(3.26) \qquad\qquad J_V^\top = \nabla\kappa\,(\boldsymbol{n}^0)^\top \quad \text{and} \quad (J_V^\top + J_V)\boldsymbol{n}^0 = \nabla\kappa + \big\langle\nabla\kappa, \boldsymbol{n}^0\big\rangle \boldsymbol{n}^0.$$

Inserting (3.24)–(3.26) into (3.23), we obtain

$$\dot{\boldsymbol{n}} = -J_V^\top \boldsymbol{n}^0 + \frac{1}{2}\big\langle(J_V^\top + J_V)\boldsymbol{n}^0, \boldsymbol{n}^0\big\rangle \boldsymbol{n}^0 = -\nabla\kappa + \big\langle\nabla\kappa, \boldsymbol{n}^0\big\rangle \boldsymbol{n}^0 = -\nabla_{\Gamma^0}\kappa,$$

finishing the proof of the lemma. □

3.3. Shape derivative of solutions of transmission problem. In this subsection, we shall discuss the existence of material and shape derivatives of the solutions of transmission problems on perturbed interfaces. Consider a deterministic problem with respect to the reference interface Γ^0:

$$(3.27a) \qquad\qquad\qquad -\alpha\triangle u^0 = f \quad \text{in } D_-^0 \cup D_+^0,$$

$$(3.27b) \qquad\qquad\qquad [u^0] = 0 \quad \text{on } \Gamma^0,$$

$$(3.27c) \qquad\qquad\qquad \Big[\alpha\frac{\partial u^0}{\partial \boldsymbol{n}}\Big] = 0 \quad \text{on } \Gamma^0,$$

$$(3.27d) \qquad\qquad\qquad u^0(\boldsymbol{x}) = \mathcal{O}(|\boldsymbol{x}|^{-1}) \quad \text{when } |\boldsymbol{x}| \to \infty.$$

The perturbed problem corresponding to the perturbed interface Γ^ϵ is given by

$$(3.28a) \qquad\qquad\qquad -\alpha^\epsilon\triangle u^\epsilon = f \quad \text{in } D_-^\epsilon \cup D_+^\epsilon,$$

$$(3.28b) \qquad\qquad\qquad [u^\epsilon] = 0 \quad \text{on } \Gamma^\epsilon,$$

$$(3.28c) \qquad\qquad\qquad \Big[\alpha^\epsilon\frac{\partial u^\epsilon}{\partial \boldsymbol{n}}\Big] = 0 \quad \text{on } \Gamma^\epsilon,$$

$$(3.28d) \qquad\qquad\qquad u^\epsilon(\boldsymbol{x}) = \mathcal{O}(|\boldsymbol{x}|^{-1}) \quad \text{when } |\boldsymbol{x}| \to \infty,$$

where (cf. (2.10))

$$\alpha^\epsilon(\boldsymbol{x}) = \begin{cases} \alpha_-, & \boldsymbol{x} \in D_-^\epsilon \\ \alpha_+, & \boldsymbol{x} \in D_+^\epsilon. \end{cases}$$

LEMMA 3.10. *Suppose $f \in L^2(\mathbb{R}^3) \cap W_0^*$ and $\kappa \in C^1(\Gamma^0)$, then*

$$(3.29) \qquad \lim_{\epsilon \to 0} \left\| u^\epsilon \circ T^\epsilon - u^0 \right\|_{W_0} = 0.$$

Here, W_0^* denotes the dual space of W_0 with respect to the L^2-inner product.

Proof. By multiplying both sides of (3.28a) with an arbitrary function $v \in C_0^\infty(\mathbb{R}^3)$ and integrating over $D_-^\epsilon \cup D_+^\epsilon$, we obtain

$$(3.30) \qquad \int_{\mathbb{R}^3} fv \, d\boldsymbol{x} = -\alpha_- \int_{D_-^\epsilon} \triangle u^\epsilon(\boldsymbol{x}) \, v(\boldsymbol{x}) \, d\boldsymbol{x} - \alpha_+ \int_{D_+^\epsilon} \triangle u^\epsilon(\boldsymbol{x}) \, v(\boldsymbol{x}) \, d\boldsymbol{x}.$$

Applying Green's identity and noting (3.28c), we obtain

$$(3.31) \qquad \int_{D_+^\epsilon \cup D_-^\epsilon} \alpha^\epsilon(\boldsymbol{x}) \, \nabla u^\epsilon(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) = \langle f, v \rangle_{L^2(\mathbb{R}^3)} \quad \forall v \in C_0^\infty(\mathbb{R}^3).$$

Since the space $C_0^\infty(\mathbb{R}^3)$ is dense in W_ϵ (see [17, Remark 2.9.3]), there holds

$$(3.32) \qquad \int_{D_+^\epsilon \cup D_-^\epsilon} \alpha^\epsilon(\boldsymbol{x}) \, \nabla u^\epsilon(\boldsymbol{x}) \cdot \nabla v^\epsilon(\boldsymbol{x}) = \langle f, v^\epsilon \rangle_{L^2(\mathbb{R}^3)} \quad \forall v^\epsilon \in W_\epsilon.$$

Choosing $v^\epsilon = u^\epsilon$ gives

$$|u^\epsilon|_{W_\epsilon}^2 \simeq \langle f, u^\epsilon \rangle_{L^2(\mathbb{R}^3)} \leq \|f\|_{W_\epsilon^*} \|u^\epsilon\|_{W_\epsilon}.$$

It follows from Lemma 2.2 that

$$(3.33) \qquad \|u^\epsilon\|_{W_\epsilon} \lesssim \|f\|_{W_\epsilon^*} \simeq \|f\|_{W_0^*}.$$

On the other hand, using the change of variables $\boldsymbol{x} = T^\epsilon(\boldsymbol{y})$ in (3.32), we have (noting that $\alpha^\epsilon(T^\epsilon(\boldsymbol{y})) = \alpha(\boldsymbol{y})$)

$$(3.34) \qquad \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \, (\nabla w(\boldsymbol{y}))^\top A(\epsilon, \boldsymbol{y}) \, \nabla(u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \, d\boldsymbol{y} = \int_{D_+^0 \cup D_-^0} f(T^\epsilon(\boldsymbol{y})) \, w(\boldsymbol{y}) \gamma(\epsilon, \boldsymbol{y}) \, d\boldsymbol{y},$$

for any $w \in W_0$. We also obtain from problem (3.27a)–(3.27d)

$$(3.35) \qquad \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \, (\nabla w(\boldsymbol{y}))^\top \nabla u^0(\boldsymbol{y}) \, d\boldsymbol{y} = \int_{D_+^0 \cup D_-^0} f(\boldsymbol{y}) \, w(\boldsymbol{y}) \, d\boldsymbol{y},$$

for any $w \in W_0$. Subtracting (3.35) from (3.34) we deduce

$$\int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla w(\boldsymbol{y})^\top \nabla \Big((u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) - u^0(\boldsymbol{y}) \Big) \, d\boldsymbol{y}$$

$$= -\int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \big(\nabla w(\boldsymbol{y}) \big)^\top \Big(A(\epsilon, \boldsymbol{y}) - I \Big) \nabla(u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$(3.36) \qquad + \int_{D_+^0 \cup D_-^0} \Big(\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y}) \Big) w(\boldsymbol{y}) \, d\boldsymbol{y} \qquad \forall w \in W_0.$$

