ROBUST POLE ASSIGNMENT IN SYSTEMS SUBJECT TO

STRUCTURED PERTURBATIONS

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(N/A Report 7/90)

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Abstract

The problem of robust pole assignment by feedback in a linear, multivariable, time-invariant system which is subject to structured perturbations is investigated. A measure of robustness, or sensitivity, of the poles to a given class of perturbations is derived, and a reliable and efficient computational algorithm is presented for constructing a feedback which assigns the prescribed poles and optimizes the robustness measure.

Keywords: control system, pole assignment, robustness, structured perturbations.

1. Introduction

For robust pole placement by feedback in a linear, multivariable, time-invariant system, the prescribed poles, or eigenvalues, of the closed loop system are required to be as insensitive as possible to perturbations. It is known [9] that the sensitivities of the eigenvalues of a matrix are dependent on the corresponding eigenvectors. It has also been established [7] that the freedom to assign the eigenvectors of a system corresponds directly to the degrees of freedom available in the feedback gain matrix. In effect, therefore, the feedback can be parameterized directly in terms of the eigenstructure of the closed loop system, which can then be selected to ensure robustness.

By a worst case analysis, it can be shown [9] that if arbitrary, unstructured perturbations are allowed in all components of the closed loop system matrix, then the maximum perturbation to an eigenvalue λ of the matrix is directly proportional to

$$c(\lambda) = \underline{y}^T\underline{x}/\|\underline{y}\|\|\underline{x}\| ,$$

where $\underline{x}$, $\underline{y}$ are the right and left eigenvectors corresponding to λ, and $\|\cdot\|$ denotes the ℓ_2-vector norm. A bound on the variation $\delta\lambda$ in an eigenvalue λ due to a general arbitrary perturbation E in the system matrix is given by

$$|\delta\lambda| \leq \kappa(X)\|E\| ,$$

where $\kappa(X) = \|X\|\|X^{-1}\|$, X is the modal matrix having the system eigenvectors as its columns and $\|\cdot\|$ denotes a matrix norm consistent with the ℓ_2-vector norm. The condition number $\kappa(X)$ of X then gives a global measure of the sensitivity of the eigenvalues of the system matrix. Methods for minimizing $\kappa(X)$ are described in [5] [4] [2].

In many practical systems, the worst case analysis is not applicable, however, since all elements of the system matrix may not be subject to arbitrary perturbations. For example, the state-space form of the second order system

$$\ddot{\underline{z}} - A_1\dot{\underline{z}} - A_2\underline{z} = B_1\underline{u}$$

is generally given by the equations

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

where

$$\underline{x} = \begin{bmatrix} \underline{z} \\ \dot{\underline{z}} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ A_2 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}. \tag{2}$$

The zero block in A is structural and is not subject to perturbations. Similarly, decentralized systems contain many blocks of structural zeros not subject to disturbances, parameter variations, or other uncertainties. For such systems it is necessary to select feedback gains to ensure that the prescribed closed loop poles are insensitive to restricted, or structured, perturbations.

In this paper we give a measure of pole sensitivity to certain classes of structured perturbations, and describe reliable numerical methods for finding feedback gains which assign prescribed poles with minimum sensitivity. The computational algorithms are based on methods developed in [5] for minimizing the pole sensitivity to unstructured perturbations. The measure of sensitivity is similar to that used in [8], but the numerical methods presented here are expected to be more efficient and reliable.

2. Formulation of the Problem

The problem of pole placement by state feedback can be stated as follows:

Given the linear, time-invariant system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\mathrm{rank}[B] = m$, and given a self-conjugate set of (distinct) complex scalars λ_j, $j = 1,2,...n$, find feedback gain matrix $K \in \mathbb{R}^{m\times n}$, such that the closed loop system matrix $A + BK$ has eigenvalues equal to λ_j, $j = 1,2,...n$.

