POINTWISE LOGARITHM-FREE ERROR ESTIMATES FOR FINITE ELEMENTS ON LINEAR TRIANGLES

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NUMERICAL ANALYSIS REPORT NO. 6/84

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ABSTRACT

We consider two pointwise error bounds connected with the piecewise linear Ritz approximation to Poisson's equation on a uniformly triangulated square. One of them - (maximum displacement error) = $O(h^2|\log h|)$ - has been a subject of speculation for some time; the other - (maximum difference in gradients between Ritz approximation and interpolant) = $O(h^2|\log h|)$ - is newer and has its origins in gradient superconvergence. For both bounds we give a simple numerical example of $O(h^2|\log h|)$ convergence and show that the $|\log h|$ term can be dropped if the smoothness of the unknown function is slightly increased.

1. THE DISPLACEMENT ERROR

Finite element error bounds depend on the smoothness parameter (m) of the unknown function u. Their variation is typically of the abstract form

$$\left.\begin{aligned} |E(u)| \;&\leq\; c\gamma_m(h)\;\|u\|_{m,p} \\ \text{with}\qquad \gamma_m(h) &= \begin{cases} h^{k+m} & (m < s) \\ h^{k+s}\,|\log h|^{\beta} & (m = s) \\ h^{k+s} & (m > s) \end{cases} \end{aligned}\right\} \qquad (1.1)$$

for fixed p; β will be either 0 or 1; β, k and s will depend on E. (See, for instance, (1.7) below or Oganesjan & Ruchovec, 1969). Here $\|\cdot\|_{m,p}$ denotes the usual Sobolev norm in W_p^m. In this section and the next we give two examples of this pattern for which β = 1, on a model problem domain. Then in section 3 we present numerical evidence that β = 1 is actually necessary for the bounds under consideration.

Let Ω be the unit square, uniformly partitioned into squares of side h and thence into triangles with diagonals all of the same orientation (see Fig. 1). Let S^h be the space of continuous piecewise linear functions

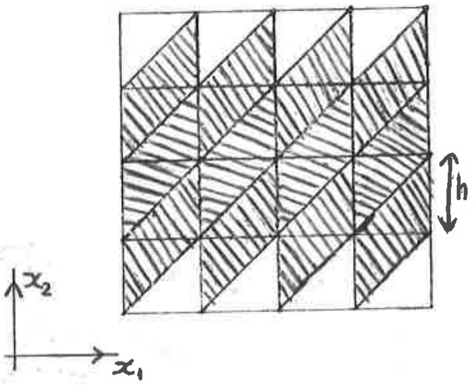

Figure 1

(In this example the twelve pairs A_k are shaded; there are eight triangles left over on $\partial\Omega$. See Section 2).

on this triangulation which vanish on the boundary $\partial\Omega$. Then for any $u \in \overset{o}{W}{}_2^1$ we define the Ritz projection $R_h u \in S^h$ by

$$(\nabla R_h u, \nabla \phi_h) = (\nabla u, \nabla \phi_h) \qquad \forall \phi_h \in S^h, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the L_2 inner product.

It is well-known that

$$\|u - R_h u\|_2 \leq ch^2 \|u\|_{2,2} \qquad \forall u \in \overset{o}{W}{}_2^1 \cap W_2^2 \tag{1.3}$$

and

$$\|\nabla(u - R_h u)\|_2 \leq ch \|u\|_{2,2} \qquad \forall u \in \overset{o}{W}{}_2^1 \cap W_2^2 , \tag{1.4}$$

where $\|\cdot\|_p$ denotes the L_p-norm. Several authors (see e.g. Nitsche, 1976; Scott, 1976; Schatz & Wahlbin, 1982) have established

$$\|u - R_h u\|_\infty \leq ch^2 |\log h| \ \|u\|_{2,\infty} \quad \forall u \in \overset{o}{W}{}_2^1 \cap W_\infty^2 \ ; \tag{1.5}$$

Rannacher and Scott (1982) showed that

$$\|\nabla(u - R_h u)\|_\infty \leq ch \ \|u\|_{2,\infty} \qquad \forall u \in \overset{o}{W}{}_2^1 \cap W_\infty^2 \ . \tag{1.6}$$

In this paper they said, "it has been considered as a challenge from the beginning to remove the logarithmic factors [in (1.5)]". This is particularly so because for higher order elements or the piecewise linear interpolant that estimate holds without the logarithm. Incidentally, the bounds (1.3) - (1.6) apply to more general finite element approximations on more general problem domains.

