MOVING FINITE ELEMENT MODELLING OF THE 2-D SHALLOW WATER EQUATIONS.

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Abstract

A mathematical model of tidal flow requires the solution of the system of shallow water equations in two dimensions.

Numerical methods are presented based on moving finite elements which are easy to implement and which dispense with some of the boundary conditions needed for an earlier scheme by allowing the grid to move.

Results show that an additional element of regularisation is likely to be needed to prevent element collapse.

Contents

1. Introduction

Power stations on coastal sites extract water from the sea for cooling purposes. When the water is returned to the sea it is heated and spreads out to form a plume. It is of economic and environmental interest to the CEGB to construct an accurate model of coastal flows in order to determine the effects of the discharge.

Detailed analysis of such flows is already possible using a finite difference model, but problems encountered with that model have motivated the desire to use schemes that can incorporate moving boundaries.

In this work we look to solve the governing equations, presented in section 2, using numerical techniques based on the Moving Finite Element method of Miller [ref. 1]. These methods are presented in section 3 and their adaptation to the model problem is discussed in section 4.

Results are presented in section 5 and conclusions given in section 6.

2. The Model Problem

2.1 The Model Equations

In this work, we wish to solve the shallow water equations in the form

$$U_t = - UU_x - VU_y - gZ_x - \frac{FU\sqrt{U^2 + V^2}}{h + Z} + \Omega V \tag{2.1}$$

$$V_t = - UV_x - VV_y \; gZ_y - \frac{FV\sqrt{U^2 + V^2}}{h + Z} - \Omega U \tag{2.2}$$

$$Z_t = U(h_x + Z_x) - V(h_y + Z_y) - (h + Z)\;(U_x + V_y) \tag{2.3}$$

where

$Z \equiv$ elevation above some specified datum (M)

$h \equiv$ bed depth below datum (M)

$U \equiv$ easterly velocity component (M/S)

$V \equiv$ northerly velocity component (M/S)

$g \equiv$ acceleration due to gravity (M/S^2)

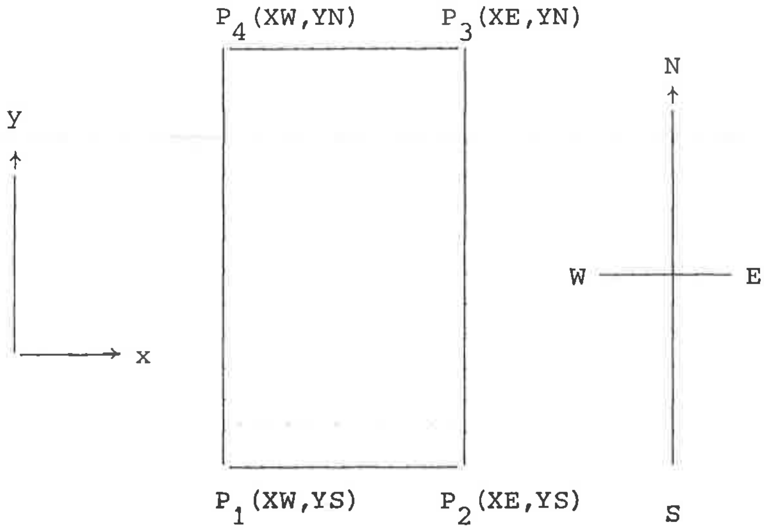
$\Omega \equiv$ Coriolis parameter ($O(10^{-4})$) (/s)

and $F \equiv$ friction factor ($O(10^{-3})$).

2.2 The 'Moving Bag' Problem

Assume an idealised geometry in which there is a straight coastline and consider a rectangular region oriented as in fig. 2.1.

fig. 2.1



P_1P_4 - coastline

P_1P_2, P_2P_3, P_3P_4 - open boundaries.

Existing numerical models of the shallow water equations use finite difference methods to produce a solution on a fixed grid. It has been found that the reliability of such schemes depend strongly on the inflow and outflow conditions applied at the open boundaries. In this work, however, we view $P_1P_2P_3P_4$ as enclosing a body of water which we track through time using a numerical scheme which allows the region to move. This approach enables the treatment of the open boundaries to be incorporated into the scheme in a natural way (see section 4). We restrict the motion of all points on P_1P_4 to the y direction in order that we follow a region that moves up or down the straight coast-line.

3. Numerical Methods

For the problem under consideration, we desire to employ both the local Moving Finite Element (MFE) and Mobile Element Method (MEM) methods ([3],[4]) for the solution of a system of hyperbolic equations in 2D. It may be useful, however, to begin this section with a brief introduction to the global MFE method for the solution of evolutionary problems in 1-D, as described by Miller [1].

3.1 Global MFE in 1-D

We wish to solve the equation

$$u_t = L(u) \tag{3.1}$$

where L is some non-linear differential operator in one space dimension.

An approximate piecewise linear MFE solution to (3.1) takes the form

$$U = \sum_{j=1}^{N} U_j(t)\ \alpha_j(x,\underline{s}(t)) \tag{3.2}$$

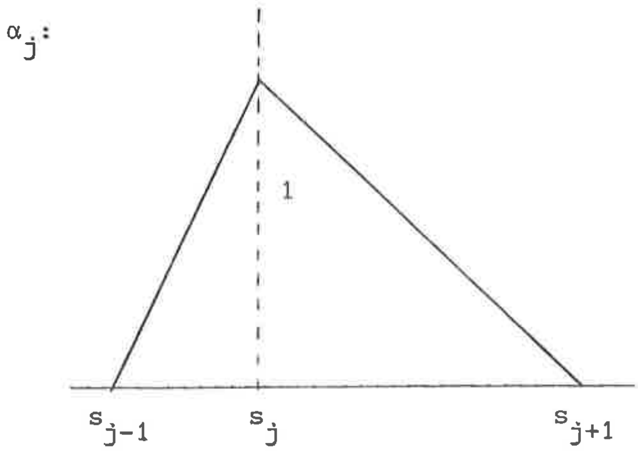
where the parameters U_j are the nodal amplitudes and the α_j essentially have the same form as the standard piecewise linear finite element basis functions (fig. 3.1), with the significant difference of being dependent on nodal positions which vary with time.

The α_j are given by

$$\alpha_j = \begin{cases} \dfrac{x - s_{j-1}}{s_j - s_{j-1}} & s_{j-1} \leq x \leq s_j \\[2ex] \dfrac{s_{j+1} - x}{s_{j+1} - s_j} & s_j \leq x \leq s_{j+1} \end{cases} \qquad (3.3)$$

and so a typical α_j looks like

fig. 3.1



Differentiating (3.2) with respect to time yields

$$U_t = \sum_{j=1}^{N} \dot{U}_j \alpha_j + \sum_{j=1}^{N} \dot{s}_j \beta_j \tag{3.4}$$

where the β_j are thought of as second basis functions and are given by

$$\beta_j = \sum_{i=j-1}^{j+1} U_j \frac{\partial \alpha_i}{\partial s_j} \tag{3.5}$$

It can readily be shown [ref. 2] that (3.3) and (3.5) imply that the β_j satisfy

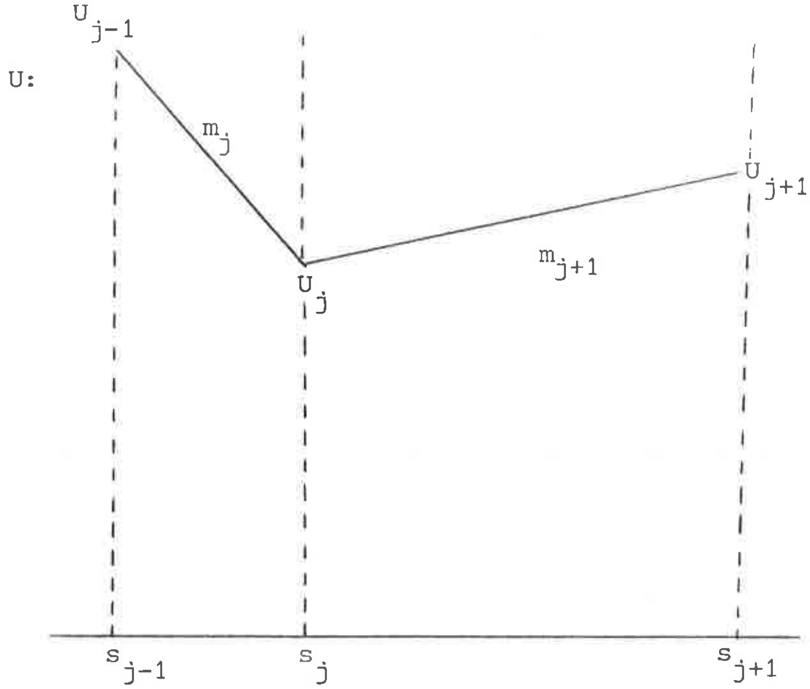
$$\beta_j = \begin{cases} -m_j \alpha_j & s_{j-1} \leq x < s_j \\ -m_{j+1} \alpha_j & s_j < x \leq s_{j-1} \end{cases} \tag{3.6}$$

where

$$m_j = \frac{U_j - U_{j-1}}{s_j - s_{j-1}}, \tag{3.7}$$

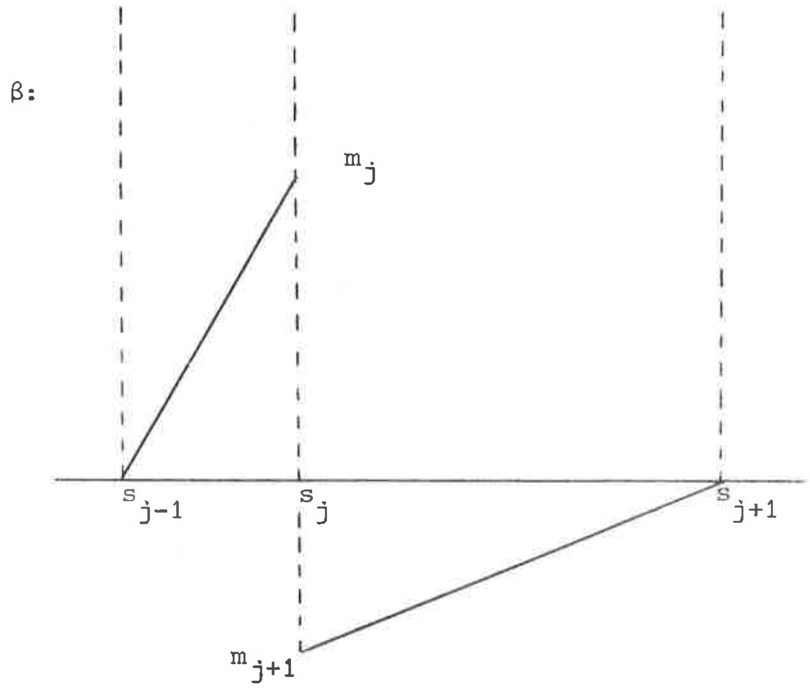
i.c. m_j is the slope in element j of the MFE approximation.

fig. 3.2



For an MFE solution U as in fig. 3.2 and an α_j as in fig. 3.1, β_j would look like

fig. 3.3



It is immediately apparent that the β_j are, in general, discontinuous at the nodes.