For robustness the prescribed eigenvalues, or poles, must be insensitive to a given class of perturbations. We assume, that the perturbed system has the form

$$\dot{\underline{x}} = (A + BK + FEG^T)\underline{x} \, . \tag{4}$$

where $A + BK$ is the nominal closed loop system matrix with the prescribed poles, E is an unknown disturbance matrix, and F, G are real scaling matrices which define the structure of the perturbations (see also [3]). For example, in the second-order system (1)-(2), if we assume A_1, A_2 are uncertain, then we take $F^T = [0, I]$ and $G^T = I$, with $E = [E_1, E_2]$; if only A_2 is subject to perturbation, we take $F^T = [0, I]$ and $G^T = [I, 0]$, with $E = [E_2]$.

The <u>robust</u> pole assignment problem then becomes:

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times m_F}$, $G \in \mathbb{R}^{n\times m_G}$ and a self-conjugate set of (distinct) complex conjugate scalars λ_j, $j = 1,2,\ldots n$, find $K \in \mathbb{R}^{m\times n}$ such that

(i) the eigenvalues of $A + BK$ are equal to λ_j, $j = 1,2,\ldots n$;

(ii) the eigenvalues of $A + BK + FEG^T$ are 'close' to λ_j, $j = 1,2,\ldots n$, for <u>any</u> matrix $E \in \mathbb{R}^{m_F\times m_G}$ of (small) perturbations.

In order to formalize the definition of 'closeness' used in the statement of the problem, it is necessary to establish a global measure of the sensitivity, or <u>robustness</u>, of the poles to perturbations having the given structure. A measure of this robustness is derived in the next section.

We remark that not all perturbations to the nominal system can be represented by the form FEG^T. More possibilities can be covered by considering perturbations of the form $\sum_{s=1}^{\ell} F_s E_s G_s^T$. An upper bound on the pole sensitivity to perturbations of this type is given, however, by the robustness measure for perturbations of the form FEG^T where $F = [F_1, F_2, \ldots F_\ell]$ and $G = [G_1, G_2, \ldots G_\ell]$. The algorithm for optimizing robustness described here can thus be applied to quite general systems.

3. Measure of Robustness

To establish a global measure of the sensitivity of the eigenvalues λ_j of the matrix $A + BK$ to structured perturbations FEG^T, we begin by observing from [9] that the first order change $\delta\lambda_j$ to λ_j due to the perturbation E is given by

$$\delta\lambda_j = \underline{y}_j^T F E G^T \underline{x}_j , \tag{5}$$

where $\underline{x}_j$, $\underline{y}_j$ are, respectively, the right and left eigenvectors of the nominal system associated with λ_j. Consequently, we may estimate

$$|\delta\lambda_j| \leq c_j \|E\| , \tag{6}$$

where we have denoted

$$c_j = \|\underline{y}_j^T F\| \|G^T \underline{x}_j\| \tag{7}$$

as the coefficient of sensitivity.

We now take $\nu = c_1^2 + c_2^2 + \ldots + c_n^2$ as the global measure of robustness. If we let $X = [\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n]$ be the matrix of right eigenvectors of $A + BK$, and let $Y = (X^{-1})^T = [\underline{y}_1, \underline{y}_2, \ldots \underline{y}_n]$ be the matrix of left eigenvectors, normalized such that $\underline{y}_j^T \underline{x}_j = 1$, $j = 1, 2, \ldots n$, and assume that the right eigenvectors are scaled such that $\|G^T \underline{x}_j\| = 1$, $j = 1, 2, \ldots n$, then the robustness measure can be written

$$\nu = \sum_{j=1}^{n} \|y_j^T F\|^2 = \|Y^T F\|_F^2 = \|X^{-1} F\|_F^2 . \tag{8}$$

Here $\|\cdot\|_F$ denotes the Frobenius matrix norm.