Choosing in (3.36) $w = u^\epsilon \circ T^\epsilon - u^0$ gives

$$\int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \left| \nabla \left((u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) - u^0(\boldsymbol{y}) \right) \right|^2 d\boldsymbol{y}$$

$$= - \int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \left(\nabla \left((u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) - u^0(\boldsymbol{y}) \right) \right)^\top \left(A(\epsilon, \boldsymbol{y}) - I \right) \nabla (u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$+ \int_{D^0_+ \cup D^0_-} \sqrt{1 + |\boldsymbol{y}|^2} \left(\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y}) \right) \frac{(u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) - u^0(\boldsymbol{y})}{\sqrt{1 + |\boldsymbol{y}|^2}} \, d\boldsymbol{y}$$

$$\lesssim \left\| \left(A(\epsilon, \cdot) - I \right) \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla (u^\epsilon \circ T^\epsilon) \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \left(u^\epsilon \circ T^\epsilon - u^0 \right) \right\|_{L^2(\mathbb{R}^3)}$$

$$+ \left\| \sqrt{1 + |\cdot|^2} \left(\gamma(\epsilon, \cdot) f \circ T^\epsilon - f \right) \right\|_{L^2(\mathbb{R}^3)} \left\| \frac{u^\epsilon \circ T^\epsilon - u^0}{\sqrt{1 + |\cdot|^2}} \right\|_{L^2(\mathbb{R}^3)}$$

implying

$$\left\| u^\epsilon \circ T^\epsilon - u^0 \right\|_{W_0} \lesssim \left\| A(\epsilon, \cdot) - I \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla (u^\epsilon \circ T^\epsilon) \right\|_{L^2(\mathbb{R}^3)}$$

$$+ \left\| \sqrt{1 + |\cdot|^2} \left(\gamma(\epsilon, \cdot) f \circ T^\epsilon - f \right) \right\|_{L^2(\mathbb{R}^3)}.$$

Hence, applying Lemma 3.2, noting (3.33) and Lemma 3.3, we obtain

$$\lim_{\epsilon \to 0} \left\| u^\epsilon \circ T^\epsilon - u^0 \right\|_{W_0} = 0,$$

finishing the proof of this lemma. $\square$

LEMMA 3.11. *Assume that $f \in H^1(\mathbb{R}^3) \cap W_0^*$ and $\kappa \in C^1(\Gamma^0)$. Then, u^ϵ has a material derivative belonging to W_0 which is the solution to the following equation with unknown z:*

$$\int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \nabla z(\boldsymbol{y}) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y} = - \int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \nabla u^0(\boldsymbol{y}) A'(0, \boldsymbol{y}) \left(\nabla w(\boldsymbol{y}) \right)^\top d\boldsymbol{y}$$

(3.37)
$$+ \int_{D^0_+ \cup D^0_-} \operatorname{div} \left(V(\boldsymbol{y}) f \right) w(\boldsymbol{y}) \, d\boldsymbol{y} \qquad \forall w \in W_0.$$

Proof. The uniqueness and existence of the solution $z \in W_0$ to the above equation is confirmed by [17, Theorem 2.10.14]. Let $z^\epsilon := (u^\epsilon \circ T^\epsilon - u^0)/\epsilon$. Our task is to prove that $\lim_{\epsilon \to 0} \| z^\epsilon - z \|_{W_0} = 0$. Dividing (3.36) by ϵ we obtain

$$\int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \nabla z^\epsilon(\boldsymbol{y}) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y} = - \int_{D^0_+ \cup D^0_-} \alpha(\boldsymbol{y}) \nabla (u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \frac{A(\epsilon, \boldsymbol{y}) - I}{\epsilon} \left(\nabla w(\boldsymbol{y}) \right)^\top d\boldsymbol{y}$$

(3.38)
$$+ \int_{D^0_+ \cup D^0_-} \frac{\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y})}{\epsilon} w(\boldsymbol{y}) \, d\boldsymbol{y} \qquad \forall w \in W_0.$$

Subtracting (3.37) from (3.38) yields

$$\int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla (z^\epsilon(\boldsymbol{y}) - z(\boldsymbol{y})) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$= -\int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \left(\nabla(u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \frac{A(\epsilon, \boldsymbol{y}) - I}{\epsilon} - \nabla u^0(\boldsymbol{y}) A'(0, \boldsymbol{y}) \right) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$+ \int_{D_+^0 \cup D_-^0} \left(\frac{\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y})}{\epsilon} - \mathrm{div}\left(V(\boldsymbol{y}) f(\boldsymbol{y}) \right) \right) w(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$(3.39) \qquad =: I_1(w) + I_2(w).$$

The first integral in the right hand side of (3.39) can be written as

$$I_1(w) = \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla(u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) \left(\frac{A(\epsilon, \boldsymbol{y}) - I}{\epsilon} - A'(0, \boldsymbol{y}) \right) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y}$$

$$+ \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla\left((u^\epsilon \circ T^\epsilon)(\boldsymbol{y}) - u^0(\boldsymbol{y})\right) A'(0, \boldsymbol{y}) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y},$$

which converges to 0 due to (3.29) and Lemma 3.2. The second integral in the right hand side of (3.39) also converges to 0 due to Lemma 3.4. Therefore, we have

$$(3.40) \qquad \lim_{\epsilon \to 0} \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla (z^\epsilon(\boldsymbol{y}) - z(\boldsymbol{y})) \cdot \nabla w(\boldsymbol{y}) \, d\boldsymbol{y} = 0 \quad \forall w \in W_0.$$

We choose in (3.39) $w = z^\epsilon - z$. Then the absolute value of the first integral on the right hand side of (3.39) can be estimated as

$$|I_1(z^\epsilon - z)| = \left| \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla u^\epsilon(\boldsymbol{y}) \left(\frac{A(\epsilon, \boldsymbol{y}) - I}{\epsilon} - A'(0, \boldsymbol{y}) \right) \cdot \nabla\left(z^\epsilon(\boldsymbol{y}) - z(\boldsymbol{y})\right) d\boldsymbol{y} \right.$$

$$+ \left. \int_{D_+^0 \cup D_-^0} \alpha(\boldsymbol{y}) \nabla\left(u^\epsilon(\boldsymbol{y}) - u^0(\boldsymbol{y})\right) A'(0, \boldsymbol{y}) \cdot \nabla\left(z^\epsilon(\boldsymbol{y}) - z(\boldsymbol{y})\right) d\boldsymbol{y} \right|,$$

$$\lesssim \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^3)} \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} - A'(0, \cdot) \right\|_{L^\infty(\mathbb{R}^3)} \|\nabla(z^\epsilon - z)\|_{L^2(\mathbb{R}^3)}$$

$$(3.41) \qquad + \left\| \nabla\left(u^\epsilon - u^0\right) \right\|_{L^2(\mathbb{R}^3)} \|A'(0, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|\nabla(z^\epsilon - z)\|_{L^2(\mathbb{R}^3)}.$$

The absolute value of the second integral in (3.39) when $w = z^\epsilon - z$ is bounded by

$$(3.42)$$
$$|I_2(z^\epsilon - z)| \leq \left\| \sqrt{1 + |\cdot|^2} \left(\frac{\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y})}{\epsilon} - \mathrm{div}\left(V(\boldsymbol{y}) f(\boldsymbol{y}) \right) \right) \right\|_{L^2(\mathbb{R}^3)} \|z^\epsilon - z\|_{W_0}.$$

Inequalities (3.41) and (3.42) give

$$\|z^\epsilon - z\|_{W_0} \leq \|\nabla u^\epsilon\|_{L^2(\mathbb{R}^3)} \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} - A'(0, \cdot) \right\|_{L^\infty(\mathbb{R}^3)}$$

$$+ \left\| \nabla\left(u^\epsilon - u^0\right) \right\|_{L^2(\mathbb{R}^3)} \|A'(0, \cdot)\|_{L^\infty(\mathbb{R}^3)}$$

$$(3.43) \qquad + \left\| \sqrt{1 + |\cdot|^2} \left(\frac{\gamma(\epsilon, \boldsymbol{y}) f(T^\epsilon(\boldsymbol{y})) - f(\boldsymbol{y})}{\epsilon} - \mathrm{div}\left(V(\boldsymbol{y}) f(\boldsymbol{y}) \right) \right) \right\|_{L^2(\mathbb{R}^3)}.$$

Using this together with (3.29) and Lemma 3.2, we can deduce from (3.43)

$$(3.44) \qquad \lim_{\epsilon \to 0} \|z^\epsilon - z\|_{W_0} = 0.$$

$\square$

Hence, we have shown that the solution of the transmission problem (3.28) has a material derivative, and thus a shape derivative. The latter turns out to be the solution of a transmission problem on the nominal interface Γ^0.