To obtain a system of ordinary differential equations for U_j and s_j we demand that the residual $R = U_t - L(U)$ be perpendicular to the space spanned by the α and β basis functions. This is equivalent to the requirement that the L_2 norm of the residual be minimised with respect to variations of the $\dot{U}_j$ and $\dot{s}_j$.

Now,

$$\|R\|^2_{L_2} = \langle R,R\rangle = \langle U_t,U_t\rangle - 2\langle U_t,L(U)\rangle + \langle L(U),L(U)\rangle \tag{3.8}$$

where $\langle .,.\rangle$ denotes the usual L_2 inner product.

Substituting (3.4) into (3.8) yields

$$\|R\|^2_{L_2} = \sum_{j,k} (\langle\alpha_j,\alpha_k\rangle\dot{U}_j\dot{U}_k + 2\langle\alpha_j,\beta_k\rangle\dot{U}_j\dot{s}_k + \langle\beta_j,\beta_k\rangle\dot{s}_j\dot{s}_k)$$

$$- 2\sum_j (\langle\alpha_j,L(U)\rangle\dot{U}_j + \langle\beta_j,L(U)\rangle\dot{s}_j) + \langle L(U),L(U)\rangle \tag{3.9}$$

Differentiating (3.9) with respect to $\dot{U}_i$ and $\dot{s}_i$ gives the 2N ordinary differential equations for U_j and s_j

$$\sum_{j=1}^{N} \langle\alpha_i,\alpha_j\rangle\dot{U}_j + \sum_{j=1}^{N} \langle\alpha_i,\beta_j\rangle\dot{s}_j = \langle\alpha_i,L(U)\rangle \tag{3.10a}$$

$$\sum_{j=1}^{N} \langle\beta_i,\alpha_j\rangle\dot{U}_j + \sum_{j=1}^{N} \langle\beta_i,\beta_j\rangle\dot{s}_j = \langle\beta_i,L(U)\rangle \tag{3.10b}$$

$i = 1, \ldots, N.$

In matrix form, if we write

$$\underline{y} = (\dot{U}_1,\dot{s}_1,\dot{U}_2,\dot{s}_2,\ldots,\dot{U}_N,\dot{s}_N)^T, \tag{3.11}$$

then equations (3.10a) and (3.10b) imply the system

$$A\dot{\underline{y}} = \underline{g} \tag{3.12}$$

where

$$\underline{g} = \{g_{2i},\ g_{2i+1}\}$$

$$\left.\begin{aligned} g_{2i} &= \langle\alpha_i,L(U)\rangle \\ g_{2i+1} &= \langle\beta_i,L(U)\rangle \end{aligned}\right\} i = 1, \ldots, N \tag{3.13}$$

and A is block tridiagonal, i.e.

$$A = \begin{bmatrix} A_1 & B_1 & & & \\ C_2 & A_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & B_{N-1} \\ & & & C_N & A_N \end{bmatrix} \tag{3.14}$$

Where

$$A_i = \begin{bmatrix} \langle\alpha_i,\alpha_i\rangle & \langle\alpha_i,\beta_i\rangle \\ \langle\beta_i,\alpha_i\rangle & \langle\beta_i,\beta_i\rangle \end{bmatrix} \quad i = 1, \ldots, N \tag{3.14a}$$

$$B_i = \begin{bmatrix} \langle\alpha_i,\alpha_{i+1}\rangle & \langle\alpha_i,\beta_{i+1}\rangle \\ \langle\beta_i,\alpha_{i+1}\rangle & \langle\beta_i,\beta_{i+1}\rangle \end{bmatrix} \quad i = 1, \ldots, N-1 \tag{3.14b}$$

$$C_i = \begin{bmatrix} \langle\alpha_i,\alpha_{i-1}\rangle & \langle\alpha_i,\beta_{i-1}\rangle \\ \langle\beta_i,\alpha_{i-1}\rangle & \langle\beta_i,\beta_{i-1}\rangle \end{bmatrix} \quad i = 2, \ldots, N \tag{3.14c}$$

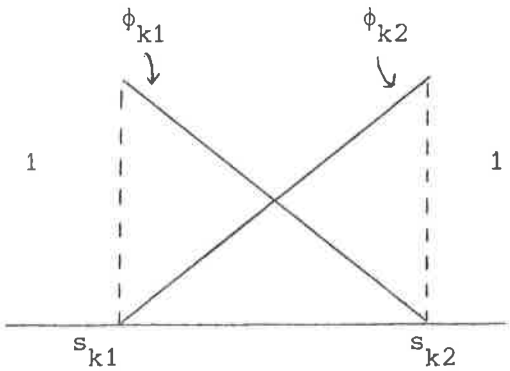
It can be seen that $C_i = B^T_{i-1}$ and so, from (3.14), it follows that the MFE matrix A is symmetric. Consequently, simplifications may be made when solving the system by some iterative technique.

However, we now consider an elementwise, or local, formulation of the MFE method which results in a decoupling of equations (3.10 a,b) and obviates the need for a numerical matrix solver.

3.2 <u>Local MFE in 1-D</u>

Consider an element k, with nodal positions s_{k1} and s_{k2} and nodal amplitudes U_{k1} and U_{k2}. Introduce the element basis functions ϕ_{k1} and ϕ_{k2} (fig. 3.4).

fig. 3.4



The $\phi_{k\nu}$ (ν=1,2 : k = 1, ..., K = no. of elements) span the space of all piecewise linear discontinuous functions on the finite element grid. In one dimension, this is equivalent to the space spanned by the α_j and β_j basis functions (j = 1, ..., N). Consequently, in this exceptional case, it is trivially true from (3.4) that we may write U_t in the form

$$U_t = \sum_{k=1}^{K} \sum_{\nu=1}^{2} W_{k\nu} \phi_{k\nu} \tag{3.15}$$

(where $K = N$ here).

We now insist that the square of the L_2 norm of the residual be minimised with respect to variations of the W's. We have

$$\| \sum_{p=1}^{K} (W_{p1}\phi_{p1} + W_{p2}\phi_{p2}) - L(U) \|^2$$

$$= \sum_{p,q} (<\phi_{p1},\phi_{q1}>W_{p1}W_{q1} + 2<\phi_{p1},\phi_{q2}>W_{p1}W_{q2} + <\phi_{p2}\phi_{q2}>W_{p2}W_{q2})$$

$$- 2 \sum_{p} (<\phi_{p1},L(U)>W_{p1} + <\phi_{p2},L(U)>W_{p2}) + <L(U),L(U)>$$

$$= \sum_{p} (<\phi_{p1},\phi_{p1}>W^2_{p1} + 2<\phi_{p1},\phi_{p2}>W_{p1}W_{p2} + <\phi_{p2},\phi_{p2}> W^2_{p2})$$

$$- \sum_{p} (<\phi_{p1},L(U)>W_{p1} + <\phi_{p2},L(U)>W_{p2}) + <L(U),L(U)> \tag{3.16}$$

since $\langle\phi_{p\nu},\phi_{q\eta}\rangle = 0$ except when $p = q$.

Differentiating (3.16) with respect to W_{k1} and W_{k2} yields the 2 x 2 system

$$C_k \underline{W}_k = \underline{b}_k \tag{3.17}$$

where

$$C_k = \begin{pmatrix} \langle\phi_{k1},\phi_{k1}\rangle & \langle\phi_{k1},\phi_{k2}\rangle \\ \langle\phi_{k1},\phi_{k2}\rangle & \langle\phi_{k2},\phi_{k2}\rangle \end{pmatrix} \tag{3.17a}$$

$$\underline{W}_k = (W_{k1}\ W_{k2})^T \tag{3.17b}$$

and

$$\underline{b}_k = \begin{pmatrix} \langle\phi_{k1},L(U)\rangle \\ \langle\phi_{k2},L(U)\rangle \end{pmatrix} \tag{3.17c}$$

Assume now that node j lies between elements $k-1$ and k. We have

$$\alpha_j = \phi_{k-1,2} + \phi_{k1}$$

$$\beta_j = - m_{k-1} \phi_{k-1,2} - m_k \phi_{k1}$$

and so

$$\sum_j (\dot{U}_j \alpha_j + \dot{s}_j \beta_j) = \sum_k \{(\dot{U}_j - m_{k-1} \dot{s}_j) \phi_{k-1,2} + (\dot{U}_j - m_k \dot{s}_j) \phi_{k1} \tag{3.18}$$

From (3.4), (3.15) and (3.18) we obtain, for interior nodes,

$$\begin{pmatrix} 1 & - & m_{k-1} \\ 1 & - & m_k \end{pmatrix} \begin{pmatrix} \dot{U}_j \\ \dot{s}_j \end{pmatrix} = \begin{pmatrix} W_{k-1,2} \\ W_{k1} \end{pmatrix} \tag{3.19}$$

Provided that $m_{k-1} \neq m_k$, the system (3.19) may be solved to give

$$\dot{U}_j = \frac{m_k W_{k-1,2} - m_{k-1} W_{k1}}{m_k - m_{k-1}} \tag{3.20}$$

and

$$\dot{s}_j = \frac{W_{k-1,2} - W_{k1}}{m_k - m_{k-1}} \tag{3.21}$$

The local MFE procedure is to solve the system (3.17) in each element and then to transfer the solution back on to the nodes via (3.20) and (3.21).