With a result which appears to have gone unappreciated in the literature, Bramble and Thomée (1974) showed by a Finite Difference analysis that, on the uniformly triangulated square:

$$\left.\begin{array}{l} \underline{\text{Theorem I}} \qquad \|u - R_h u\|_\infty \leq ch^2 \, |\log h|^\beta \, \|u\|_{2+\varepsilon,\infty} \qquad \forall u \in \overset{o}{W}{}^1_2 \cap W^{2+\varepsilon}_\infty \\ \qquad \text{with } \beta = \begin{cases} 1 & \text{when } \varepsilon = 0 \\ 0 & \text{when } \varepsilon > 0 \end{cases} \end{array}\right\} \quad (1.7)$$

We will now establish (1.7) by Finite Element Methods.

Let z be any point which is bounded away from $\partial\Omega$ and in the interior of one of the triangles (T_z say) and let $\delta = \delta(x;z) = \delta(|x-z|) \in C_0(T_z)$ be such that

$$\sup_{T_z} |\delta| \leq ch^{-2}, \quad \int_{T_z} \delta(x;z)dx = 1$$

$$\text{and} \qquad \phi_h(z) = (\phi_h,\delta) \qquad \forall \phi_h \in S^h. \qquad (1.8)$$

We define $G = G(x;z) \in \overset{o}{W}{}^1_2 \cap W^2_2$ by

$$(\nabla u, \nabla G) = (u,\delta) \qquad \forall u \in \overset{o}{W}{}^1_2(\Omega). \qquad (1.9)$$

G is a smoothed Green's function with "singularity" at z, for Poisson's equation with homogeneous Dirichlet boundary data. By a simple though lengthy argument (see Levine, 1985), there exists a (harmonic) function $V=V(x;z)$ which is normed in W^2_2 independently of h and for which

$$\left.\begin{array}{ll} G - V = \frac{1}{2\pi} \log|x-z| & \text{if } |x-z| > h \\ \text{and} \quad |\nabla_m(G-V)| \leq ch^{-m}|\log h| & \text{if } |x-z| < h, \end{array}\right\} \quad (1.10)$$

where ∇_m denotes the tensor of m^{th} derivatives, $(m \geq 0)$. We remark here that the results that follow can be applied to points on or near element edges (or vertices) of $\partial\Omega$ by adding one (or three) suitable image functions to G.

Let $I_h u$ denote the nodal interpolant of u. From (1.2), (1.8) and (1.9) we obtain

$$(u - R_h u)(z) = (u - I_h u)(z) - F_{z;h}(u) \qquad (1.11)$$

$$\begin{aligned} \text{where} \qquad F_{z;h}(u) &= (R_h - I_h)u(z) \\ &= ((R_h - I_h)u,\delta) \\ &= (\nabla(R_h - I_h)u,\nabla G) \\ &= (\nabla(u - I_h u), \nabla R_h G) \end{aligned}$$

We use a result of Rannacher and Scott (1982). Let $\sigma = \sigma(x;z)$ be defined for some fixed $k > 0$ by

$$\sigma^2 = |x - z|^2 + k^2h^2 \quad ; \tag{1.12}$$

then for any function $\phi \in \overset{o}{W}{}^1_2 \cap W^2_2$,

$$\int \sigma^{2+\alpha}|\nabla(\phi - R_h\phi)|^2 \leq ch^2 \int \sigma^{2+\alpha}|\nabla_2\phi|^2 \tag{1.13}$$

where α is any fixed, strictly positive number, Recalling (1.7), we take α to have any (small) value if $\varepsilon = 0$ and $\alpha = \varepsilon/2$ if $\varepsilon > 0$. From the Holder inequality, (1.13) and (1.10) we obtain

$$\begin{aligned}\|\sigma^{\varepsilon}\nabla(G - R_hG)\|_1 &\leq c \left(\int\sigma^{-2-\alpha+2\varepsilon}\right)^{\frac{1}{2}} \left(\int\sigma^{2+\alpha}|\nabla(G - R_hG)|^2\right)^{\frac{1}{2}} \\ &\leq c \ . \ ch^{-\alpha\beta/2} . \ ch \\ &\leq ch \ |\log h|^{\beta} .\end{aligned}$$

Also from (1.10),

$$\|\sigma^{\varepsilon}\nabla G\|_1 \leq ch$$

whence

$$\|\sigma^{\varepsilon}\nabla R_hG\|_1 \leq ch \ |\log h|^{\beta} . \tag{1.14}$$