LEMMA 3.12. *Under the assumption of Lemma 3.11, the shape derivative u' of u^ϵ exists and is the solution of the transmission problem*

$$(3.45) \qquad \begin{cases} \Delta u' & = 0 \quad in \ D^0_- \cup D^0_+ \\ [u'] & = g_D \quad on \ \Gamma^0 \\ \left[\alpha \dfrac{\partial u'}{\partial \boldsymbol{n}}\right] & = g_N \quad on \ \Gamma^0 \\ |u'(\boldsymbol{x})| & = \mathcal{O}\left(|\boldsymbol{x}|^{-1}\right) \quad as \ |\boldsymbol{x}| \to \infty, \end{cases}$$

where

$$g_D := -\left[\frac{\partial u^0}{\partial \boldsymbol{n}}\right] \kappa \quad and \quad g_N := \nabla_{\Gamma^0} \cdot \left(\kappa \left[\alpha \nabla_{\Gamma^0} u^0\right]\right).$$

Proof. Existence of u' is confirmed by Lemma 3.11. In this proof only, for notational convenience, we use $\boldsymbol{n}^\epsilon_{\pm}$ to indicate the normal vector to Γ^ϵ pointing outwards $D^\epsilon_{\pm}$, respectively. Note here that $\boldsymbol{n}^\epsilon = \boldsymbol{n}^\epsilon_- = -\boldsymbol{n}^\epsilon_+$. From (3.32) we deduce

$$(3.46) \qquad \alpha_- \int_{D^\epsilon_-} \nabla u^\epsilon_- \cdot \nabla v \, d\boldsymbol{x} + \alpha_+ \int_{D^\epsilon_+} \nabla u^\epsilon_+ \cdot \nabla v \, d\boldsymbol{x} = \langle f, v \rangle_{L_2(\mathbb{R}^3)} \qquad \forall v \in C_0^\infty(\mathbb{R}^3).$$

Denoting

$$J(D^\epsilon_{\pm}) := \alpha_{\pm} \int_{D^\epsilon_{\pm}} \nabla u^\epsilon_{\pm}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, d\boldsymbol{x}$$

and using Green's formula, we obtain

$$J(D^\epsilon_{\pm}) = -\alpha_{\pm} \int_{D^\epsilon_{\pm}} u^\epsilon_{\pm}(\boldsymbol{x}) \triangle v(\boldsymbol{x}) \, d\boldsymbol{x} + \alpha_{\pm} \int_{\Gamma^\epsilon} u^\epsilon_{\pm}(\boldsymbol{x}) \frac{\partial v}{\partial \boldsymbol{n}_{\pm}} \, d\sigma =: \mathcal{J}_1(D^\epsilon_{\pm}) + \mathcal{J}_2(D^\epsilon_{\pm}).$$

By Lemma 3.8, $u' \triangle v$ is the shape derivative of $u^\epsilon \triangle v$. On the other hand, by Lemmas 3.8–3.9, the shape derivative of $\dfrac{\partial v}{\partial \boldsymbol{n}}\Big|_{\Gamma^\epsilon} = \nabla v \cdot \boldsymbol{n}^\epsilon$ is $-\nabla_{\Gamma^0} v \cdot \nabla_{\Gamma^0} \langle V, \boldsymbol{n}^0 \rangle$, so that the shape derivative of $u^\epsilon \dfrac{\partial v}{\partial \boldsymbol{n}}\Big|_{\Gamma^\epsilon}$ is $u' \dfrac{\partial v}{\partial \boldsymbol{n}}\Big|_{\Gamma^0} - u^0 \left(\nabla_{\Gamma^0} v \cdot \nabla_{\Gamma^0} \langle V, \boldsymbol{n}^0 \rangle\right)$. Using Lemma 3.8, we deduce

$$d\mathcal{J}_1(D^\epsilon_{\pm})|_{\epsilon=0} = -\alpha_{\pm} \int_{D^0_{\pm}} u'_{\pm}(\boldsymbol{x}) \triangle v \, d\boldsymbol{x} - \alpha_{\pm} \int_{\Gamma^0} u^0 \triangle v \langle V, \boldsymbol{n}^0_{\pm} \rangle \, d\sigma$$

and

$$
d\mathcal{J}_2(D_\pm^\epsilon)|_{\epsilon=0} = \alpha_\pm \int_{\Gamma^0} \left(u'_\pm \frac{\partial v}{\partial \boldsymbol{n}_\pm} - u^0 \left(\nabla_{\Gamma^0} v \cdot \nabla_{\Gamma^0} \langle V, \boldsymbol{n}^0 \rangle \right) \right) d\sigma + \alpha_\pm \int_{\Gamma^0} \frac{\partial}{\partial \boldsymbol{n}_\pm} \left(u^0 \frac{\partial v}{\partial \boldsymbol{n}_\pm} \right) \langle V, \boldsymbol{n}_\pm^0 \rangle d\sigma
$$

$$
+ \alpha_\pm \int_{\Gamma^0} \mathrm{div}_{\Gamma^0}(\boldsymbol{n}_\pm^0) u^0 \frac{\partial v}{\partial \boldsymbol{n}_\pm} \langle V, \boldsymbol{n}_\pm^0 \rangle \, d\sigma,
$$

since $u_-^0 = u_+^0$ on the interface Γ^0 by (3.27b). Therefore, differentiating by ϵ both sides of (3.46), using Green's formula, the jump condition (3.27c) and noting that $\triangle v = \triangle_{\Gamma^0} v + \mathrm{div}_{\Gamma^0}(\boldsymbol{n}^0)\partial v/\partial\boldsymbol{n} + \partial^2 v/\partial\boldsymbol{n}^2$, we obtain

$$
(3.47) \qquad 0 = \alpha_- \int_{D_-^0} \nabla u' \cdot \nabla v \, d\boldsymbol{x} + \alpha_+ \int_{D_+^0} \nabla u' \cdot \nabla v \, d\boldsymbol{x}
$$

$$
- \alpha_- \int_{\Gamma^0} u \langle V, \boldsymbol{n}_-^0 \rangle \triangle_{\Gamma^0} v \, d\sigma - \alpha_+ \int_{\Gamma^0} u \langle V, \boldsymbol{n}_+^0 \rangle \triangle_{\Gamma^0} v \, d\sigma
$$

$$
- \alpha_- \int_{\Gamma^0} u \nabla_{\Gamma^0} v \cdot \nabla_{\Gamma^0} \langle V, \boldsymbol{n}_-^0 \rangle - \alpha_+ \int_{\Gamma^0} u \nabla_{\Gamma^0} v \cdot \nabla_{\Gamma^0} \langle V, \boldsymbol{n}_+^0 \rangle .
$$

Applying the tangential Green formula on the third and the fourth integrals on the right hand side of the above identity and the product rule, the above identity can be written as

$$
(3.48)
$$
$$
0 = \alpha_- \int_{D_-^0} \nabla u' \cdot \nabla v \, d\boldsymbol{x} + \alpha_+ \int_{D_+^0} \nabla u' \cdot \nabla v \, d\boldsymbol{x} + \int_{\Gamma^0} (\alpha_- \nabla_{\Gamma^0} u_-^0 - \alpha_+ \nabla_{\Gamma^0} u_+^0) \cdot \nabla_{\Gamma^0} v \langle V, \boldsymbol{n}_-^0 \rangle \, d\sigma.
$$

We choose in (3.48) $v \in C_0^\infty(D_\pm)$ to obtain

$$
(3.49) \qquad\qquad\qquad \alpha \triangle u'(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D_\pm^0.
$$

We now choose $v \in C_0^\infty(\mathbb{R}^3)$ and applying the Green's identity to the first two integrals on the right hand side of (3.48), noting (3.49), to obtain

$$
(3.50) \quad 0 = \alpha_- \int_{\Gamma^0} v \frac{\partial u'_-}{\partial \boldsymbol{n}_-} d\sigma + \alpha_+ \int_{\Gamma^0} v \frac{\partial u'_+}{\partial \boldsymbol{n}_+} + \int_{\Gamma^0} (\alpha_- \nabla_{\Gamma^0} u_-^0 - \alpha_+ \nabla_{\Gamma^0} u_+^0) \cdot \nabla_{\Gamma^0} v \langle V, \boldsymbol{n}_-^0 \rangle \, d\sigma.
$$