It can be shown that (3.17a) implies

$$C_k = \Delta s_k \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \tag{3.22}$$

where $\Delta s_k = s_{k2} - s_{k1}$ is the length of the element k. Therefore

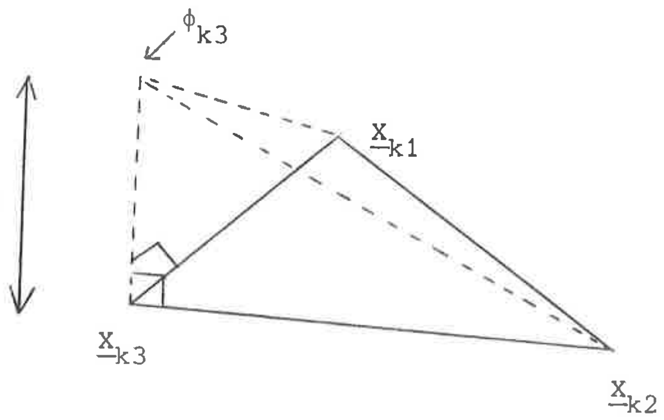
$$\underline{W}_k = \frac{1}{\Delta s_k} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \underline{b}_k . \tag{3.23}$$

We see from (3.23), (3.20) and (3.21) that there are only two types of singularity that occur in the method and that these occur when the area of an element becomes zero, or when the gradients in neighbouring elements are equal (parallelism).

3.3 Local MFE in 2-D

The elementwise formulation of the MFE method can be extended to higher dimensions. In 2-D, consider a triangular element k with nodal positions (X_{k1}, Y_{k1}), (X_{k2}, Y_{k2}) and (X_{k3}, Y_{k3}) and nodal amplitudes U_{k1}, U_{k2} and U_{k3}. Define the element basis functions $\phi_{k\nu}$ $(\nu = 1,2,3)$ to be the linear functions which take a value 1 at node ν and zero at the other two nodes of the element (e.g. fig. 3.5)

Fig. 3.5



As before, we seek to minimise

$$\left\| \sum_{k=1}^{k} \sum_{\nu=1}^{3} W_{k\nu}\phi_{k\nu} - L(U) \right\|_{L_2}^{2} \tag{3.24}$$

over the W's. This gives the system

$$C_k \underline{W}_k = \underline{b}_k \tag{3.25}$$

where

$$C_k = \begin{pmatrix} \langle\phi_{k1},\phi_{k1}\rangle & \langle\phi_{k1},\phi_{k2}\rangle & \langle\phi_{k1},\phi_{k3}\rangle \\ \langle\phi_{k1},\phi_{k2}\rangle & \langle\phi_{k2},\phi_{k2}\rangle & \langle\phi_{k2},\phi_{k3}\rangle \\ \langle\phi_{k1},\phi_{k3}\rangle & \langle\phi_{k2},\phi_{k3}\rangle & \langle\phi_{k3},\phi_{k3}\rangle \end{pmatrix} \tag{3.25a}$$

$$\underline{W}_k = (W_{k1} \quad W_{k2} \quad W_{k3})^T \tag{3.25b}$$

and

$$\underline{b}_k = \begin{pmatrix} \langle \phi_{k1}, L(U) \rangle \\ \langle \phi_{k2}, L(U) \rangle \\ \langle \phi_{k3}, L(U) \rangle \end{pmatrix} \tag{3.25c}$$

We now need to link the $W_{k\nu}$ to the nodal velocities.

In the 2-D global formulation, U_t is expressed in the form

$$U_t = \sum_j \left(\dot{U}_j - \frac{\partial U}{\partial x}\dot{X}_j - \frac{\partial U}{\partial y}\dot{Y}_j\right)\alpha_j \tag{3.26}$$

where α_j is the linear function that takes a value 1 at node j and 0 at all neighbouring nodes, and $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ are constant in each element.

Let $\frac{\partial U}{\partial x} = m_k$ and $\frac{\partial U}{\partial y} = n_k$ in element k.

Then

$$\left\{\dot{U}_j - \frac{\partial U}{\partial x}\dot{X}_j - \frac{\partial U}{\partial y}\dot{Y}_j\right\}\alpha_j = \sum_{k=k_1}^{k_\ell}\left\{\dot{U}_j - m_k\dot{X}_j - n_k\dot{Y}_j\right\}\phi_{k\nu_k} \tag{3.27}$$

where $k_1, \ldots, k_\ell$ are the elements which contain node j and ν_k is the local number of the node in element k which corresponds to node j.

Therefore the relation

$$\sum_j (\dot{U}_j - \frac{\partial U}{\partial x} \dot{X}_j - \frac{\partial U}{\partial y} \dot{Y}_j)\alpha_j = \sum_{k=1}^{k} \sum_{\nu=1}^{3} W_{k\nu} \phi_{k\nu} \tag{3.28}$$

is satisfied if

$$W_{k\nu} = \dot{U}_j - m_k \dot{X}_j - n_k \dot{Y}_j \tag{3.29}$$

where j is the global number of the node corresponding to node ν.

Since, in general, there are more than 3 elements surrounding node j, equations (3.29) define a rectangular system of the form

$$M_j \underline{\dot{y}}_j = \underline{W}_j \tag{3.30}$$

where

$$M_j = \begin{pmatrix} 1 & -m_{k_1} & -n_{k_1} \\ 1 & -m_{k_2} & -n_{k_2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & -m_{k_\ell} & -n_{k_\ell} \end{pmatrix} \tag{3.30a}$$

$$\dot{\underline{y}}_j = (\dot{U}_j \ \dot{X}_j \ \dot{Y}_j)^T \tag{3.30b}$$

and

$$\underline{W}_j = (W_{k_1}\nu_{k_1} \ \cdots\cdots\cdots \ W_{k_\ell}\nu_{k_\ell})^T \tag{3.30c}$$

A least squares minimisation is used to produce a solution to the overdetermined system (3.30). This takes the form

$$M_j^T \ \Delta_j M_j \ \dot{\underline{y}}_j = M^T \ \Delta_j \ \underline{W}_j \tag{3.31}$$

where

$$\Delta_j = \begin{bmatrix} A_{k_1} & & & \bigcirc \\ & \ddots & & \\ & & \ddots & \\ \bigcirc & & & A_{k_\ell} \end{bmatrix} \tag{3.31a}$$

(A_k = area of element k) is introduced to ensure conservation (see Edwards and Baines [3]). We now have a 3 x 3 system to solve in order to transfer the elementwise solution of (3.25) back on to the nodes.

3.4 Extension to a system of 3 equations

Suppose we now have a system of evolutionary partial differential equations in 2-D

$$u_t^{(p)} = L^{(p)} \left(u^{(1)}, u^{(2)}, u^{(3)}\right) \quad , \quad p = 1,2,3 \tag{3.32}$$

If we choose tc solve this system on a common grid, then the essential stages of the method are as before, but with superscripts over relevant quantities.

We minimise

$$\left\| \sum_{k=1}^{K} \sum_{\nu=1}^{3} W_{k\nu}^{(p)} \phi_{k\nu} - L^{(p)} (\underline{U}) \right\|_{L_2}^2 , \quad p = 1,2,3 \tag{3.33}$$

$[\underline{U} = (U^{(1)}, U^{(2)}, U^{(3)})^T]$ over the W's, giving 3 systems of the form

$$C_k \underline{W}_k^{(p)} = \underline{b}^{(p)} \qquad p = 1,2,3 \tag{3.34}$$

where C_k is given by (3.25a) and

$$\underline{W}_k^{(p)} = (W_{k1}^{(p)} \quad W_{k2}^{(p)} \quad W_{k3}^{(p)}) \tag{3.34a}$$

$$\underline{b}_k^{(p)} = \begin{bmatrix} \langle\phi_{k1}, L^{(p)} (\underline{U})\rangle \\ \langle\phi_{k2}, L^{(p)} (\underline{U})\rangle \\ \langle\phi_{k3}, L^{(p)} (\underline{U})\rangle \end{bmatrix} \tag{3.34b}$$

For the $W_{k\nu}^{(p)}$ in terms of the nodal velocities, we now have the relation

$$W_{k\nu}^{(p)} = \dot{U}_j^{(p)} - m_k^{(p)} \dot{X}_j - n_k^{(p)} \dot{Y}_j \tag{3.35}$$

This gives the system

$$M_j \dot{\underline{y}}_j = \underline{W}_j \tag{3.36}$$

where

$$M_j = \begin{pmatrix} 1 & 0 & 0 & -m_{k_1}^{(1)} & -n_{k_1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & -m_{k_\ell}^{(1)} & -n_{k_\ell}^{(1)} \\ 0 & 1 & 0 & -m_{k_1}^{(2)} & -n_{k_1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & -m_{k_\ell}^{(2)} & -n_{k_\ell}^{(2)} \\ 0 & 0 & 1 & -m_{k_1}^{(3)} & -n_{k_1}^{(3)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -m_{k_\ell}^{(3)} & -n_{k_\ell}^{(3)} \end{pmatrix} \tag{3.36a}$$

$$\dot{\underline{y}}_j = (\dot{U}_j^{(1)}, \dot{U}_j^{(2)}, \dot{U}_j^{(3)}, \dot{X}_j, \dot{Y}_j)^T \tag{3.36b}$$

and $$\underline{W}_j = (W^{(1)}_{k_1 \nu_{k_1}} \ldots W^{(1)}_{k_\ell \nu_{k_\ell}} \quad W^{(2)}_{k_1 \nu_{k_1}} \ldots W^{(2)}_{k_\ell \nu_{k_\ell}} \quad W^{(3)}_{k_1 \nu_{k_1}} \ldots W^{(3)}_{k_\ell \nu_{k_\ell}}) \tag{3.36c}$$