This bound is the source of $O(h^2|\log h|)$ estimates. To prove (1.7), we must weight the integral of ∇R_hG so that contributions either decay radially, cancel transversely or vanish altogether. We will now therefore decompose u into the parts which combine differently with ∇R_hG. Since $u \in W^{2+\varepsilon}_{\infty}$ for some $\varepsilon \geq 0$, we can write

$$u = q + r$$

where q is a quadratic and

$$\left.\begin{aligned} &\nabla_m q(z) = \nabla_m u(z) \qquad m = 0,1,2 \\ \text{and} \qquad &|\nabla_2 r(x)| \leq \|u\|_{2+\varepsilon,\infty} |x - z|^{\varepsilon} \end{aligned}\right\} \tag{1.15}$$

It is one of the principal features of Ω as triangulated that

$$(\nabla(Q - I_h Q), \nabla\phi_h) = 0 \qquad \forall \text{ quadratics } Q, \quad \forall\phi_h \in S^h. \tag{1.16}$$

(This is because the interpolant of any quadratic satisfies the Ritz equations (1.2), i.e. $I_h Q \equiv R_h Q$. For an alternative proof see Section 2). Thus

$$\begin{aligned} |F_{z;h}(u)| &= |(\nabla(r - I_h r), \nabla R_h G)| \\ &\leqq \|\sigma^{-\varepsilon}\nabla(r - I_h r)\|_\infty \quad \|\sigma^{\varepsilon}\nabla R_h G\|_1 \quad . \end{aligned}$$

We apply the Bramble-Hilbert Lemma (Bramble & Hilbert, 1970) and (1.15) separately to each element. From (1.14) we then obtain

$$|F_{z;h}(u)| \leqq ch \; \|u\|_{2+\varepsilon,\infty} \; \cdot \; ch \; |\log h|^{\beta} \quad .$$

Finally the Bramble-Hilbert lemma applied element by element gives

$$\|u - I_h u\|_\infty \leqq ch^2 \; |u|_{2,\infty}$$

and (1.7) now follows from (1.11).

2. THE GRADIENT DIFFERENCE

We prove below that

$$\|\nabla(I_h - R_h)u\|_2 \leqq ch^2 \; \|u\|_{3,2} \qquad \forall u \in \overset{o}{W}{}^1_2 \cap W^3_2 \quad . \tag{2.1}$$

This result should be compared with (1.4). It leads to "gradient superconvergence": ∇u can be estimated from $\nabla R_h u$ to $O(h^2)$ by any algorithm which recovers derivatives from $\nabla I_h u$ to that accuracy. (See Levine, 1983 and 1985; for related results on quadrilateral elements see Le-Saint & Zlámal, 1979). We will also prove the corresponding pointwise result:

Theorem II

$$\left.\begin{aligned} &\|\nabla(I_h - R_h)u\|_\infty \leqq ch^2 |\log h|^{\beta} \|u\|_{3+\varepsilon,\infty} \quad \forall u \in \overset{o}{W}{}^1_2 \cap W^{3+\varepsilon}_\infty \\ &\text{with } \beta = \begin{cases} 1 & \text{when } \varepsilon = 0 \\ 0 & \text{when } \varepsilon > 0 \quad . \end{cases} \end{aligned}\right\} \tag{2.2}$$

We consider first the products

$$F_i = \left(\frac{\partial}{\partial x_i}(I_h u - u), \frac{\partial \phi h}{\partial x_i}\right), \qquad \phi_h \in S^h, \; i = 1,2 ,$$

where (x_1, x_2) are co-ordinate axes parallel to the sides of Ω. We bound F_1, noting that F_2 can clearly be treated identically. We partition Ω into pairs of triangles which have common edges parallel to the x_1-axis, denoting these pairs by A_k $(k = 0,\ldots,k_{max})$, with a number of single elements on $\partial\Omega$ which cannot be paired. (See Fig. 1). Since $\phi_h = 0$ on $\partial\Omega$ $\forall \phi_h \in S^h$, $\partial\phi_h/\partial x_1$ is not only constant on each A_k but vanishes in the unpaired elements and we can write

$$F_1 = \sum_k \left(\int_{A_k} \frac{\partial}{\partial x_1}(I_h u - u)\right)\left[\frac{\partial \phi h}{\partial x_1}\right]_{A_k}$$

$$= \sum_k C_{1,k}(u) \left[\frac{\partial \phi h}{\partial x_1}\right]_{A_k} \text{(say)} . \qquad (2.3)$$