Applying the tangential Green formula on the surface Γ^0 to the last term on the right hand side of the above identity, we deduce

$$
\int_{\Gamma^0} v \left[\alpha \frac{\partial u'}{\partial \boldsymbol{n}} \right] d\sigma = \int_{\Gamma^0} v \nabla_{\Gamma^0} \cdot \left(\langle V, \boldsymbol{n}_-^0 \rangle [\alpha \nabla_{\Gamma^0} u^0] \right) d\sigma,
$$

yielding

$$
(3.51) \qquad\qquad \left[\alpha \frac{\partial u'}{\partial \boldsymbol{n}} \right] = \nabla_{\Gamma^0} \cdot \left(\langle V, \boldsymbol{n}_-^0 \rangle [\alpha \nabla_{\Gamma^0} u^0] \right) \quad \text{on } \Gamma^0.
$$

Recalling the transmission conditions (3.28b), we have for any smooth function v

$$
\int_{\Gamma^\epsilon} [u^\epsilon] v \, d\sigma = 0.
$$

Differentiating by ϵ both sides, applying Lemma 3.8 we have

$$
\begin{aligned}
0 = d\left(\int_{\Gamma^\epsilon} [u^\epsilon]\, v\, d\sigma\right) &= d\left(\int_{\Gamma^\epsilon_-} u^\epsilon_-\, v\, d\sigma - \int_{\Gamma^\epsilon_+} u^\epsilon_+\, v\, d\sigma\right) \\
&= \int_{\Gamma^0} (u^0_- v)' + \int_{\Gamma^0} \left(\frac{\partial(u^0_- v)}{\partial \boldsymbol{n}_-} + \operatorname{div}_{\Gamma^0}(\boldsymbol{n}^0_-)(u^0_- v)\right) \langle V, \boldsymbol{n}^0_- \rangle\, d\sigma \\
&\quad - \int_{\Gamma^0} (u^0_+ v)' - \int_{\Gamma^0} \left(\frac{\partial(u^0_+ v)}{\partial \boldsymbol{n}_+} + \operatorname{div}_{\Gamma^0}(\boldsymbol{n}^0_+)(u^0_+ v)\right) \langle V, \boldsymbol{n}^0_+ \rangle\, d\sigma \\
&= \int_{\Gamma^0} [u']\, v\, d\sigma + \int_{\Gamma^0} \left[\frac{\partial u^0}{\partial \boldsymbol{n}}\right] v \langle V, \boldsymbol{n}^0_- \rangle\, d\sigma \\
&\quad + \int_{\Gamma^0} [u^0] \left(\frac{\partial v}{\partial \boldsymbol{n}_-} + \operatorname{div}_{\Gamma^0}(\boldsymbol{n}^0_-)\, v\right) \langle V, \boldsymbol{n}^0_- \rangle\, d\sigma \\
&= \int_{\Gamma^0} [u']\, v\, d\sigma + \int_{\Gamma^0} \left[\frac{\partial u^0}{\partial \boldsymbol{n}_-}\right] v \langle V, \boldsymbol{n}^0_- \rangle\, d\sigma,
\end{aligned}
$$

noting that $[u^0] = 0$. Hence, there holds

(3.52) $$[u'] = -\left[\frac{\partial u^0}{\partial \boldsymbol{n}}\right] \langle V, \boldsymbol{n}^0_- \rangle =: g_D.$$

Hence, from (3.49), (3.51) and (3.52), the shape derivative $u' \in H^1(D^0_-) \times H^1_w(D^0_+)$ is the weak solution of the transmission problem (3.45). □

3.4. Random interfaces. In Subsection 3.2, we have defined material and shape derivatives in which the quantity $\kappa(\boldsymbol{x})$ does not contain uncertainty. Since the transmission problem (2.11) is posed on a domain with a random interface (see (2.7)), the shape derivative also depends on ω, and it is necessary to approximate the mean and the covariance fields of the random solutions. The result is given in the following lemma, where we recall the notation $\mathcal{H}^1(D^0_\pm)$ indicating $H^1(D^0_-)$ or $H^1_w(D^0_+)$.

LEMMA 3.13. *Let $u^\epsilon(\omega)$ be the solution of the transmission problem (2.11a)–(2.11d) with the random interface $\Gamma^\epsilon(\omega)$ given by (2.7), and let u^0 denote the solution of the transmission problem with the reference interface Γ^0. Assume that the perturbation function κ belongs to $L^k(\Omega, C^1(\Gamma^0))$ for an integer k and $f \in H^1(\mathbb{R}^3) \cap W^*_0$. Then, for any compact subset $K \subset\subset D^0_\pm$, the expectation and the k-th order central moments of the solution $u^\epsilon(\omega)$ can be approximated, respectively, by*

(3.53) $$\mathbb{E}[u^\epsilon] = u^0 + o(\epsilon) \quad in \quad H^1(K)$$

and

(3.54) $$\mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]] = \epsilon^k \mathcal{M}^k[u'] + o(\epsilon^k) \quad in \quad H^1_{\mathrm{mix}}(K^k).$$

Moreover

(3.55) $$\mathcal{M}^k[u^\epsilon - u^0] = \epsilon^k \mathcal{M}^k[u'] + o(\epsilon^k) \quad in \quad H^1_{\mathrm{mix}}(K^k).$$

Proof. It follows from Lemmas 3.6 and 3.12 that

$$(3.56) \qquad u^\epsilon(\boldsymbol{x}, \omega) = u^0(\boldsymbol{x}) + \epsilon u'(\boldsymbol{x}, \omega) + \epsilon h(\epsilon, \boldsymbol{x}, \omega) \quad \text{in} \quad H^1(K),$$

where h satisfies $\lim_{\epsilon \to 0} \|h(\epsilon, \cdot, \cdot)\|_{L^k(\Omega, H^1(K))} = 0$. This implies

$$\mathbb{E}[u^\epsilon(\boldsymbol{x}, \cdot)] = u^0(\boldsymbol{x}) + \epsilon \mathbb{E}[u'(\boldsymbol{x}, \cdot)] + \epsilon \mathbb{E}[h(\epsilon, \boldsymbol{x}, \cdot)] \quad \text{in} \quad H^1(K).$$

Here, u' is the solution of (3.45) in which the function κ defining g_D and g_N depends on ω and satisfies $\mathbb{E}[\kappa] = 0$; see (2.9). Since u' depends linearly on κ, there also holds $\mathbb{E}[u'] = 0$, yielding (3.53).

By the definition of the statistical moments (2.4) we have

$$\mathcal{M}^k[u^\epsilon - u^0] - \epsilon^k \mathcal{M}^k[u'] = \epsilon^k \big(\mathcal{M}^k[u' + h] - \mathcal{M}^k[u'] \big)$$

and by [23, Corollary V.5.1]

$$\|\mathcal{M}^k[u' + h] - \mathcal{M}^k[u']\|_{H^1_{\mathrm{mix}}(K^k)} \leq \mathbb{E}\Big[\|(u' + h) \otimes \cdots \otimes (u' + h) - u' \otimes \cdots \otimes u'\|_{H^1_{\mathrm{mix}}(K^k)} \Big] =: \mathcal{E}.$$

Then by the triangle inequality, binomial formula and Hölder's inequality with $p = \dfrac{k}{j}$ and $q = \dfrac{k}{k-j}$

$$\begin{aligned}
\mathcal{E} = \mathbb{E}\Bigg[\Big\| \sum_{\substack{v_i = u' \text{ or } h, \\ (v_1, \ldots, v_k) \neq (u', \ldots, u')}} v_1 \otimes \cdots \otimes v_k \Big\|_{H^1_{\mathrm{mix}}(K^k)} \Bigg] \\
\leq \sum_{\substack{v_i = u' \text{ or } h, \\ (v_1, \ldots, v_k) \neq (u', \ldots, u')}} \mathbb{E}\Big[\|v_1 \otimes \cdots \otimes v_k\|_{H^1_{\mathrm{mix}}(K^k)} \Big] \\
= \sum_{\substack{v_i = u' \text{ or } h, \\ (v_1, \ldots, v_k) \neq (u', \ldots, u')}} \mathbb{E}\Big[\|v_1\|_{H^1(K)} \cdots \|v_k\|_{H^1(K)} \Big] \\
= \sum_{j=1}^{k} \binom{k}{j} \mathbb{E}\Big[\|h\|_{H^1(K)}^j \|u'\|_{H^1(K)}^{k-j} \Big] \\
\leq \sum_{j=1}^{k} \binom{k}{j} \mathbb{E}\Big[\|h\|_{H^1(K)}^{jp} \Big]^{\frac{1}{p}} \mathbb{E}\Big[\|u'\|_{H^1(K)}^{(k-j)q} \Big]^{\frac{1}{q}} \\
= \sum_{j=1}^{k} \binom{k}{j} \mathbb{E}\Big[\|h\|_{H^1(K)}^k \Big]^{\frac{j}{k}} \mathbb{E}\Big[\|u'\|_{H^1(K)}^k \Big]^{\frac{k-j}{k}} \\
= \sum_{j=1}^{k} \binom{k}{j} \|h\|_{L^k(\Omega, H^1(K))}^j \|u'\|_{L^k(\Omega, H^1(K))}^{k-j} \\
= o(1)
\end{aligned}$$

and (3.55) follows. An analogous estimate holds for

$$\mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]] - \epsilon^k \mathcal{M}^k[u'] = \epsilon^k \big(\mathcal{M}^k[u' + (h - \mathbb{E}[h])] - \mathcal{M}^k[u'] \big).$$