The least squares minimisation now takes the form

$$M_j^T \Delta_j M_j \dot{\underline{y}}_j = M_j^T \Delta_j \underline{W}_j \tag{3.37}$$

where

$$\Delta_j = \text{diag}\,\{A_{k_1}, \ldots, A_{k_\ell}, A_{k_1}, \ldots, A_{k_\ell}, A_{k_1}, \ldots, A_{k_\ell}\} \tag{3.37a}$$

This produces the system

$$\begin{pmatrix} d & 0 & 0 & a_1 & b_1 \\ 0 & d & 0 & a_2 & b_2 \\ 0 & 0 & d & a_3 & b_3 \\ a_1 & a_2 & a_3 & a_4 & b_4 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} \dot{\underline{y}}_j = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} \tag{3.38}$$

where

$$d = \sum_{k=k_1}^{k_\ell} A_k \tag{3.38a}$$

$$a_p = -\sum_{k=k_1}^{k_\ell} m_k^{(p)} A_k \qquad p = 1,2,3 \tag{3.38b}$$

$$b_p = -\sum_{k=k_1}^{k_\ell} n_k^{(p)} A_k \qquad p = 1,2,3 \tag{3.38c}$$

$$a_4 = \sum_{k=k_1}^{k_\ell} \sum_{p=1}^{3} m_k^{(p)^2} A_k \tag{3.38d}$$

$$b_4 = \sum_{k=k_1}^{k_\ell} \sum_{p=1}^{3} m_k^{(p)} n_k^{(p)} A_k \qquad (3.38e)$$

$$b_5 = \sum_{k=k_1}^{k_\ell} \sum_{p=1}^{3} n_k^{(p)^2} A_k \qquad (3.38f)$$

$$r_p = \sum_{k=k_1}^{k_\ell} W_{k\nu_k}^{(p)} A_k \qquad p = 1,2,3 \qquad (3.38g)$$

$$r_4 = - \sum_{k=k_1}^{k_\ell} \sum_{p=1}^{3} m_k^{(p)} W_{k\nu_k}^{(p)} A_k \qquad (3.38h)$$

and

$$r_5 = - \sum_{k=k_1}^{k_\ell} \sum_{p=1}^{3} n_k^{(p)} W_{k\nu_k}^{(p)} A_k \; . \qquad (3.38i)$$

3.5 Local MEM in 2-D

In the Mobile Element Method, we seek instead to minimise the L_2 norm using the mobile operator

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \dot{x} \frac{\partial u}{\partial x} + \dot{y} \frac{\partial u}{\partial y} \qquad (3.39)$$

in order to obtain a grid velocity with the best possible L_2 stability (see Edwards and Baines [3]).

So, given the equation

$$u_t = L(u) \tag{3.40}$$

we minimise

$$\left\| \left(\dot{x}\frac{\partial u}{\partial x} + \dot{y}\frac{\partial u}{\partial y}\right) + L(u) \right\|_{L_2} \tag{3.41}$$

Using the global MFE discretisation, (3.41) can be written as

$$\left\| \left(\sum_j \dot{X}_j\beta_j + \sum_j \dot{Y}_j\gamma_j\right) - L(u) \right\|_{L_2} \quad . \tag{3.42}$$

In the global approach to MEM, the square of (3.42) would be minimised over the $\dot{X}_j$ and $\dot{Y}_j$. In the local formulation, we write

$$\sum_j \dot{X}_j\beta_j + \sum_j \dot{Y}_j\gamma_j = \sum_{k=1}^{k} \sum_{\nu=1}^{3} W_{k\nu}\phi_{k\nu} \tag{3.43}$$

where the $W_{k\nu}$ satisfy

$$W_{k\nu} = -(m_k\dot{X}_j + n_k\dot{Y}_j) . \tag{3.44}$$

As in section 3.3, we minimise over the W's to give

$$C_k\underline{W}_k = \underline{b}_k \tag{3.45}$$

where C_k, $\underline{W}_k$ and $\underline{b}_k$ are given by (3.25a), (3.25b) and (3.25c) respectively. Equations (3.44) define the rectangular system

$$M_j \underline{\dot{X}}_j = \underline{W}_j \qquad (3.46)$$

where, in the notation of section 3.3,

$$M_j = \begin{pmatrix} -m_{k_1} & -n_{k_1} \\ -m_{k_2} & -n_{k_2} \\ \vdots & \vdots \\ -m_{k_\ell} & -n_{k_\ell} \end{pmatrix} \qquad (3.46a)$$

$$\underline{\dot{X}}_j = (\dot{X}_j \; . \; \dot{Y}_j)^T \qquad (3.46b)$$

and $\underline{W}_j$ is given by (3.36c).

The system (3.46) is solved in the form

$$M_j^T \Delta_j M_j \underline{\dot{X}}_j = M^T \Delta_j \underline{W}_j \qquad (3.47)$$

where Δ_j is given by (3.31a). This gives a 2 x 2 system for the nodal velocities $\dot{X}_j$ and $\dot{Y}_j$.

It remains to find the rate of change of nodal amplitude $\dot{U}_j$. We return to the residual of the full equation

$$\| \sum_j (\dot{U}_j - \dot{X}_j \frac{\partial u}{\partial x} - \dot{Y}_j \frac{\partial u}{\partial y})\alpha_j - L(U) \|_{L_2}$$

$$= \| \sum_k \sum_\nu W^*_{k\nu} \phi_{k\nu} - L(U) \|_{L_2} \tag{3.48}$$

where $\dot{X}_j$ and $\dot{Y}_j$ are known from (3.47) and

$$W^*_{k\nu} = \dot{U}_j - (m_k \dot{X}_j + n_k \dot{Y}_j). \tag{3.49}$$

The norm on the right hand side of (3.48) is minimised over the $W^*_{k\nu}$, and so the $W^*_{k\nu}$ correspond to the $W_{k\nu}$ of (3.45).

We now minimise the norm

$$\| \Delta_j^{\frac{1}{2}} \{ \underline{L}_j \dot{U}_j - (\underline{W}_j + \underline{M}_j \dot{X}_j + \underline{N}_j \dot{Y}_j)\} \|_{l_2} \tag{3.50}$$

over the $\dot{U}_j$, where $\underline{L}_j$ is a vector of 1's with length equal to the number of elements surrounding node j, $\underline{W}_j$ is given by (3.36c) and $\underline{M}_j$ and $\underline{N}_j$ are the columns of M_j.

This process of finding $\dot{U}_j$ is equivalent to substituting the values of $\dot{X}_j$ and $\dot{Y}_j$ from (3.47) into the equation for $\dot{U}_j$ arising from the local MFE system (3.31) for node j.

3.6 Local MEM for 2-D Systems

As in the MFE case, the stages in the formulation are the same as for a single PDE, but with superscripts over appropriate quantities.

For a system of 3 PDE's, we look to minimise the L_2 norms using the mobile operators

$$\frac{Du^{(p)}}{Dt} = \frac{\partial u^{(p)}}{\partial t} + \dot{x}\,\frac{\partial u^{(p)}}{\partial x} + \dot{y}\,\frac{\partial u^{(p)}}{\partial y} \qquad p = 1,2,3 \tag{3.51}$$

This demands the solution of the systems

$$C_k\,\underline{W}_k^{(p)} = \underline{b}_k^{(p)} \qquad p = 1,2,3 \tag{3.52}$$

where C_k, $\underline{W}_k^{(p)}$ and $\underline{b}_k^{(p)}$ are given by (3.25a), (3.34a) and (3.34b) respectively.

The $W_{k\nu}^{(p)}$ satisfy

$$W_{k\nu}^{(p)} = -\,(m_k^{(p)}\dot{X}_j + n_k^{(p)}\dot{Y}_j) \tag{3.53}$$

yielding the system

$$M_j \underline{X}_j = \underline{W}_j \tag{3.54}$$

where

$$M_j = \begin{bmatrix} -m_{k_1}^{(1)} & -n_{k_1}^{(1)} \\ & \\ -m_{k_\ell}^{(1)} & -n_{k_\ell}^{(1)} \\ -m_{k_1}^{(2)} & -n_{k_1}^{(2)} \\ & \\ -m_{k_\ell}^{(2)} & -n_{k_\ell}^{(2)} \\ -m_{k_1}^{(3)} & -n_{k_1}^{(3)} \\ & \\ -m_{k_\ell}^{(3)} & -n_{k_\ell}^{(3)} \end{bmatrix} \tag{3.54a}$$

and $\dot{\underline{X}}_j$ and $\dot{\underline{W}}_j$ are given by (3.46b) and (3.36c) respectively.

The system (3.54) is solved in a least squares manner in the form

$$M_j^T \Delta_j M_j \dot{\underline{X}}_j = M_j^T \Delta_j \underline{W}_j \tag{3.55}$$

where Δ_j is given by (3.37a). This gives the system

$$\begin{pmatrix} a_4 & b_4 \\ b_4 & b_5 \end{pmatrix} \begin{pmatrix} \dot{X}_j \\ \dot{Y}_j \end{pmatrix} = \begin{pmatrix} r_4 \\ r_5 \end{pmatrix} \qquad (3.56)$$

where a_4, b_4 and b_5 are given by (3.38d), (3.38e) and (3.38f) respectively.

The process for finding the $\dot{U}_j^{(p)}$ is equivalent to substituting the solution of (3.56) into the $\dot{U}_j^{(1)}$, $\dot{U}_j^{(2)}$ and $\dot{U}_j^{(3)}$ equations of the local MFE system (3.38).