It is easily verified (Levine, 1982 or 1984) that $C_{1,k}(u)$ vanishes for quadratic u, $\forall k$. Therefore, by the Bramble-Hilbert lemma in W_p^3,

$$\left.\begin{aligned} C_{1,k}(u) &= c_{1,k}(u)\, h^{4-2/p} \\ \text{where} \quad |c_{1,k}| &\leq c \;\|u\|_{W_p^3(A_k)} \end{aligned}\right\} \forall k. \qquad (2.4)$$

For each k, let z_k be the centroid of A_k. Since all the A_k are congruent, $c_{1,k}$ depends only on the variation of u within A_k, thus:

$$c_{1,k}(u(x)) = c_{1,0}(u(x + z_k - z_0)). \qquad (2.5)$$

Now (2.1) can be obtained directly from (2.3) and (2.4) with $p = 2$. For

$$|F_1| \leq \sum_k ch^3 \;\|u\|_{W_2^3(A_k)} \cdot ch^{-1} \;\|\nabla\phi_h\|_{L_2(A_n)}$$

$$\leq ch^2 \;\|u\|_{3,2} \;\|\nabla\phi_h\|_2 .$$

We take $\phi_h = (I_h - R_h)u$ and apply (1.2):

$$\|\nabla(I_h - R_h)u\|_2^2 = (\nabla(I_h u - u), \nabla\phi_h)$$

$$\leq |F_1| + |F_2|$$

$$\leq ch^2 \;\|u\|_{3,2} \;\|\nabla(I_h - R_h)u\|_2 ,$$

whence (2.1). (Note that since $C_{i,k}(Q) = 0$ for all quadratics Q, we have also derived (1.16)).

To establish the pointwise result we must again introduce a Green's function. Let A_0 be one of the triangle pairs introduced above, bounded away from $\partial\Omega$. We let $\delta = \delta(x;z_0)$ be such that

$$\sup_{A_0} |\delta| \leq ch^{-2} \quad \text{and} \int_{A_0} \delta \, dx = 1 \quad ;$$

therefore

$$\left[\frac{\partial \phi_h}{\partial x_1}\right]_z = \left(\frac{\partial \phi h}{\partial x_1}, \delta\right) \qquad \forall \phi_h \in S^h, \ \forall z \in A_0. \tag{2.6}$$

We define $g = g(x;z) \in \overset{\circ}{W}{}^1_2 \cap W^2_2$ to be the smoothed derivative Green's function of Rannacher and Scott:

$$(\nabla u, \nabla g) = \left(\frac{\partial u}{\partial x_1}, \delta\right) \qquad \forall u \in \overset{\circ}{W}{}^1_2 . \tag{2.7}$$

Again (see Levine, 1984 for details) we have a (harmonic) function $v=v(x;z)$, normed independently of h in W^2_2, for which

$$\left.\begin{aligned} g - v &= \frac{(x - z_0)_1}{2\pi|x - z_0|^2} && \text{if } |x - z_0| > h \\ \text{and} \quad |\nabla_m(g - v)| &\leq ch^{-1-m} \ (m \geq 0) && \text{if } |x - z_0| < h. \end{aligned}\right\} \tag{2.8}$$

(If A_0 is not bounded away from $\partial\Omega$ as $h \to 0$, we modify (2.8) by adding one (or three) suitable image functions to g.) This decomposition implies that

$$||\sigma^{\varepsilon}\nabla g||_1 \quad \leq \quad c|\log h|^{\beta} \quad .$$

Also, Rannacher and Scott (1982) have proved from (1.13) that

$$||\nabla(R_h g - g)||_1 \leq c \qquad ; \tag{2.9}$$

hence

$$||\sigma^{\varepsilon}\nabla R_h g||_1 \leq c \ |\log h|^{\beta} \qquad . \tag{2.10}$$

Now, by (2.6), (2.7), (1.2), (2.3) and (2.4) with $p = \infty$,

$$\left[\frac{\partial}{\partial x_1}(I_h - R_h)u\right]_{A_0} = (\nabla(I_h - R_h)u, \nabla g)$$

$$= (\nabla(I_h u - u), \nabla R_h g)$$

$$= \sum_{i=1,2} \sum_k c_{i,k}(u)h^4 \left[\frac{\partial R_h g}{\partial x_i}\right]_{A_k} . \tag{2.11}$$