□

The above lemma states in particular that $\mathcal{M}^k[u^\epsilon - u^0]$, $\mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]]$ and $\epsilon^k \mathcal{M}^k[u']$ coincide in the limit $\epsilon \to 0$, indicating that $\epsilon^k \mathcal{M}^k[u']$ may be a good approximation for $\mathcal{M}^k[u^\epsilon - u^0]$ and $\mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]]$ if ϵ is small. On the other hand, the task of approximation of $\epsilon^k \mathcal{M}^k[u']$ is significantly simpler than approximation of $\mathcal{M}^k[u^\epsilon - u^0]$ or $\mathcal{M}^k[u^\epsilon - \mathbb{E}[u^\epsilon]]$ and reduces to solving the homogeneous transmission problem (3.45).

4. Boundary reduction. In this section we briefly recall boundary integral equation methods to solve (3.45). We rewrite here this problem for convenience.

Find $u' \in H^1(D^0_-) \times H^1_w(D^0_+)$ satisfying

(4.1)
$$\begin{cases} \triangle u' & = 0 \quad \text{in } D^0_\pm \\ [u'] & = g_D(\omega) \quad \text{on } \Gamma^0 \\ \left[\alpha \dfrac{\partial u'}{\partial \boldsymbol{n}} \right] & = g_N(\omega) \quad \text{on } \Gamma^0 \\ |u'(\boldsymbol{x})| & = \mathcal{O}\left(|\boldsymbol{x}|^{-1} \right) \quad \text{as } |\boldsymbol{x}| \to \infty. \end{cases}$$

The single and double layer potentials are given by

(4.2) $$\tilde{\mathcal{V}}w(\boldsymbol{x}) = \int_{\Gamma^0} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} w(\boldsymbol{y}) \, d\sigma_{\boldsymbol{y}}, \qquad \mathcal{W}v(\boldsymbol{x}) = \int_{\Gamma^0} \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{y}}} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} v(\boldsymbol{y}) \, d\sigma_{\boldsymbol{y}}, \quad \boldsymbol{x} \in D^0_\pm$$

for $w \in H^{-1/2}(\Gamma^0)$ and $v \in H^{1/2}(\Gamma^0)$. The limits of these potentials for $\boldsymbol{x}$ approaching Γ^0 are given by (see [15, page 14])

(4.3)
$$\mathcal{V}u(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in D^0_\pm}} \tilde{\mathcal{V}}u(\boldsymbol{y}) \quad \text{for } \boldsymbol{x} \in \Gamma^0,$$

(4.4)
$$\mathcal{K}u(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in D^0_\pm}} \mathcal{W}u(\boldsymbol{y}) \mp \frac{1}{2} u(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma^0,$$

(4.5)
$$\mathcal{K}'u(\boldsymbol{x}) := \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in D^0_\pm}} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{y}} \tilde{\mathcal{V}}u(\boldsymbol{y}) \pm \frac{1}{2} u(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma^0,$$

(4.6)
$$\mathcal{D}u(\boldsymbol{x}) := - \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in D^0_\pm}} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{y}} \mathcal{W}u(\boldsymbol{y}) \quad \text{for } \boldsymbol{x} \in \Gamma^0.$$

The solution of (4.1) is given by

(4.7)
$$u'(\boldsymbol{x}) = \begin{cases} \tilde{\mathcal{V}}(\frac{\partial u'_-}{\partial \boldsymbol{n}})(\boldsymbol{x}) - \mathcal{W}u'_-(\boldsymbol{x}), & \boldsymbol{x} \in D^0_-, \\ \mathcal{W}u'_+(\boldsymbol{x}) - \tilde{\mathcal{V}}(\frac{\partial u'_+}{\partial \boldsymbol{n}})(\boldsymbol{x}), & \boldsymbol{x} \in D^0_+; \end{cases}$$

see e.g. [15]. The Dirichlet-to-Neumann operators are

(4.8)
$$\mathcal{S}_- u'_- := \frac{\partial u'_-}{\partial \boldsymbol{n}} = \mathcal{V}^{-1}(\frac{1}{2}I + \mathcal{K})u'_-,$$

(4.9)
$$\mathcal{S}_+ u'_+ = \frac{\partial u'_+}{\partial \boldsymbol{n}} = \mathcal{V}^{-1}(\mathcal{K} - \frac{1}{2}I)u'_+.$$

These equalities together with (4.7) imply

$$(4.10) \qquad u'(\boldsymbol{x}) = \begin{cases} (\tilde{\mathcal{V}}\mathcal{S}_- - \mathcal{W})(u'_-)(\boldsymbol{x}) =: E_-(u'_-)(\boldsymbol{x}), & \boldsymbol{x} \in D^0_-, \\ (\mathcal{W} - \tilde{\mathcal{V}}\mathcal{S}_+)(u'_+)(\boldsymbol{x}) =: E_+(u'_+)(\boldsymbol{x}), & \boldsymbol{x} \in D^0_+. \end{cases}$$

The randomness of the interface $\Gamma(\omega)$ which is given via the randomness of the vector field $V(\epsilon, \boldsymbol{x}, \omega)$ implies the randomness in the solution u. From (4.10), we have

$$u'(\boldsymbol{x}, \omega) = \begin{cases} E_-(u'_-(\omega)|_{\Gamma^0})(\boldsymbol{x}), & \boldsymbol{x} \in D^0_-, \\ E_+(u'_+(\omega)|_{\Gamma^0})(\boldsymbol{x}), & \boldsymbol{x} \in D^0_+. \end{cases}$$

Tensorizing and integrating both sides of the above equation, we deduce

$$(4.11) \qquad \mathrm{Cov}[u'](\boldsymbol{x}_1, \boldsymbol{x}_2) = \begin{cases} (E_{-,\boldsymbol{x}_1} \otimes E_{-,\boldsymbol{x}_2})\mathrm{Cor}[u'_-|_{\Gamma^0}](\boldsymbol{x}_1, \boldsymbol{x}_2), & \boldsymbol{x}_1, \boldsymbol{x}_2 \in D^0_-, \\ (E_{+,\boldsymbol{x}_1} \otimes E_{+,\boldsymbol{x}_2})\mathrm{Cor}[u'_+|_{\Gamma^0}](\boldsymbol{x}_1, \boldsymbol{x}_2), & \boldsymbol{x}_1, \boldsymbol{x}_2 \in D^0_+, \end{cases}$$

and in general

$$(4.12) \quad \mathcal{M}^k[u'](\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \begin{cases} (E_{-,\boldsymbol{x}_1} \otimes \cdots \otimes E_{-,\boldsymbol{x}_k})\mathcal{M}^k[u'_-|_{\Gamma^0}](\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k), & \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in D^0_-, \\ (E_{+,\boldsymbol{x}_1} \otimes \cdots \otimes E_{+,\boldsymbol{x}_k})\mathcal{M}^k[u'_+|_{\Gamma^0}](\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k), & \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in D^0_+. \end{cases}$$

Equation (4.11) suggests that the covariance of the solution u' in $D^0_\pm$ can be computed from the correlation function of the Dirichlet data $u'_\pm|_{\Gamma^0}$ on the transmission interface.