Consequently, to produce an MEM solution at node j we must solve the system

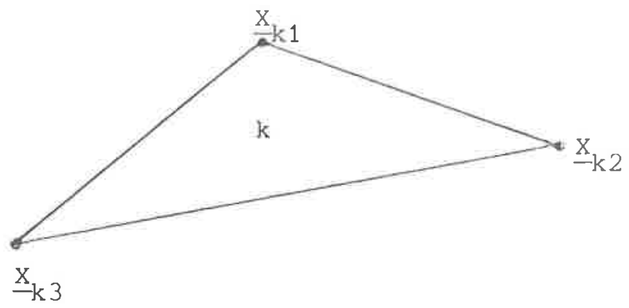
$$\begin{pmatrix} d & 0 & 0 & a_1 & b_1 \\ 0 & d & 0 & a_2 & b_2 \\ 0 & 0 & d & a_3 & b_3 \\ 0 & 0 & 0 & a_4 & b_4 \\ 0 & 0 & 0 & b_4 & b_5 \end{pmatrix} \dot{\underline{y}}_j = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} \qquad (3.57)$$

where $\dot{\underline{y}}_j$ is given by (3.36b) and all other quantities are as defined in section 3.4.

3.7 Treatment of the Inner Products

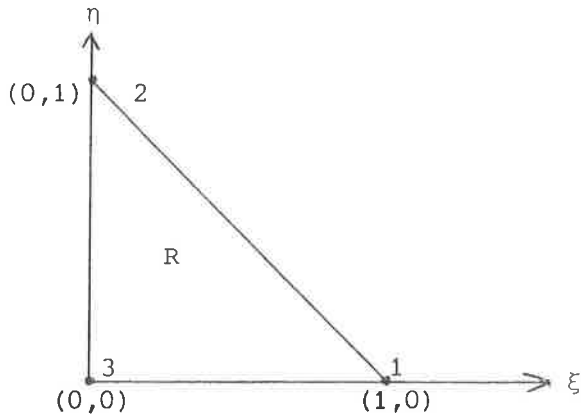
We take a typical element k (fig 3.6)

Fig. 3.6



and map onto a reference element R in (ξ,η) space (fig. 3.7),

Fig. 3.7



thus enabling integrals to be calculated on a standard region.

The transformation is

$$\underline{x} = \xi \underline{x}_{k1} + \eta \underline{x}_{k2} + (1 - \xi - \eta)\, \underline{x}_{k3} \tag{3.58}$$

and the basis functions ζ_i and element R are

$$\zeta_1 = \xi, \quad \zeta_2 = \eta, \quad \zeta_3 = 1 - \xi - \eta \tag{3.58a}$$

With this transformation, we have the result

$$\iint_k f(x,y)\ dxdy = \iint_R F\ (\xi,\eta)\ \frac{\partial \underline{x}}{\partial \underline{\xi}}\ d\xi\ \ d\eta \tag{3.59}$$

where

$$f\ (x,y) = F\ (\xi,\eta)$$

and $\frac{\partial \underline{x}}{\partial \underline{\xi}}$ is the Jacobian, J_k, of the transformation

from k to R, given by

$$J_k = \det \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \tag{3.60}$$

Using (3.58), we have

$$J_k = (X_{k1} - X_{k3})(Y_{k2} - Y_{k3}) - (X_{k2} - X_{k3})(Y_{k1} - Y_{k3}) \tag{3.61}$$

and so J_k is constant in each element. Therefore, (3.59) implies that

$$\iint_k f(x,y)\ dxdy = J_k \iint_R F(\xi,\eta)\ d\xi\ d\eta \tag{3.62}$$

This result enables us to evaluate the inner products in the matrix C_k analytically via the ζ basis fuctions. We obtain the matrix

$$C_k = \frac{J_k}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \quad (3.63)$$

By setting $F(x,y) = F(\xi,\eta) = 1$ we obtain the result

$$A_k = \frac{J_k}{2}. \quad (3.64)$$

The integrals in $\underline{b}_k$ cannot generally be expressed in closed form. We therefore employ 7 point Gauss Quadrature on the region R and then transorm to global variables using (3.62).

3.8 Element Gradients

Again these are evaluated using the reference element R. For a function U, taking the values U_{k1}, U_{k2} and U_{k3} at the nodes of element k, the transformation is

$$U = \xi U_{k1} + \eta U_{k2} + (1 - \xi - \eta)U_{k3} \quad . \quad (3.65)$$

We have the chain rule

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} \quad . \tag{3.66}$$

Using the identity

$$\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = I_2 \tag{3.67}$$

we have

$$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = \frac{1}{J_k}\begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix} \tag{3.68}$$

Substituting in (3.66) gives

$$\frac{\partial U}{\partial x} = \frac{1}{J_k}\left\{\frac{\partial U}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial U}{\partial \eta}\frac{\partial y}{\partial \xi}\right\} . \tag{3.69}$$

Using (3.58) and (3.65), and denoting $\frac{\partial U}{\partial x}$ in the k^{th} element as m_k, we have

$$m_k = \frac{1}{J_k}\left[(U_{k1} - U_{k3})(Y_{k2} - Y_{k3}) - (U_{k2} - U_{k3})(Y_{k1} - Y_{k3})\right] . \tag{3.70}$$

It can similarly be shown that, in element k,

$$\frac{\partial U}{\partial y} = n_k = -\frac{1}{J_k}\left[(U_{k1} - U_{k3})(X_{k2} - X_{k3}) - (U_{k2} - U_{k3})(X_{k1} - X_{k3})\right]. \quad (3.71)$$

3.7 Element Folding and Parallelism

Clearly, the local MFE (or MEM) process for the solution of a system of 3 equations in 2-D will break down if the matrix C_k becomes singular for any element k or if the 5×5 matrix in the system to determine the speeds for node j becomes singular for any j.

It can be shown that

$$C_k^{-1} = \frac{1}{J_k} \begin{pmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{pmatrix} . \quad (3.72)$$

Consequently, C_k will only become singular if the element Jacobian becomes zero and, from (3.64), we see that this corresponds to the element k having zero area.

We have, at time level $n + 1$

$$\left.\begin{aligned} X_{k\nu}^{n+1} &= X_{k\nu}^{n} + \Delta t\ \dot{X}_{k\nu}^{n} \\ & \qquad\qquad\qquad\qquad = 1,2,3 \\ Y_{k\nu}^{n+1} &= Y_{k\nu}^{n} + \Delta t\ \dot{Y}_{k\nu}^{n} \end{aligned}\right\} \tag{3.73}$$

Substituting (3.73) into (3.61) gives an expression for J_k^{n+1} which is quadratic in Δt. Evaluation of the roots of this quadratic gives the timestep Δt_{FOLD} for which J^{n+1} would equal zero. In order to avoid the singularity of C_k, we must take a timestep

$$\Delta t = \theta\ \Delta t_{FOLD} \tag{3.74}$$

where $0<\theta<1$ (typically, we take a value $\theta = 0.5$).

The local MFE system (3.38) may be written as

$$\begin{pmatrix} D & P \\ P^T & Q \end{pmatrix} \begin{pmatrix} \underline{U}_j \\ \underline{X}_j \end{pmatrix} = \begin{pmatrix} \underline{R}_1 \\ \underline{R}_2 \end{pmatrix} \tag{3.75}$$

where

$$D = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \tag{3.75a}$$

$$P = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \tag{3.75b}$$

$$Q = \begin{pmatrix} a_4 & b_4 \\ b_4 & b_5 \end{pmatrix} \tag{3.75c}$$

$$\dot{\underline{U}}_j = (\dot{U}_j^{(1)} \ \dot{U}_j^{(2)} \ \dot{U}_j^{(3)})^T \tag{3.75d}$$

$$\dot{\underline{X}}_j = (\dot{X}_j \ \dot{Y}_j)^T \tag{3.75e}$$

$$\underline{R}_1 = (r_1 \ r_2 \ r_3)^T \tag{3.75f}$$

and

$$\underline{R}_2 = (r_4 \ r_5)^T \tag{3.75g}$$

With this partitioning, we have

$$\dot{\underline{U}}_j = D^{-1} \underline{R}_1 - D^{-1} P \dot{\underline{X}}_j \tag{3.76}$$

and

$$Q\dot{\underline{X}}_j = \underline{R}_2 - P^T \underline{U}_j. \tag{3.77}$$

Substitution of (3.76) into (3.77) gives

$$(Q - P^T D^{-1} P)\underline{X}_j = \underline{R}_2 - P^T D^{-1} \underline{R}_1. \tag{3.78}$$

Any possible singularity of the matrix D is prevented by the restriction (3.74) which ensures that element areas are non-zero.

Therefore, the 5x5 system will always have a solution unless the 2x2 matrix $(Q - P^T D^{-1} P)$ is singular.

Now

$$(Q - P^T D^{-1} D) = \begin{pmatrix} a_4 - \frac{1}{d} \sum_{i=1}^{3} a_i^2 & b_4 - \frac{1}{d} \sum_{i=1}^{3} a_i b_i \\ b_4 - \frac{1}{d} \sum_{i=1}^{3} a_i b_i & b_5 - \frac{1}{d} \sum_{i=1}^{3} b_i^2 \end{pmatrix} \quad (3.79)$$

$$= \begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix} = Q^*, \text{ say.}$$

In the **MFE** program, we test for singularity by checking if the eigenvalues λ_1, λ_2 of Q^* are widely spaced in modulus.

It is easily shown that

$$\left| \frac{\lambda_1}{\lambda_2} \right| = \left| \frac{1 + \sqrt{1 - 4\sigma}}{1 - \sqrt{1 - 4\sigma}} \right| \quad (3.80)$$

where

$$\sigma = \frac{\det Q^*}{(q_1 + q_3)^2} . \quad (3.81)$$

Consequently, we check to see whether the quantity $|\sigma|$ is small. (In the event that $q_1 + q_3 = 0$, we test the magnitude of det (Q^*)).

If this is so then we adopt the values of $\dot{X}_j$ and $\dot{Y}_j$ that were calculated at the previous timestep and insert these into (3.76) to determine $\underline{\dot{U}}_j$.

For MEM, the test is the same, but now there is no matrix P^T in the partitioning of the system and so $Q^* = Q$.