As in section 1, we treat this integral of $\nabla R_h g$ carefully. Since $u \in W_\infty^{3+\varepsilon}$ for some $\varepsilon \geqq 0$, we can write

$$u = q + r$$

where q is now a cubic and

$$\left.\begin{aligned} &\nabla_m q(z_0) = \nabla_m u(z_0) \qquad m = 0,\ldots,3 \\ \text{and} \qquad &|\nabla_3 r(x)| \leqq \|u\|_{3+\varepsilon,\infty} |x - z_0|^\varepsilon \end{aligned}\right\} \tag{2.12}$$

Since q is a cubic, we have from (2.5) and (1.16):

$$\left|\sum_{i,k} c_{i,k}(q)h^4 \left[\frac{\partial R_h g}{\partial x_i}\right]_{A_k}\right| = \left|\sum_{i,k} c_{i,0}(q)h^4 \left[\frac{\partial R_h g}{\partial x_i}\right]_{A_k}\right|$$

$$\leqq ch^2 \|q\|_{3,\infty} \left|\int_\Omega \nabla R_h g\right|$$

$$= ch^2 \|q\|_{3,\infty} \left|\int_\Omega \nabla(R_h g - g)\right|, \tag{2.13}$$

because $\int_\Omega \nabla g = \oint_{\partial\Omega} g = 0$. Also, by (2.4) and (2.12),

$$\left|\sum_{i,k} c_{i,k}(r)h^4 \left[\frac{\partial R_h g}{\partial x_i}\right]_{A_k}\right| \leqq ch^2 \sum_k |r|_{W_\infty^3(A_k)} \cdot \left|\int_{A_k} \nabla R_h g\right|$$

$$\leqq ch^2 \|u\|_{3+\varepsilon,\infty} \|\sigma^\varepsilon \nabla R_h g\|_1 . \tag{2.14}$$

Combining (2.9) - (2.14), we conclude that since A_0 is an arbitrary element pair,

$$\left\|\frac{\partial}{\partial x_1}(I_h - R_h)u\right\|_\infty \leqq ch^2 |\log h|^\beta \|u\|_{3+\varepsilon,\infty} .$$

The x_2-derivative is bounded identically and (2.2) follows immediately.

3. NUMERICAL EXAMPLES OF LOGARITHMIC CONVERGENCE

We illustrate the above results with examples of functions with $\varepsilon = 0$ for which $\beta = 1$ is necessary in estimates (1.7) & (2.2). The examples were derived by an inuitive matching of the directions of ∇u with ∇G (or ∇g). Their existence implies that some pointwise estimates do indeed require the extra part-derivative smoothness (i.e.$\varepsilon > 0$) if they are to be logarithm-free. Note that there exist other pointwise estimates, such as (1.6), which do not involve $|\log h|$ even when $\varepsilon = 0$.

We start with (1.7). Let (x_1,x_2) be rectangular Cartesian co-ordinates such that the square $\Omega = (-\frac{1}{2},\frac{1}{2})^2$. We set

$$u = x_1^4 x_2^2 / |x|^4 - |x|^3/5 \qquad (3.1)$$

and define $R_h u$ by

$$(\nabla R_h u, \nabla\phi_h) = -\overline{(\nabla^2 u, \phi_h)} \qquad \forall\phi_h \in S^h$$

where $\overline{(\cdot,\cdot)}$ denotes the use of the centroid rule in each element to approximate $(\cdot,\cdot)$. The $|x|^3$ term is in W^3_∞ - i.e. $\varepsilon >> 0$ - and so does not contribute to the $|\log h|$ behaviour. This term, with the factor 1/5, is chosen to highlight the asymptotic behaviour of $(u - R_h u)$ for computationally practical values of h. For the same reason we sample

$$E(h;u;z) = (u - R_h u)(z)$$

at one point - the origin $z = (0,0)$ - rather than taking its supremum over all z in Ω. In Table I, values of $|E|$, $|E/h^2|$, $|E/h^2 \log h|$ are tabulated for $h = \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots, \frac{1}{22}$; in addition we give the relative differences between rows in the columns marked Δ. The $|\log h|$ factor is clear for $h < \frac{1}{10}$.

For (2.2) we take Ω as above and

$$u = x_1^3 x_2^2 / |x|^2. \qquad (3.2)$$

Again, we do not maximise

$$e(h;u;z) = \nabla(R_h - I_h)u(z)$$

over all z: we sample the x_1 component of e at the point $z = (0,-h/2)$. We take h to have the values $\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{15}$. The results are given in Table II; this time the $|\log h|$ factor is very clear for $h < \frac{1}{7}$.