The jump conditions in (4.1) gives

$$(4.13) \qquad u'_-(\omega) = u'_+(\omega) + g_D(\omega) \quad \text{on} \quad \Gamma^0,$$

and

$$(4.14) \qquad \underbrace{(\alpha_-\mathcal{S}_- - \alpha_+\mathcal{S}_+)}_{=:[\alpha\mathcal{S}]} u'_+(\omega) = g_N(\omega) - (\alpha_-\mathcal{S}_-)g_D(\omega) \quad \text{on} \quad \Gamma^0.$$

We note that for a fixed $\omega \in \Omega$, the right hand side $g_N(\omega) - (\alpha_-\mathcal{S}_-)g_D(\omega) \in H^{-1/2}(\Gamma^0)$. The solution $u'_+(\omega)$ of (4.14) belongs to $H^{1/2}(\Gamma^0)$. The variational form for (4.14) is: Find $u'_+(\omega) \in H^{1/2}(\Gamma^0)$ satisfying

$$(4.15) \qquad B(u'_+(\omega), v) = \langle g_N(\omega) - (\alpha_-\mathcal{S}_-)g_D(\omega), v \rangle \qquad \forall v \in H^{1/2}(\Gamma^0),$$

with the bilinear form $B(\cdot, \cdot)$ and the duality pairing $\langle \cdot, \cdot \rangle$ given by
(4.16)
$$B(v, w) := \int_{\Gamma^0} ([\alpha\mathcal{S}] v) w \, d\sigma \quad \text{and} \quad \langle g, v \rangle := \int_{\Gamma^0} g v \, d\sigma \quad \forall v, w \in H^{1/2}(\Gamma^0), \quad g \in H^{-1/2}(\Gamma^0).$$

We next show the continuity and ellipticity of the operator $[\alpha\mathcal{S}]$ which confirms existence of the unique solution of equation (4.14) for a fixed arbitrary ω.

LEMMA 4.1. *The bilinear form* $B(\cdot, \cdot) : H^{1/2}(\Gamma^0) \times H^{1/2}(\Gamma^0) \to \mathbb{R}$ *is bounded, i.e.*

$$(4.17) \qquad |B(v, w)| \le C_1 \|v\|_{H^{1/2}(\Gamma^0)} \|w\|_{H^{1/2}(\Gamma^0)} \qquad \forall v, w \in H^{1/2}(\Gamma^0),$$

and $H^{1/2}(\Gamma)$-elliptic, i.e.

$$(4.18) \qquad B(v,v) \geq C_2 \, \|v\|^2_{H^{1/2}(\Gamma^0)} \qquad \forall v \in H^{1/2}(\Gamma^0),$$

where the positive constants C_1 and C_2 are independent of v.

Proof. The boundedness of the bilinear form B is derived directly from the boundedness of $\mathcal{V}^{-1}$ and $\mathcal{K}$. To prove ellipticity we first note that the hypersingular operator $\mathcal{D}$ is $H^{1/2}(\Gamma^0)$-semi-elliptic for all closed interface Γ^0, i.e.,

$$(4.19) \qquad \langle \mathcal{D}v, v \rangle_{L_2(\Gamma^0)} \geq C \, |v|_{H^{1/2}(\Gamma^0)} \quad \forall v \in H^{1/2}(\Gamma^0);$$

see e.g. [21, Corollary 6.25]. The Cauchy data $(u_-, \dfrac{\partial u_-}{\partial \boldsymbol{n}})$ on Γ^0 satisfy

$$(4.20) \qquad \begin{pmatrix} u_- \\ \dfrac{\partial u_-}{\partial \boldsymbol{n}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2}I - \mathcal{K} & \mathcal{V} \\ \mathcal{D} & \dfrac{1}{2}I + \mathcal{K}' \end{pmatrix} \begin{pmatrix} u_- \\ \dfrac{\partial u_-}{\partial \boldsymbol{n}} \end{pmatrix}.$$

Substituting (4.8) into the second equation of (4.20) gives

$$\frac{\partial u_-}{\partial \boldsymbol{n}} = \mathcal{D} \, u_- + (\tfrac{1}{2}I + \mathcal{K}')\mathcal{V}^{-1}(\tfrac{1}{2}I + \mathcal{K}) \, u_- \quad \text{on} \quad \Gamma^0.$$

This equation and (4.8) yield

$$\mathcal{S}_- = \mathcal{D} + (\tfrac{1}{2}I + \mathcal{K}')\mathcal{V}^{-1}(\tfrac{1}{2}I + \mathcal{K}).$$

Noting that $\mathcal{K}'$ is the adjoint operator of $\mathcal{K}$, we have

$$(4.21) \qquad \langle \mathcal{S}_- v, v \rangle = \langle \mathcal{D}v, v \rangle + \left\langle \mathcal{V}^{-1}(\tfrac{1}{2}I + \mathcal{K})v, (\tfrac{1}{2}I + \mathcal{K})v \right\rangle \quad \forall v \in H^{1/2}(\Gamma^0).$$

Similarly, the exterior Dirichlet-to-Neumann operator $\mathcal{S}_+$ satisfies

$$\mathcal{S}_+ = -\mathcal{D} - (\tfrac{1}{2}I - \mathcal{K}')\mathcal{V}^{-1}(\tfrac{1}{2}I - \mathcal{K})$$

and

$$(4.22) \qquad \langle \mathcal{S}_+ v, v \rangle = -\langle \mathcal{D}v, v \rangle - \left\langle \mathcal{V}^{-1}(\tfrac{1}{2}I - \mathcal{K})v, (\tfrac{1}{2}I - \mathcal{K})v \right\rangle \quad \forall v \in H^{1/2}(\Gamma^0).$$

From (4.21), (4.22), (4.19) and noting the $H^{1/2}$-ellipticity of the inverse operator of $\mathcal{V}$, we derive

$$\langle [\alpha \mathcal{S}] \, v, v \rangle = (\alpha_- + \alpha_+) \, \langle \mathcal{D}v, v \rangle + \alpha_- \left\langle \mathcal{V}^{-1}(\tfrac{1}{2}I + \mathcal{K})v, (\tfrac{1}{2}I + \mathcal{K})v \right\rangle_{\Gamma^0}$$

$$+ \alpha_+ \left\langle \mathcal{V}^{-1}(\tfrac{1}{2}I - \mathcal{K})v, (\tfrac{1}{2}I - \mathcal{K})v \right\rangle_{\Gamma^0}$$

$$\gtrsim (\alpha_- + \alpha_+) \, |v|^2_{H^{1/2}(\Gamma^0)} + \alpha_- \left\| (\tfrac{1}{2}I + \mathcal{K})v \right\|^2_{H^{1/2}(\Gamma^0)} + \alpha_+ \left\| (\tfrac{1}{2}I - \mathcal{K})v \right\|^2_{H^{1/2}(\Gamma^0)}$$

$$(4.23) \qquad \gtrsim |v|^2_{H^{1/2}(\Gamma^0)} + \left\| (\tfrac{1}{2}I + \mathcal{K})v \right\|^2_{H^{1/2}(\Gamma^0)} + \left\| (\tfrac{1}{2}I - \mathcal{K})v \right\|^2_{H^{1/2}(\Gamma^0)}.$$

Applying the triangle inequality to the last two terms on the right hand side of the inequality above, we obtain

$$\langle [\alpha\mathcal{S}]\, v, v\rangle \gtrsim |v|^2_{H^{1/2}(\Gamma^0)} + \|v\|^2_{H^{1/2}(\Gamma^0)} \gtrsim \|v\|^2_{H^{1/2}(\Gamma^0)} \quad \forall v \in H^{1/2}(\Gamma^0),$$

completing the proof of the lemma. $\square$ We consider the tensor product operator $[\alpha\mathcal{S}]^{(k)} := [\alpha\mathcal{S}] \otimes \cdots \otimes [\alpha\mathcal{S}]$ which is a linear mapping

$$[\alpha\mathcal{S}]^{(k)} : H^{1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0) \to H^{-1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0),$$

see [22, Proposition 2.4] for more details. Tensorization of equation (4.14) yields for almost all $\omega \in \Omega$
(4.24)
$$[\alpha\mathcal{S}]^{(k)} \left(u'_+(\omega) \otimes \cdots \otimes u'_+(\omega) \right) = \otimes^k_{i=1} \left(g_N(\omega) - (\alpha_- \mathcal{S}_-) g_D(\omega) \right) \quad \text{in} \quad H^{-1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0).$$