The problem of robustness with respect to structured perturbations can now be stated precisely as:

Problem 1 Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times m_F}$, $G \in \mathbb{R}^{n \times m_G}$ and a self-conjugate set of (distinct) complex scalars λ_j, $j = 1,2,\ldots n$, find a matrix $K \in \mathbb{R}^{m\times n}$ and a non-singular matrix X such that

(i) $(A+BK)X = X\Lambda$ where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots \lambda_n\}$;

(ii) $\|G^T X e_j\| = 1$, $j = 1,2,\ldots n$;

(iii) $\nu = \|X^{-1}F\|_F^2$ is minimized.

□

We remark that multiple eigenvalues can be assigned, with a minor restriction that ensures the existence of a non-singular matrix X. If a non-singular matrix X does not exist satisfying (i) of Problem 1, then the nominal closed loop system must be defective and hence is necessarily less robust than a non-defective system (see [9] and [5]).

In the next section we describe a computational method for selecting the matrix X of eigenvectors of the closed loop system to optimize the robustness measure.

4. Numerical Method

We now derive a computational technique for minimizing the global measure of pole sensitivity ν to the given class of perturbations. The method is a generalization of Method 1 of [5] and is iterative in nature. Initially an arbitrary set of eigenvectors is selected, and then each eigenvector is up-dated in turn, so as to minimize ν over all possible choices for that vector, whilst the remaining eigenvectors are kept fixed. This procedure is repeated iteratively until the measure ν ceases to decrease significantly. Each eigenvector is required to belong to a specified subspace, (see [7] and [5]) and, as in Method 1 of [5], the minimization of ν in the update step can be performed explicitly using numerically stable QR decompositions to find the required least square solutions.

4.1 Basic steps of the algorithm

We begin by describing the subspace $\mathcal{S}_j$ to which an eigenvector $\underline{x}_j$ corresponding to a prescribed pole λ_j of the nominal closed loop system $A + BK$ must belong for $j = 1,2,\ldots n$. The following Lemma gives the required result.

Lemma 1 The eigenvector $\underline{x}_j$ of $A + BK$ corresponding to the assigned eigenvalue λ_j must belong to the space

$$\mathcal{S}_j = \mathcal{N}\{U_1^T(A - \lambda_j I)\} , \tag{9}$$

where

$$B = [U_0, U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix} , \tag{10}$$

$U = [U_0, U_1]$ is orthogonal, Z is non-singular, and $\mathcal{N}\{\cdot\}$ denotes null space.

□

The proof is given in [5], and the subspace is equivalent to that derived in [7]. The decomposition (10) of B exists by the assumption that B is of full column rank and can be found by the QR or SVD (singular value decomposition) methods. An orthonormal basis for $\mathcal{S}_j$, comprised by the columns of matrix S_j, can also be found by QR or SVD decompositions [5].

The basic steps of the algorithm for minimizing the sensitivity measure ν are then given as follows.

<u>Algorithm 1</u>

<u>Step 1</u> Find decomposition (10) of B and a basis S_j for $\mathcal{S}_j$, $j = 1,2,\ldots n$.

<u>Step 2</u> Select an initial matrix $X = [\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n]$ such that $\underline{x}_j \in \mathcal{S}_j$, $j = 1,2,\ldots n$, and X is non-singular.

<u>Step 3</u> For $j = 1,2,\ldots n$ do

<u>Step 3.1</u> Find $\hat{\underline{x}}_j$ to solve

$$\min_{\underline{x}_j \in \mathcal{S}_j} \|X^{-1}F\|_F$$

subject to $\|G^T\underline{x}_j\| = 1$ and $\underline{x}_i$ fixed, $i = 1,2,\ldots j-1, j+1,\ldots n$.

<u>Step 3.2</u> Form updated matrix

$X = [\underline{x}_1, \underline{x}_2, \ldots \underline{x}_{j-1}, \hat{\underline{x}}_j, \underline{x}_{j+1}, \ldots \underline{x}_n]$ and CONTINUE.

<u>Step 4</u> Repeat Step 3 until $\nu = \|X^{-1}F\|_F$ is 'converged'.

<u>Step 5</u> Construct feedback matrix K from

$$K = Z^{-1}U_0^T(X\Lambda X^{-1} - A) ,$$

where $\Lambda = \text{diag}\,\{\lambda_1, \lambda_2, \ldots \lambda_n\}$.