CONCLUDING REMARKS

For a model triangulation we have justified simple conditions necessary and sufficient to guarantee logarithm-free convergence of the error bounds under consideration. On a general problem domain the proofs require modification for two reasons. Near $\partial\Omega$ the Green's functions G and g are no longer given by simple image functions. It may well be possible however to derive (1.14) and (2.10) so that G and g are nowhere expanded as explicitly as in (1.10) and (2.8); see Rannacher and Scott (1982).

In addition, the quantities

$$(\nabla(Q - I_hQ), \nabla\phi_h) \tag{3.3}$$

(Q quadratic, $\phi_h = R_hG$ or R_hg) are no longer simple to estimate; this could affect the results as well as their proofs. Now in the case of gradient superconvergence, the L_2 result (2.1) does not hold on general domains unless the triangulation is a "smooth" distortion of a uniform mesh (Levine, 1983); this condition is necessary for better-than-O(h) convergence. We have the framework and some details of a proof that (2.2) is also retained on such regions, but these are too long to be given here.

The displacement result (1.3) requires only non-geneneracy of the triangulation of Ω. The general topological prerequisite for (1.16) and (2.1) - namely that exactly six elements should meet at each internal node of Ω - is no longer necessary. On such a mesh we do not know how to treat the term (3.3) (or even where to start) and it is unclear whether we can improve upon (1.5). The example of Jespersen (1978) and Fried (1980) indicates that the $|\log h|$ term may be necessary on some triangulations, for arbitrarily smooth u. However, it is based on the reduction of Poisson's equation with cylindrical symmetry to a one-dimensional (singular) problem and is therefore not directly related to our two-dimensional approximations.

ACKNOWLEDGEMENT

I would like to thank Professor K.W. Morton for his aid in supervisions and the Science & Engineering Research Council and University of Reading for their financial support.

November, 1983

h^{-1}	$\|E(h;u;(0,0))\|$ $(\times 10^5)$	$\|E/h^2\|$ $(\times 10^2)$	$\|E/h^2 \log h\|$ $(\times 10^3)$	$\|\Delta(E/h^2)\|$	$\|\Delta(E/h^2 \log h)\|$
4	55.53	0.8885	6.409		
				10%	16%
6	27.16	0.9777	5.456		
				9%	5.6%
8	16.76	1.073	5.158		
				7.6%	2.6%
10	11.57	1.157	5.026		
				6.3%	1.3%
12	8.559	1.232	4.960		
				5.3%	0.73%
14	6.630	1.299	4.924		
				4.5%	0.42%
16	5.310	1.359	4.903		
				3.9%	0.25%
18	4.363	1.414	4.891		
				3.4%	0.14%
20	3.658	1.463	4.884		
				3.0%	0.08%
22	3.117	1.508	4.880		

TABLE I

$E(h;u;(0,0)) = (u - R_h u)(0,0) = O(h^2|\log h|)$

when $u = x_1^{\,4} x_2^{\,2}/|x|^4 - |x|^3/5$

and Ω is the uniformly triangulated square $(-\frac{1}{2},\frac{1}{2})^2$.

h^{-1}	$\lvert e_1(h;u;(0,-\frac{h}{2}))\rvert$ $(\times 10^3)$	$\lvert e_1/h^2\rvert$ $(\times 10)$	$\lvert e_1/h^2 \log h\rvert$ $(\times 10)$	$\lvert\Delta(e_1/h^2)\rvert$	$\lvert\Delta(e_1/h^2 \log h)\rvert$
3	17.70	1.593	1.450		
				26%	12%
5	8.301	2.076	1.290		
				16%	2.9%
7	4.976	2.438	1.253		
				11%	0.85%
9	3.370	2.730	1.242		
				8.4%	0.32%
11	2.245	2.970	1.238		
				6.6%	0.14%
13	1.877	3.172	1.2367		
				5.4%	0.06%
15	1.488	3.347	1.2359		

TABLE II

$e_1(h;u;(0,-\frac{h}{2}))=\frac{\partial}{\partial x_1}(I_h - R_h)u(0,-\frac{h}{2}) = O(h^2\lvert\log h\rvert)$

when $u = x_1{}^3 x_2{}^2/\lvert x\rvert^2$

and Ω is the uniformly triangulated square $(-\frac{1}{2},\frac{1}{2})^2$.

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