4. MFE/MEM Method for The Model Problem

4.1 Discretisation of equations

Denote U, V, Z as $U^{(1)}$, $U^{(2)}$, $U^{(3)}$ respectively (see §7.1) With this notation, and using the notation in section 3.4, the right hand sides $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ of equations (2.1), (2.2) and (2.3) become, in element k,

$$L^{(1)} = - U^{(1)} m_k^{(1)} - U^{(2)} n_k^{(1)} - g m_k^{(3)} - \frac{F U^{(1)} \sqrt{{U^{(1)}}^2 + {U^{(2)}}^2}}{h + U^{(3)}} + \Omega U^{(2)} \qquad (4.1)$$

$$L^{(2)} = - U^{(1)} m_k^{(2)} - U^{(2)} n_k^{(2)} - g n_k^{(3)} - \frac{F U^{(2)} \sqrt{{U^{(1)}}^2 + {U^{(2)}}^2}}{h + U^{(3)}} - \Omega U^{(1)} \qquad (4.2)$$

$$L^{(3)} = -U^{(1)}(h_x + m_k^{(3)}) - U^{(2)}(h_y + n_k^{(3)}) - (h + U^{(3)})(m_k^{(1)} + n_k^{(2)}). \quad (4.3)$$

These expressions are used in the evaluation of $<\phi_{k\nu}, L^{(i)}>$ via 7 point Gauss, with function values $U^{(i)}$ given by (3.65) with the appropriate superscripts and values of ξ and η that give the sample points. The $m_k^{(i)}$ and $n_k^{(i)}$ are given by (3.70) and (3.71) with the appropriate superscripts.

It should be noted that for the purposes of this work we take a constant value for the bed depth $h(=20\text{m})$, and so $h_x = h_y = 0$. The program does, however, allow for the introduction of a variable h.

4.2 Initial Conditions

We take as initial data an exact solution of the system

$$U_t = -gZ_x \quad (4.4)$$

$$V_t = -gZ_y \quad (4.5)$$

$$Z_t = -h(U_x + V_y) \quad (4.6)$$

which is the wave equation in 2-D.

We expect that the problem described in section 2.2 will be dominated by motion in the y-direction and so we use the solution given by

$$U = 0 \tag{4.7}$$

$$V = \frac{Ag}{\sqrt{gh}} \sin \left\{ \frac{2\pi}{P} \left(\frac{y}{\sqrt{gh}} - t \right) \right\} \tag{4.8}$$

$$Z = A \sin \left\{ \frac{2\pi}{P} \left(\frac{y}{\sqrt{gh}} - t \right) \right\} \tag{4.9}$$

where

$A \equiv$ tidal amplitude (1M)

$P \equiv$ tidal period (12.42 hrs)

and $\sqrt{gh}$ is the gravity wave speed which we take to be a constant (14M/S).

4.3 Boundary Conditions/Treatment

We look to impose a constraint on the movement of boundary nodes in order to ensure that the finite element grid moves with the flow.

One possible constraint is to assign a value to the normal velocity $\dot{N}_j$ of a boundary node j, given by

$$\dot{N}_j = \dot{X}_j \cos\theta_j + \dot{Y}_j \sin\theta_j \qquad (4.10)$$

where θ_j is the average of the angles that the portions of boundary on either side of node j make with the X-axis.

This constraint can be imposed in the form

$$\begin{aligned} \dot{N}_j &= U\cos\theta_j + V\sin\theta_j \\ &= U^{(1)}\cos\theta_j + U^{(2)}\sin\theta_j , \end{aligned} \qquad (4.11)$$

that is the normal velocity of the boundary point j is equal to the normal velocity of the flow at that point. However, when this treatment was used in the MFE/MEM program, it was found that several of the boundary nodes picked up spurious values of $\dot{Y}_j$ of orders of magnitude many times greater than V $(= U_j^{(2)})$. This was due to spurious velocities being generated in the tangential directions.

Consequently, a more rigourous constraint is adopted whereby the nodal velocities $\dot{S}_j$, $\dot{Y}_j$ of a boundary node are overwritten as $U_j^{(1)}$, $U_j^{(2)}$ respectively, and we set $\dot{X}_j = 0$ for all nodes on the left hand boundary, in order to preserve the straight coastline.

We also specify the rate of change of elevation $\dot{U}_j^{(3)}$

for a boundary node j, which is taken to be the value of Z_t given by the solution (4.9).

5. Results

The methods outlined in sections 3 and 4 have been programmed in FORTRAN and run on the NORD system at Reading.

The program sets up an initial triangular mesh on the region (XW,XE) x (YS,YN), with NX nodes along (XW,XE) and NY nodes along (YS,YN). In all the examples presented here we use values of XW = 0, XE = 10000 M, YS = 0, YN = 50000 M and NX = 5, NY = 25. This generates a mesh of 113 moving nodes and 168 elements, as in Fig. 5.1. We use initial data given by (4.7), (4.8) and (4.9). The initial functions for V and Z are shown in Figs. 5.2 and 5.3.

Fig. 5.4 shows the breakdown of the MFE method for the system of shallow water equations. It can be seen that some of the elements in the mesh have become stretched or compressed and as a result the timestep (chosen to prevent element folding) has become truncated to a value less than a specified tolerance (10^{-6}) and the solution process has ceased. Figs. 5.5, 5.6 and 5.7 display U,V and Z at the breakdown time. It is noticeable that the functions V and Z have not deformed much from their initial shape of elongated sine curves. However, the elelevation Z is lower over the whole mesh, which accounts for the expansion of the region since the amount of water enclosed should remain the same. The most striking feature of the graphs is that the horizontal velocity U, initially zero everywhere, now resembles V and Z, but is an order of magnitude lower over the region. Numerical output for this example reveals that large velocities are generated for some of the internal nodes, thus causing the folding.

Figs. 5.8 - 5.11 show that the MEM method fares no better for the same problem. Although it runs for more timesteps (66 as compared to 47), the breakdown time is shorter (114 secs as compared to 472 secs). Once again, the quantity U seems to have picked up a sine wave and the nature of the grid distortion is similar to the MFE case.

In order to assess the accuracy and capabilities of both methods, the program was used to solve the wave equation given by the system (4.4), (4.5) and (4.6).

Figs. 5.12 - 5.15 display the MEM solution to this system. It is apparent from Fig. 5.12 that the grid collapses, after just 11 iterations, in what appears to be a perfectly symmetrical way. The exact solution indicates that U should be zero everywhere and Fig. 5.13 shows that the MEM solution gives $|U|<1.4 \times 10^{-4}$ at the nodes. Comparison of numerical output for V and Z with the solutions (4.8) and (4.9) reveals 3 decimal place accuracy at all the nodes. However, scrutiny of the nodal speeds reveals that 46 of the 80 internal nodes have a vertical speed of either 14.0 M/S or 14.1 M/S. This indicates that motion is being influenced by gravity waves, which is consistent with the MEM propery of following dominant waves. Therefore, it is inevitable that the mesh will collapse, since nodes on the upper boundary are moving with speed V.

The same problems are encountered when solving the wave equation using MFE (Figs. 5.16 - 5.19). Fig. 5.16 shows that the mesh collapses in a similar way, although the method does run for several more timesteps than MEM and has a longer breakdown time. The solutions obtained for V and Z are just as accurate as MEM, but the maximum and minimum values of U have increased in modulus by an order of magnitude.

The nodes on the boundary, whose velocities are overwritten into the MFE/MEM system, pick up accurate nodal values of U,V and Z in the above examples of the wave equation. In view of this it was attempted to produce a solution of the wave equation with all internal horizontal and vertical nodal speeds prescribed as the values of U and V at the node. This solution is presented in Figs. 5.20 - 5.23 and referred to as MEM, although the systems for MFE and MEM are the same when all the speeds are specified.

Fig. 5.20 shows that the mesh is not discernably distorted after 100 timesteps. Indeed, the maximum time step of 60 seconds has been taken throughout. However, Figs. 5.21 - 5.23 display instability which is probably caused by the fact that we are forcing nodes into the wrong positions.

This instability is also apparent in the solution of the shallow water equations with prescribed speeds (Figs. 5.24 - 5.27).

6. Conclusions

In this work we have solved the 2-D system of shallow water equations using the local formulations of the Moving Finite Element method and the Mobile Element Method.

However, using the schemes presented in section 3, it is impossible to produce a solution to the system for any significant amount of time. This was due to the tendency of freely moving internal nodes to move with speeds much greater than nodes on or near the boundary, thus causing the grid to collpase.

A solution of the wave equation by these methods indicated that a probable reason for the high nodal speeds in the y-direction is that the nodes are following gravity waves.

In future work, a variant of the above methods may be developed in which the nodes are not mainly influenced by the speed of the dominant wave. It is also possible that the spurious velocities arising from other sources may be countered by the introduction of regularisation terms or implicitness into the scheme.

Acknowledgements

I would like to thank Dr M.J. Baines and Dr C.P. Please of the University of Reading and Mr R. Preston and Mr N. Edwards of the CERL for their help and assistance during the course of this project.

I acknowledge the financial support of the CEGB.

Appendix (A. Priestley)

If, after solving the system (3.57) in the MEM approach we over-write $\dot{x} = u$, $\dot{y} = v$, the solution proceeds to about 3,700 seconds. However, an improvement is obtained by putting $\dot{x} = u$, $\dot{y} = v$ before solving (3.57) so that $\dot{u}, \dot{v}, \dot{z}$ is the solution of

$$\begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} r_1 - a_1 u - b_1 v \\ r_2 - a_2 u - b_2 v \\ r_3 - a_3 u - b_3 v \end{bmatrix} \tag{A1}$$

In that case the solution continues until about 9000 seconds before breaking down.