□

In the next section we derive in detail the procedure required to carry out Step 3.1 of the algorithm.

We remark that the steps of Algorithm 1 are numerically stable provided that the computed matrix of eigenvectors X is well-conditioned (for inversion). Since the aim of the procedure is to minimize the pole sensitivity, X is expected to be reasonably well-conditioned. For better stability, the method proposed by [1] for constructing the feedback K from the eigenvectors of the closed loop system could be used in place of the direct method of Step 5. In any case, if the constructed matrix X is very badly conditioned, it is an indication that the closed loop system will necessarily be sensitive, and that an alternative selection of poles should be prescribed, allowing a robust feedback to be found.

4.2 Updating the eigenvectors

The key step of the algorithm is Step 3.1, the computation of the update to the eigenvector $\underline{x}_j$. This step is accomplished explicitly, as in Method 1 of [5], but the computation is more complicated than in the case of unstructured perturbations. To facilitate the derivation of the procedure, the following result is needed.

Lemma 2 Let A, B, C, $\underline{v}$ and $\underline{w}$ be matrices and vectors of suitable sizes. Let

$$Z_1 = \begin{bmatrix} A-B\underline{w}\underline{v}^T \\ \underline{v}^T \end{bmatrix}, \qquad Z_2 = (\underline{v}^T\underline{v})\begin{bmatrix} B \\ C \end{bmatrix}\underline{w} - \begin{bmatrix} A\underline{v} \\ 0 \end{bmatrix}. \tag{11}$$

If $\|C\underline{w}\| = 1$, then

$$\|Z_1\|_F^2 = \|Z_2\|_F^2/\underline{v}^T\underline{v} + \mathrm{trace}(A(I-\underline{v}\underline{v}^T/\underline{v}^T\underline{v})A^T) . \tag{12}$$

<u>Proof</u>: We observe first that $\|Z_1\|_F^2$ may be written

$$\|Z_1\|_F^2 = \sum_i \|\underline{e}_i^T(A - B\underline{w}\underline{v}^T)\|^2 + \|\underline{v}\|^2 . \tag{13}$$

Expanding each term in the series and adding and subtracting the term $\underline{e}_i^T A\underline{v}\underline{v}^T A^T \underline{e}_i/\underline{v}^T\underline{v}$ gives

$$\|\underline{e}_i^T(A-B\underline{w}\underline{v}^T)\|^2 = \underline{e}_i^T(A\underline{v}\underline{v}^T A^T/(\underline{v}^T\underline{v}) - 2A\underline{v}\underline{w}^T B^T + B\underline{w}(\underline{v}^T\underline{v})\underline{w}^T B^T)\underline{e}_i + \alpha_i$$

$$= (\underline{v}^T\underline{v})\,|\underline{e}_i^T(B\underline{w} - A\underline{v}/(\underline{v}^T\underline{v}))|^2 + \alpha_i , \tag{14}$$

where

$$\alpha_i = \underline{e}_i^T A(I - \underline{v}\underline{v}^T/\underline{v}^T v)A^T\underline{e}_i , \tag{15}$$

Summing the terms (14) and using $\|C\underline{w}\|^2 = 1$, then gives

$$\|Z_1\|_F^2 = (\underline{v}^T\underline{v})\|B\underline{w} - A\underline{v}/\underline{v}^T\underline{v}\|^2 + (\underline{v}^T\underline{v})\|C\underline{w}\|^2 + \sum_i \alpha_i$$

$$= \|(\underline{v}^T\underline{v})\begin{bmatrix} B \\ C \end{bmatrix}\underline{w} - \begin{bmatrix} A \\ 0 \end{bmatrix}\underline{v}\|^2/(\underline{v}^T\underline{v}) + \sum_i \alpha_i . \tag{16}$$

which establishes the result.