Taking the mean of (4.24) yields a deterministic k-th moment problem. In particular, for $k = 2$ it reads: Find $\mathrm{Cov}[u'_+](\boldsymbol{x}, \boldsymbol{y}) \in H^{1/2}_{\mathrm{mix}}(\Gamma^0 \times \Gamma^0)$ satisfying

$$
\begin{aligned}
([\alpha\mathcal{S}] \otimes [\alpha\mathcal{S}])\, \mathrm{Cov}[u'_+](\boldsymbol{x}, \boldsymbol{y}) &= (\nabla_{\Gamma,\boldsymbol{x}} \otimes \nabla_{\Gamma,\boldsymbol{y}}) \cdot \left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{x}} u^0(\boldsymbol{x})\right] \left[\alpha\nabla_{\Gamma,\boldsymbol{y}} u^0(\boldsymbol{y})\right] \right) \\
&\quad + \left((\alpha_- \mathcal{S}_-) \otimes (\alpha_- \mathcal{S}_-)\right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\frac{\partial u^0(\boldsymbol{x})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\right] \left[\frac{\partial u^0(\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right] \right) \\
&\quad - \left(\nabla_{\Gamma,\boldsymbol{x}} \cdot \otimes (\alpha_- \mathcal{S}_-)\right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{x}} u^0(\boldsymbol{x})\right] \left[\frac{\partial u^0(\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right] \right) \\
&\quad - \left((\alpha_- \mathcal{S}_-) \otimes \nabla_{\Gamma,\boldsymbol{y}} \cdot \right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{y}} u^0(\boldsymbol{y})\right] \left[\frac{\partial u^0(\boldsymbol{x})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\right] \right).
\end{aligned}
\tag{4.25}
$$

Similarly, we have

$$
\begin{aligned}
([\alpha\mathcal{S}] \otimes [\alpha\mathcal{S}])\, \mathrm{Cov}[u'_-](\boldsymbol{x}, \boldsymbol{y}) &= (\nabla_{\Gamma,\boldsymbol{x}} \otimes \nabla_{\Gamma,\boldsymbol{y}}) \cdot \left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{x}} u^0(\boldsymbol{x})\right] \left[\alpha\nabla_{\Gamma,\boldsymbol{y}} u^0(\boldsymbol{y})\right] \right) \\
&\quad + \left((\alpha_+ \mathcal{S}_+) \otimes (\alpha_+ \mathcal{S}_+)\right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\frac{\partial u^0(\boldsymbol{x})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\right] \left[\frac{\partial u^0(\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right] \right) \\
&\quad - \left(\nabla_{\Gamma,\boldsymbol{x}} \cdot \otimes (\alpha_+ \mathcal{S}_+)\right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{x}} u^0(\boldsymbol{x})\right] \left[\frac{\partial u^0(\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right] \right) \\
&\quad - \left((\alpha_+ \mathcal{S}_+) \otimes \nabla_{\Gamma,\boldsymbol{y}} \cdot \right)\left(\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) \left[\alpha\nabla_{\Gamma,\boldsymbol{y}} u^0(\boldsymbol{y})\right] \left[\frac{\partial u^0(\boldsymbol{x})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\right] \right).
\end{aligned}
\tag{4.26}
$$

Denote $g^\kappa_+ := \mathbb{E}[\otimes^k_{i=1}\left(g_N(\omega) - (\alpha_- \mathcal{S}_-) g_D(\omega)\right)]$. Recalling (4.15), the variational formulation for finding $\mathcal{M}^k[u'_+]$ reads: Given $g^\kappa_+ \in H^{-1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0)$, find $\mathcal{M}^k[u'_+] \in H^{1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0)$ satisfying

$$\mathcal{B}(\mathcal{M}^k[u'_+], v) = \langle\langle g^\kappa_+, v\rangle\rangle \qquad \forall v \in H^{1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0), \tag{4.27}$$

where $\mathcal{B}(\cdot, \cdot) = \left\langle\left\langle [\alpha\mathcal{S}]^{(k)}\cdot, \cdot\right\rangle\right\rangle$ is a bilinear form and $\langle\langle\cdot, \cdot\rangle\rangle$ is the $H^{-1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0) - H^{1/2}_{\mathrm{mix}}(\Gamma^0 \times \cdots \times \Gamma^0)$ duality pairing obtained by tensorization of $B(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle$ from (4.16). Proposition 2.4 in [22]

implies

LEMMA 4.2. *The bilinear form* $\mathcal{B}(\cdot, \cdot) : H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0) \times H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0) \to \mathbb{R}$ *is bounded and* $H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0)$-*elliptic, i.e.,*

$$(4.28) \qquad \mathcal{B}(v, w) \le C_1 \|v\|_{H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0)} \|w\|_{H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0)},$$

and

$$(4.29) \qquad C_2 \|v\|_{H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0)}^2 \le \mathcal{B}(v, v)$$

for all $v, w \in H_{\mathrm{mix}}^{1/2}(\Gamma^0 \times \cdots \times \Gamma^0)$. By Lemma 4.2 there exists a unique solution of (4.27).

5. Examples. In this section, we consider the transmission problem (2.11a)–(2.11d) where the random interface $\Gamma(\omega)$ is given by

$$\Gamma(\omega) = \{\boldsymbol{x} + \epsilon \kappa(\boldsymbol{x}, \omega) \boldsymbol{n}(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{S}\}.$$

Here, the reference interface Γ^0 is the unit sphere $\mathbb{S}$. The perturbation parameter $\kappa(\boldsymbol{x}, \omega) = a(\omega)$, where $a(\omega)$ is uniformly distributed in $[-1, 1]$. The mean value $\mathbb{E}[\kappa] = 0$ and the covariance $\mathrm{Cov}[\kappa](\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Cor}[\kappa](\boldsymbol{x}, \boldsymbol{y}) = 1/3$. The interface $\Gamma(\omega)$ is a sphere of radius $R(\omega) = 1 + \epsilon a(\omega)$.

5.1. Analytic example. Firstly, we choose the right hand side f to be

$$f(\boldsymbol{x}) = \begin{cases} (4r_{\boldsymbol{x}}^2 - 1)^2 & \text{if } 0 \le r_{\boldsymbol{x}} \le 1/2, \\ 0 & \text{if } 1/2 \le r_{\boldsymbol{x}}, \end{cases}$$

where $r_{\boldsymbol{x}} = |\boldsymbol{x}|$. Then solution of the transmission problem with respect to the random interface $\Gamma(\omega)$ can be analytically computed as follows:

$$(5.1) \quad u(\boldsymbol{x}, \omega) = \begin{cases} \frac{1}{\alpha_-}\left(\frac{8}{21}r_{\boldsymbol{x}}^6 - \frac{2}{5}r_{\boldsymbol{x}}^4 + \frac{r_{\boldsymbol{x}}^2}{6}\right) - \frac{3}{105\alpha_-}r_{\boldsymbol{x}} - \frac{23}{840\alpha_-} + \frac{\alpha_+ - \alpha_-}{105\alpha_-\alpha_+ R(\omega)} & \text{if } 0 \le r_{\boldsymbol{x}} \le \frac{1}{2}, \\ -\frac{1}{105\alpha_- r_{\boldsymbol{x}}} + \frac{\alpha_+ - \alpha_-}{105\alpha_-\alpha_+ R(\omega)} & \text{if } \frac{1}{2} \le r_{\boldsymbol{x}} \le R(\omega), \\ -\frac{1}{105\alpha_+ r_{\boldsymbol{x}}} & \text{if } R(\omega) \le r_{\boldsymbol{x}}. \end{cases}$$

In particular, the exact solution u^0 of the transmission problem on the reference interface Γ^0 is given by (5.1) where $R(\omega) = 1$, i.e.,

$$(5.2) \qquad u^0(\boldsymbol{x}) = \begin{cases} \frac{1}{\alpha_-}\left(\frac{8}{21}r_{\boldsymbol{x}}^6 - \frac{2}{5}r_{\boldsymbol{x}}^4 + \frac{r_{\boldsymbol{x}}^2}{6}\right) - \frac{3}{105\alpha_-}r_{\boldsymbol{x}} - \frac{23}{840\alpha_-} + \frac{\alpha_+ - \alpha_-}{105\alpha_-\alpha_+} & \text{if } 0 \le r_{\boldsymbol{x}} \le \frac{1}{2}, \\ -\frac{1}{105\alpha_- r_{\boldsymbol{x}}} + \frac{\alpha_+ - \alpha_-}{105\alpha_-\alpha_+} & \text{if } \frac{1}{2} \le r_{\boldsymbol{x}} \le 1, \\ -\frac{1}{105\alpha_+ r_{\boldsymbol{x}}} & \text{if } 1 \le r_{\boldsymbol{x}}. \end{cases}$$