References

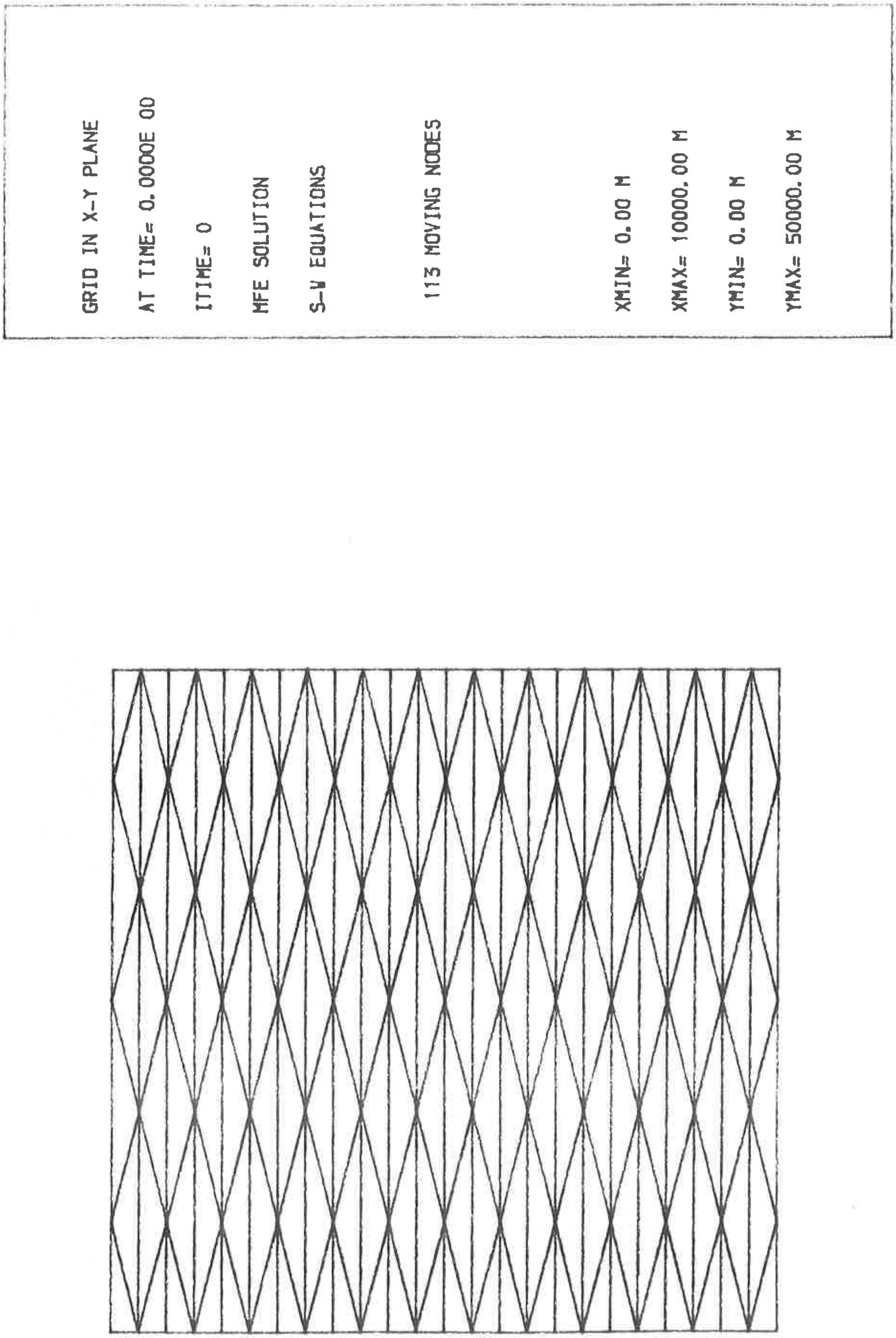
[1] K. Miller, SIAM J. Numer. Anal. 18, (1981), 1033.

[2] D.R. Lynch, J. Comput.Phys., 47, (1982), 387-411.

[3] M.J. Baines, Numerical Analysis Report 9/85, Department of Mathematics, University of Reading.

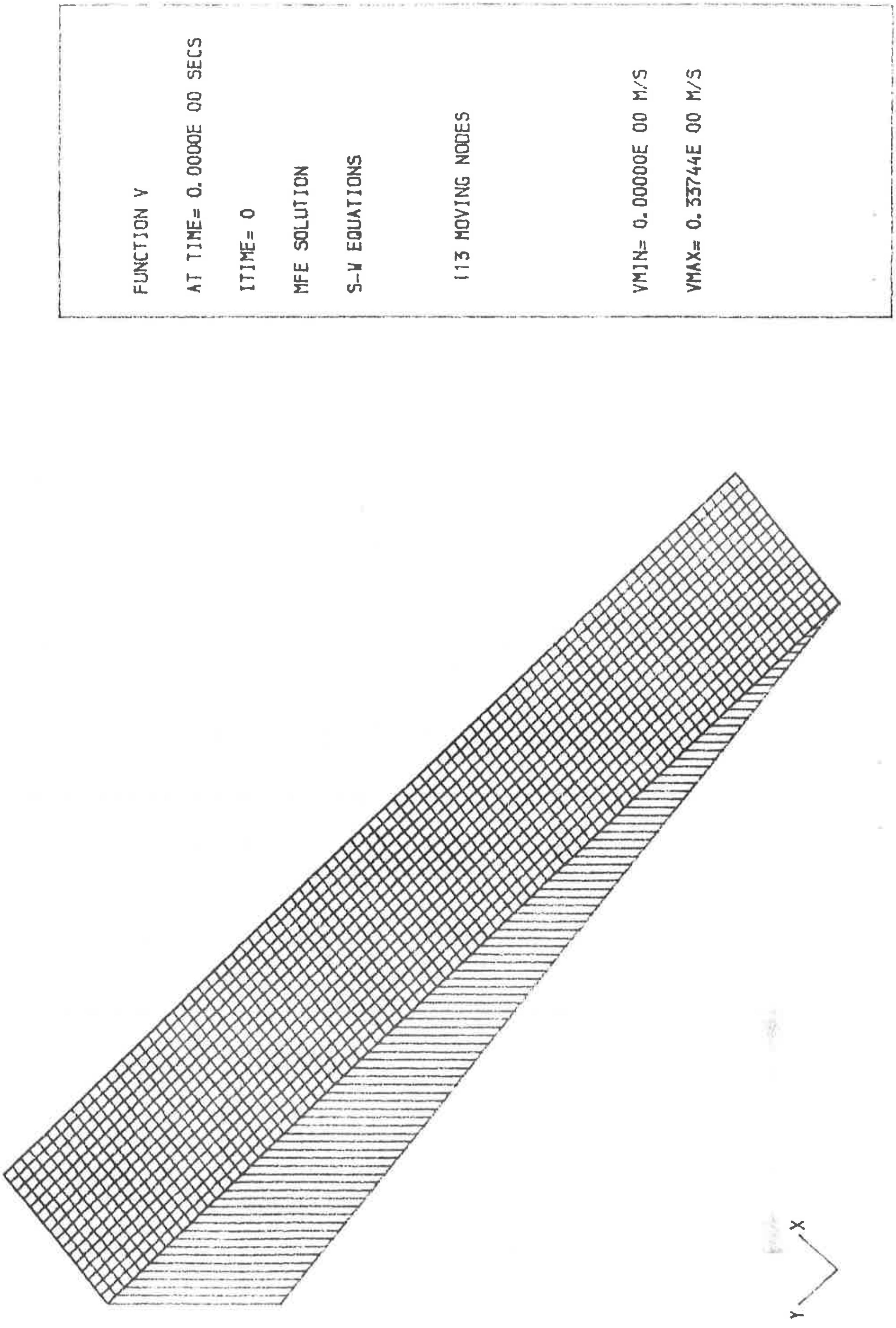
[4] M.G. Edwards and M.J. Baines, Dept. of Mathematics, University of Reading (submitted to J. Comput. Phys.).

Fig. 5.1



GRID IN X-Y PLANE
AT TIME= 0.0000E 00
ITIME= 0
MFE SOLUTION
S-W EQUATIONS
113 MOVING NODES
XMIN= 0.00 M
XMAX= 10000.00 M
YMIN= 0.00 M
YMAX= 50000.00 M

Fig. 5.2



FUNCTION V
AT TIME= 0.0000E 00 SECS
ITIME= 0
MFE SOLUTION
S-W EQUATIONS
113 MOVING NODES
VMIN= 0.00000E 00 M/S
VMAX= 0.33744E 00 M/S
X
Y