□

The result of Lemma 2 was originally applied in [2] to derive a method for robust pole assignment by <u>output</u> feedback. It should be noted that the matrices used in the statement of the lemma are generic and are not related to similarly denoted matrices in the rest of the paper.

In essence the update step, Step 3.1 of Algorithm 1, aims to orthogonalize the eigenvectors $\underline{x}_i$, $i = 1,2,\ldots n$, with respect to the matrix F, subject to the constraints. To update vector $\underline{x}_j$, the first stage of the procedure is thus to find orthogonal bases Q and $\underline{q}$ for the space spanned by the fixed eigenvectors $\{\underline{x}_i,\ i \neq j\}$ and its orthogonal complement, respectively, and to express the measure ν in terms of these bases. Next, the required eigenvector is scaled to have a fixed normalization, and the direction of the minimizing vector in the required subspace is found. The optimal normalization is then determined which satisfies the constraint. The technical details are as follows.

We denote $X_j = [\underline{x}_1, \underline{x}_2, \ldots \underline{x}_{j-1}, \underline{x}_{j+1}, \ldots \underline{x}_n]$ and let S_j represent an orthonormal basis for $\mathcal{S}_j$. Then $\underline{x}_j = S_j\underline{w} \in \mathcal{S}_j$ for any $\underline{w}$. The QR decomposition of X_j is written

$$X_j = [Q,\ \underline{q}] \begin{bmatrix} R \\ \underline{0}^T \end{bmatrix}, \tag{17}$$

where $[Q,\ \underline{q}]$ is orthogonal and R is invertible. It follows that we may write $\nu = \|X^{-1}F\|_F = \|Z\|_F$, where

$$Z = [X_j,\ S_j\underline{w}]^{-1}F = \begin{bmatrix} R^{-1} & -\rho R^{-1}Q^TS_j\underline{w} \\ \underline{0}^T & \rho \end{bmatrix} \begin{bmatrix} Q^TF \\ \underline{q}^TF \end{bmatrix}$$

$$= \begin{bmatrix} R^{-1}Q^TF - R^{-1}Q^TS_j\underline{w}(\rho\underline{q}^TF) \\ \rho\underline{q}^TF \end{bmatrix}, \tag{18}$$

and $\rho = (\underline{q}^T S_j \underline{w})$.

Applying Lemma 2 with $\|G^T S_j \underline{w}\| = 1$ we obtain

$$\|Z\|_F = \delta^2 \left\| \begin{bmatrix} R^{-1}Q^T S_j \\ G^T S_j \end{bmatrix} (\rho \underline{w}) - \begin{bmatrix} R^{-1}Q^T FF^T \underline{q}/\delta^2 \\ \underline{0} \end{bmatrix} \right\|_F + c \,, \qquad (19)$$

where $\delta^2 = \underline{q}^T FF^T \underline{q}$ and

$$c = \text{trace}(R^{-1}Q^T F(I - F^T \underline{q}\underline{q}^T F/\delta^2)(R^{-1}Q^T F)^T)$$

are constant with respect to $\underline{w}$. In order to fix the scaling of the vector $\rho\underline{w}$ we find the orthogonal Householder transformation P such that

$$\underline{q}^T S_j P = \sigma \underline{e}_m^T \qquad (20)$$

and define $\hat{\underline{w}}$ such that

$$\begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix} = \sigma P^T (\rho \underline{w}) \,. \qquad (21)$$

The minimization problem then becomes:

$$\min_{\hat{\underline{w}}} \left\| \begin{bmatrix} R^{-1}Q^T \\ G^T \end{bmatrix} S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix} - \begin{bmatrix} \sigma \underline{h} \\ \underline{0} \end{bmatrix} \right\|_F \,, \qquad (22)$$

where $\underline{h} = R^{-1}Q^T FF^T \underline{q}/\delta^2$. This is a standard least squares problem that can be solved by a QR (or SVD) method. To reconstruct the scaling we observe that

$$\|G^T S_j \underline{w}\| = \|G^T S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix}\|^2/\sigma^2\rho^2 = 1 , \qquad (23)$$

using (21), and hence the required update is

$$\underline{x}_j = S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix} / \|G^T S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix}\| , \qquad (24)$$

where $\hat{\underline{w}}$ solves the least square problem (22).