Noting (5.1) and using simple calculation, we obtain

$$(5.3) \qquad \mathbb{E}[u(\boldsymbol{x}, \cdot)] = \begin{cases} u^0(\boldsymbol{x}) + \frac{\alpha_+ - \alpha_-}{105\alpha_-\alpha_+} \frac{\ln(1+\epsilon) - \ln(1-\epsilon)}{2\epsilon} & \text{if } 0 \le r_{\boldsymbol{x}} < 1, \\ u^0(\boldsymbol{x}) & \text{if } 1 < r_{\boldsymbol{x}}. \end{cases}$$

Elementary calculus reveals that $\frac{\ln(1+\epsilon) - \ln(1-\epsilon)}{2\epsilon} = \sum_{n=1}^{\infty} \frac{\epsilon^{2n}}{2n+1}$. Therefore, the mean value $\mathbb{E}[u]$ in (5.3) agrees with our result (3.53) in Lemma 3.13. The linearized error appears in this example to be $\mathcal{O}(\epsilon^2)$.

We then compute the covariance of the solution u by elementary calculations, noting (5.1), to obtain

$$(5.4) \qquad \mathrm{Cov}_u(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} \frac{1}{3} \frac{[\alpha]^2}{(105\alpha_- \alpha_+)^2} \, \epsilon^2 + \mathcal{O}(\epsilon^4) & \text{if } r_{\boldsymbol{x}} < 1 \text{ and } r_{\boldsymbol{y}} < 1, \\ 0 & \text{if } r_{\boldsymbol{x}} > 1 \text{ or } r_{\boldsymbol{y}} > 1. \end{cases}$$

We test accuracy of our shape calculus method by computing the covariance of u via covariance of the shape derivative. Noting (5.2), we first solve equations (4.25) and (4.26) to obtain $\mathrm{Cov}[u'_+]$ and $\mathrm{Cor}_{u'_-}$. In this example, these equations can be solved exactly and

$$\mathrm{Cov}[u'_-] = \frac{1}{3} \frac{[\alpha]^2}{(105\alpha_- \alpha_+)^2} \quad \text{and} \quad \mathrm{Cov}[u'_+] = 0.$$

Applying (4.11), we obtain

$$\mathrm{Cov}[u'](\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} \frac{1}{3} \frac{[\alpha]^2}{(105\alpha_- \alpha_+)^2} & \text{if } r_{\boldsymbol{x}} < 1 \text{ and } r_{\boldsymbol{y}} < 1 \\ 0 & \text{if } r_{\boldsymbol{x}} > 1 \text{ or } r_{\boldsymbol{y}} > 1. \end{cases}$$

This and (5.4) agree with our theoretical result (3.54) and the linearized error in this example is $\mathcal{O}(\epsilon^4)$.

5.2. Numerical example. Secondly, we solve the problem (2.11a)–(2.11d) where the right hand side f is given by

$$\begin{aligned}(5.5) \qquad f(\boldsymbol{x}) = {}& 2\,[x_1^2 + x_2^2 + (x_3 - 1)^2]^{-1/2}\,(1 - |\boldsymbol{x}|^2) \\ & - 4\,[x_1^2 + x_2^2 + (x_3 - 1)^2]^{-1/2}\,(|\boldsymbol{x}|^2 - x_3) - 6[x_1^2 + x_2^2 + (x_3 - 1)^2]^{1/2}.\end{aligned}$$

The deterministic solution of the transmission problem with the reference interface $\Gamma^0 = \mathbb{S}$ is then

$$\begin{aligned}(5.6) \qquad u_-(\boldsymbol{x}) &= \frac{1}{\alpha_-}\,[x_1^2 + x_2^2 + (x_3 - 1)^2]^{1/2}\,(1 - |\boldsymbol{x}|^2), \quad \boldsymbol{x} \in D_-^0 \\ u_+(\boldsymbol{x}) &= \frac{1}{\alpha_+}\,[x_1^2 + x_2^2 + (x_3 - 1)^2]^{1/2}\,(1 - |\boldsymbol{x}|^2), \quad \boldsymbol{x} \in D_+^0.\end{aligned}$$

Following the method discussed in Section 3, covariance of the solution is approximated by covariance of the shape derivative (see Lemma 3.13), which can be obtained by solving the equations (4.25) and (4.26). Note here that these equations are given on the reference interface $\Gamma^0 = \mathbb{S}$. The right hand sides and the solutions of these equations belong to the tensor space $H_{\mathrm{mix}}^{2-\sigma}(\Gamma^0 \times \Gamma^0)$ for any $\sigma > 0$. To solve these equations numerically we use the hyperbolic cross tensor approximation spaces of spherical harmonics which are defined by

$$(5.7) \qquad S_p^\delta := \mathrm{span}\big\{ \boldsymbol{Y}_{\boldsymbol{\ell},\boldsymbol{m}} : \boldsymbol{\ell} \in \delta_p, \ m_i = -\ell_i, \ldots, \ell_i \text{ for } i = 1, 2 \big\},$$

where

$$(5.8) \qquad \delta_p := \left\{ \boldsymbol{\ell} = (\ell_1, \ell_1) \in \mathbb{N}^2 : \prod_{i=1}^{2} (1 + \ell_i) \leq 1 + p \right\}.$$
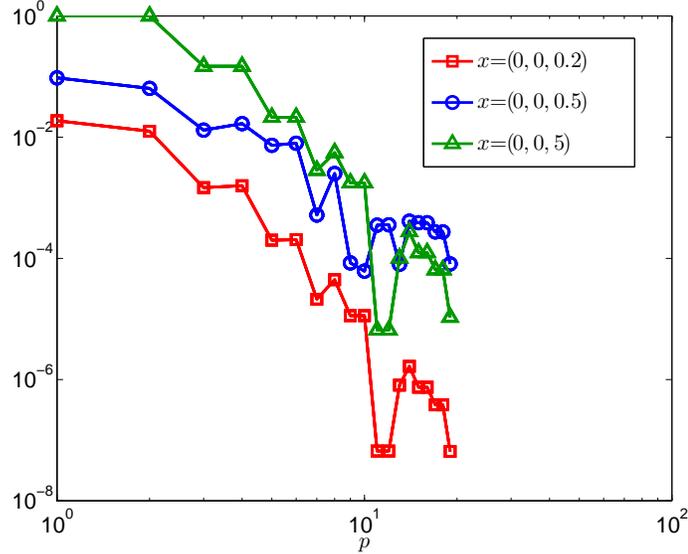
FIGURE 1.   *Convergence of the absolute error* $|\mathrm{Var}[u'](\boldsymbol{x}) - \mathrm{Var}[u'_p](\boldsymbol{x})|$ *for three points $\boldsymbol{x}$ inside and outside the unit sphere with respect to the order of the hyperbolic cross p.*

The Galerkin method was used to find the approximate solutions $u'_p \in S^\delta_p$ of (4.25) and (4.26). It has been shown in [5] that the use of the space S^δ_p yields the convergence rate of $p^{-(2-\sigma-t)}$ and demands only $\mathcal{O}(p^2 \log p)$ unknowns, where t is the order of the Sobolev norm in which the errors are computed. The same convergence rate $p^{-(2-\sigma-t)}$ is achieved when using the standard full tensor product approximation of degree p which meanwhile requires $\mathcal{O}(p^4)$ unknowns. We then compute the variance of $u'(\boldsymbol{x})$ at three points $\boldsymbol{x} = (0,0,0.2)$, $(0,0,0.5)$ and $(0,0,5)$ inside and outside the unit sphere. The convergence curves for the absolute error

$$|\mathrm{Var}[u'](\boldsymbol{x}) - \mathrm{Var}[u'_p](\boldsymbol{x})|$$

with respect to the order of the hyperbolic cross p are presented in Fig 1.

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