We may summarize the update step of the algorithm as follows:

Algorithm 1: Step 3.1

Step 3.1.1 Form matrix X_j and find its QR decomposition (17) to obtain R, Q, and $\underline{q}$. Form vector $\underline{q}^T S_j$ and find the Householder matrix P satisfying (20).

Step 3.1.2 Form vector $\underline{h} = R^{-1}Q^T FF^T \underline{q}/\delta^2$, where $\delta^2 = \underline{q}^T FF^T \underline{q}$, and solve the least square problem (22) for $\hat{\underline{w}}$.

Step 3.1.3 Construct the update

$$\hat{\underline{x}}_j = S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix} / \|G^T S_j P \begin{bmatrix} \hat{\underline{w}} \\ 1 \end{bmatrix}\| .$$

□

We remark that the QR decomposition of X_j can be found by inexpensive up-dating techniques from the QR decomposition of X_{j-1}. The solution of the least squares problem (22) requires the decomposition of a matrix of order m-1, which may be small even where the order n of the system is large, and the procedure can thus be very efficient.

In the case $\underline{x}_j$ corresponds to a real eigenvalue λ_j, the method generates a real up-date. In the case λ_j is complex, a complex eigenvector is generated, and in order to ensure that the computed feedback K is real, it is necessary aso to update the eigenvector corresponding to $\bar{\lambda}_j$, using $\bar{\hat{\underline{x}}}_j$, the complex conjugate of the computed up-date. In practice the real and imaginary parts of $\hat{\underline{x}}_j$ can be generated independently, and hence complex arithmetic can be avoided. The optimization, however, is no longer precise (since both $\underline{x}_j$ and $\bar{\underline{x}}_j$ are not selected simultaneously), and a reduction in ν cannot be guaranteed at every step of the process. Experience indicates that this is not a drawback, and overall convergence of the algorithm is obtained in practice. Improved techniques for treating the complex poles are currently being investigated.

5. Applications

We consider the application of Algorithm 1 to two examples given in [8].

5.1 Example 1

As a simple example we consider the third-order, linear system with two inputs where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{25}$$

The poles to be prescribed are $\mathcal{L} = \{-1, -2, -3\}$, and it is assumed that the (1, 2) and (2, 2) components of the closed loop system are subject to perturbations of equal magnitude. The matrices describing the structure of the perturbations are therefore

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{26}$$

Algorithm 1 is implemented on this example using the software package MATLAB [6]. The initial set of vectors X generated by the process has condition number $\kappa(X) = 166.0$ and the value of the sensitivity measure is $\nu = 45.71$ The feedback K corresponding to these eigenvectors is

$$K = \begin{bmatrix} -19.9265 & -9.8564 & 13.6998 \\ 12.0377 & 3.1321 & -9.1813 \end{bmatrix},$$

to four decimal accuracy. To test the sensitivity of the prescribed poles, random errors of order $0(0.01)$ are introduced into the (1, 2) and (2, 2) components of the corresponding closed loop system matrix. The maximum variation in the closed loop poles then has magnitude 0.19, about twenty times the size of the system perturbations.

After one iteration of the algorithm, the sensitivity measure is reduced to $\nu = 2.490$. The feedback K is given by

$$K = \begin{bmatrix} -2.6477 & -4.7917 & 2.0846 \\ -0.3507 & 0.0477 & -1.8576 \end{bmatrix}$$

and the conditioning of the eigenvectors X is $\kappa(X) = 6.083$. Random perturbations of order $0(0.01)$ in the (1, 2) and (2, 2) components of the closed loop system lead to a maximum perturbation in the eigenvalues equal to 0.0090. The perturbations in the eigenvalues are thus of the same order of magnitude as those introduced into the system matrix, and the solution is <u>robust</u> with respect to the given class of perturbations.

After three iterations of the procedure, the improvement in the sensitivity measure is less than 0.01, and the algorithm is stopped. The sensitivity is then $\nu = 2.4716$, and the conditioning of the eigenvectors is $\kappa(X) = 6.1121$. The gain matrix is

$$K = \begin{bmatrix} -2.6923 & -4.7622 & 2.1695 \\ 0.0518 & 0.2332 & -2.2896 \end{bmatrix} .$$

Random perturbations of order $O(0.01)$ now lead to a maximum perturbation of size 0.0041 in the closed loop poles, which again demonstrates the robustness of the solution. Of course, a different set of random perturbations of order $O(0.01)$ could lead to larger errors in the poles, but since the sensitivity measure is $O(1)$, the perturbations in the poles would be expected always to be of the same order of magnitude as the system perturbations.

5.2 <u>Example 2</u>

As a second example we consider a linear-perturbation model describing the lateral dynamics of a F8-C aircraft with system matrices

$$A = \begin{bmatrix} -1.38 & 0.223 & -33.0 & 0 \\ -0.00371 & -0.196 & 6.71 & 0 \\ 0.115 & -0.999 & -0.107 & 0.0302 \\ 0.989 & 0.149 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 11.6 & 4.43 \\ 0.209 & -1.76 \\ -0.00141 & -0.0107 \\ 0 & 0 \end{bmatrix}.$$

The state and control components are sideslip, yaw rate, roll rate, bank angle, and aileron and rudder angle deflections, respectively. Desirable pole locations are given by $\mathcal{L} = \{-0.01, -2.75, -1.2 \pm i2.75\}$. It is assumed that the (1, 1) and (1, 3) components of the closed loop system are subject to arbitrary perturbations, and therefore we take

$$F = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The sensitivity of the initial closed loop system generated by the algorithm is $\nu = 110.1$ and the condition number of the corresponding eigenvectors is $\kappa(X) = 98.09$. The feedback gain matrix is then

$$K = \begin{bmatrix} 10.0427 & -20.4420 & 39.0198 & 36.7146 \\ -14.5990 & 29.3143 & -49.4385 & -52.1093 \end{bmatrix}.$$

Random perturbations introduced into the (1, 1) and (1, 3) components of the closed loop system matrix in this case cause perturbations which are larger by an order of magnitude in the closed loop poles, and the nominal system is not robust.

After one iteration of Algorithm 1, the sensitivity measure is reduced to $\nu = 0.7433$. The conditioning of the eigenvectors of the closed loop system is $\kappa(X) = 26.25$, and the feedback gain becomes

$$K = \begin{bmatrix} 0.1409 & -0.9014 & 3.5105 & -0.3208 \\ -0.5115 & 1.5504 & 1.1862 & 0.3555 \end{bmatrix}$$

to four decimal accuracy. Random perturbations in the (1, 1) and (1, 3) components of the nominal closed loop system matrix now lead to perturbations in the eigenvalues of only a fraction the size, at most one-third the magnitude.

Further iterations give no significant improvement to the sensitivity measure and one iteration is sufficient to give a robust solution. We observe that with the improvement in sensitivity of the system we obtain an improvement in the conditioning of the eigenvectors and a reduction in the magnitude of the gains. This is to be expected in general, but the feedback gain matrix which gives minimal sensitivity does not in general coincide with a minimum gain solution.

6. Conclusions

The problem of robust pole assignment by state feedback in systems which are subject to structured perturbations is examined here. A measure of robustness, or sensitivity of the poles to a given class of perturbations is derived and a reliable computational procedure for constructing a state feedback which assigns the prescribed poles and optimizes the measure of robustness is presented.

The numerical algorithm is similar to that derived for robust pole placement with respect to unstructured perturbations in [5], [4] and [2]. It is expected, therefore, that the procedure is readily extendable to feedback in descriptor (singular) systems and to robust pole assignment by output feedback. These extensions are currently under investigation.

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