Fig. 5.3

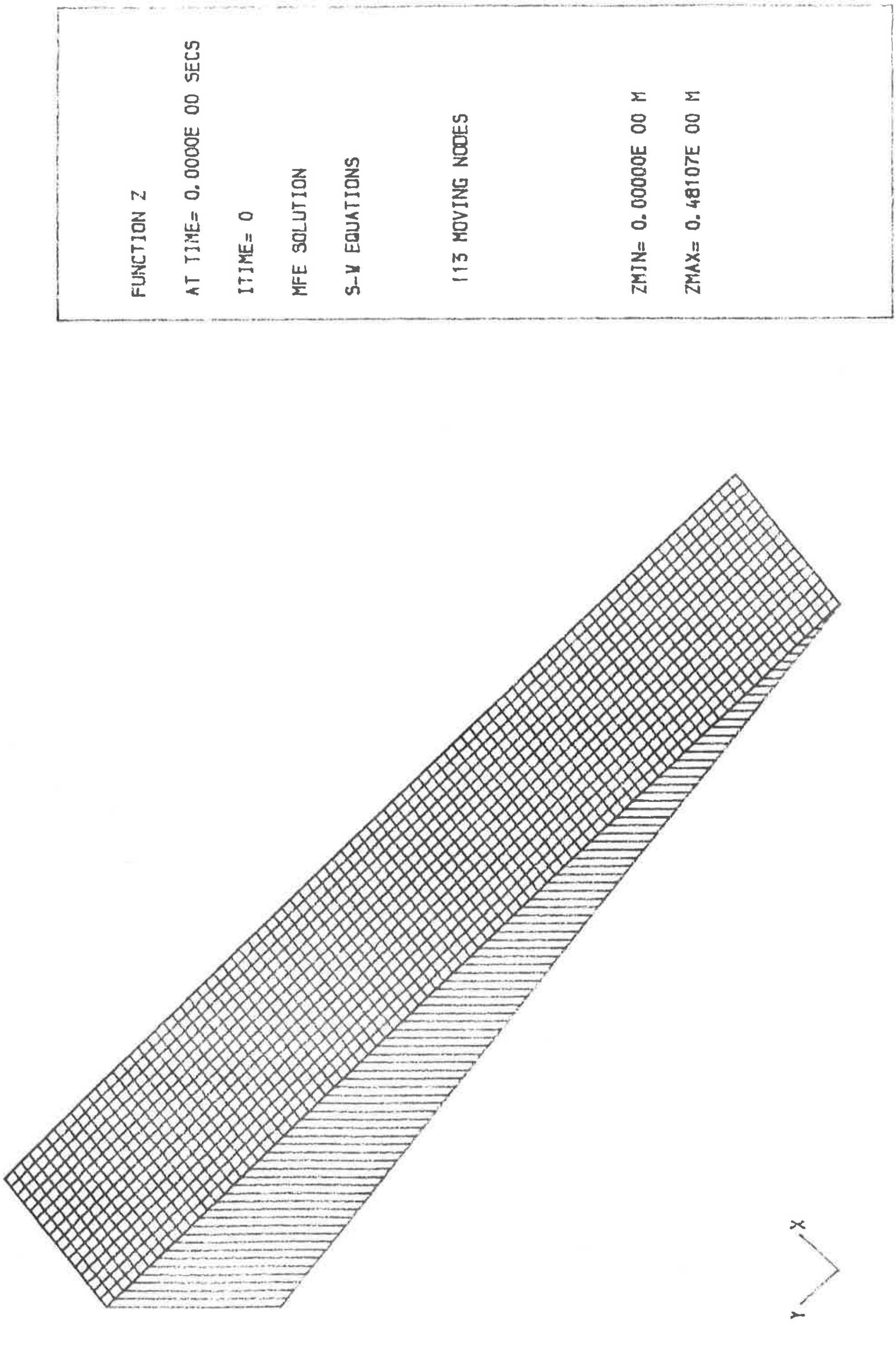

FUNCTION Z
AT TIME= 0.0000E 00 SECS
ITIME= 0
MFE SOLUTION
S-W EQUATIONS
113 MOVING NODES
ZMIN= 0.00000E 00 M
ZMAX= 0.48107E 00 M
X
Y

Fig. 5.4

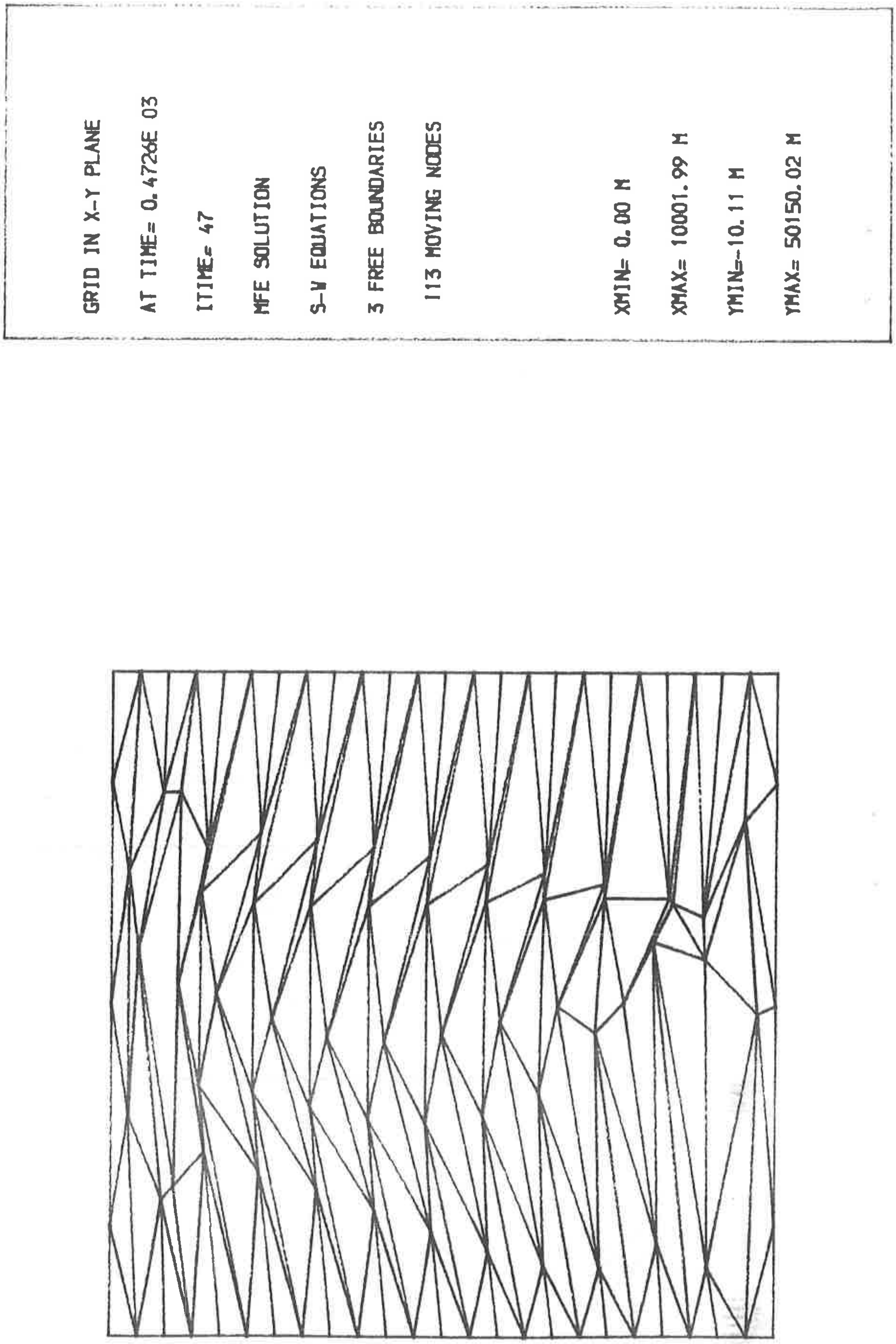

GRID IN X-Y PLANE
AT TIME= 0.4726E 03
ITIME= 47
MFE SOLUTION
S-W EQUATIONS
3 FREE BOUNDARIES
113 MOVING NODES
XMIN= 0.00 M
XMAX= 10001.99 M
YMIN=-10.11 M
YMAX= 50150.02 M

Fig. 5.5

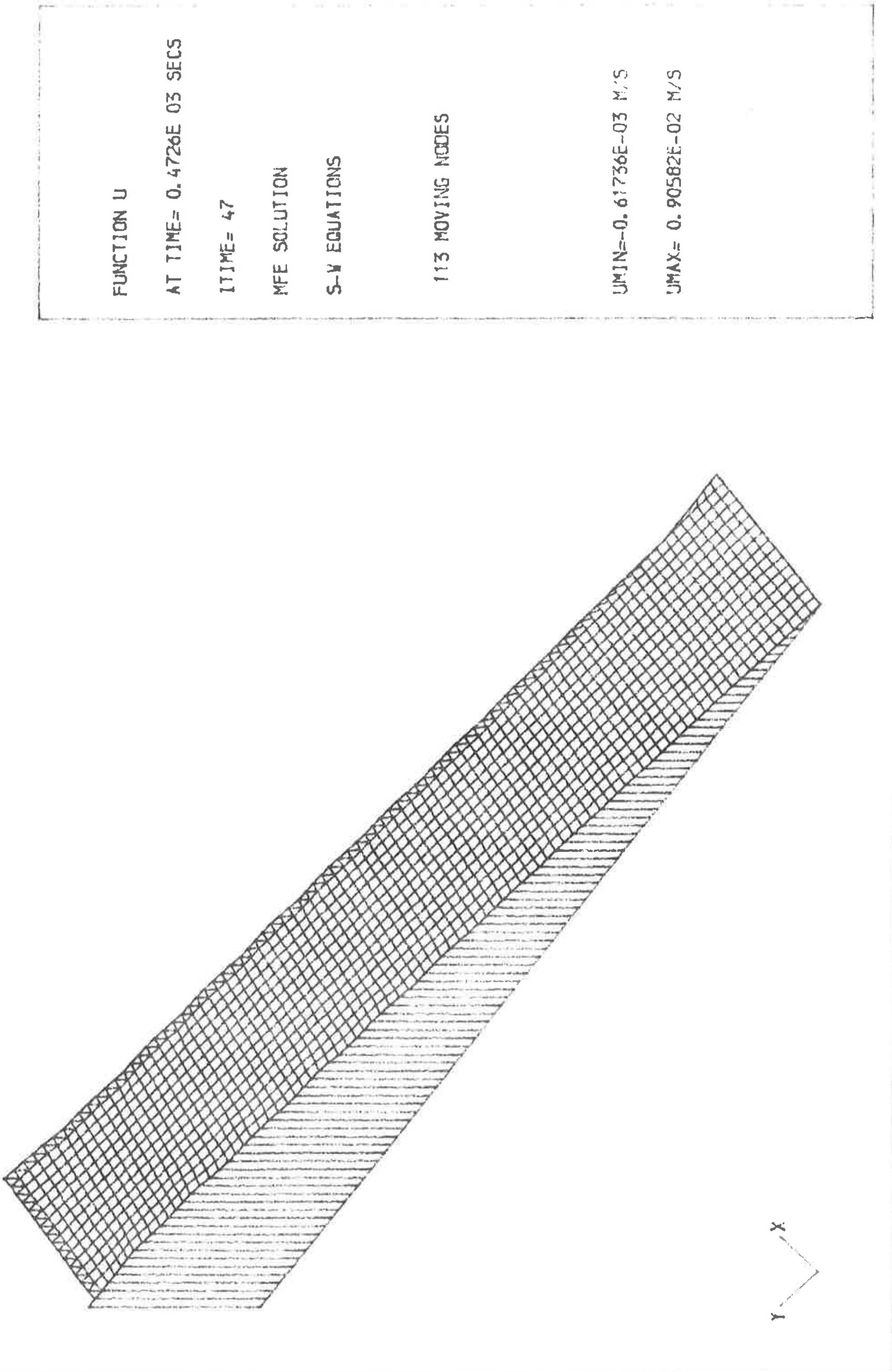
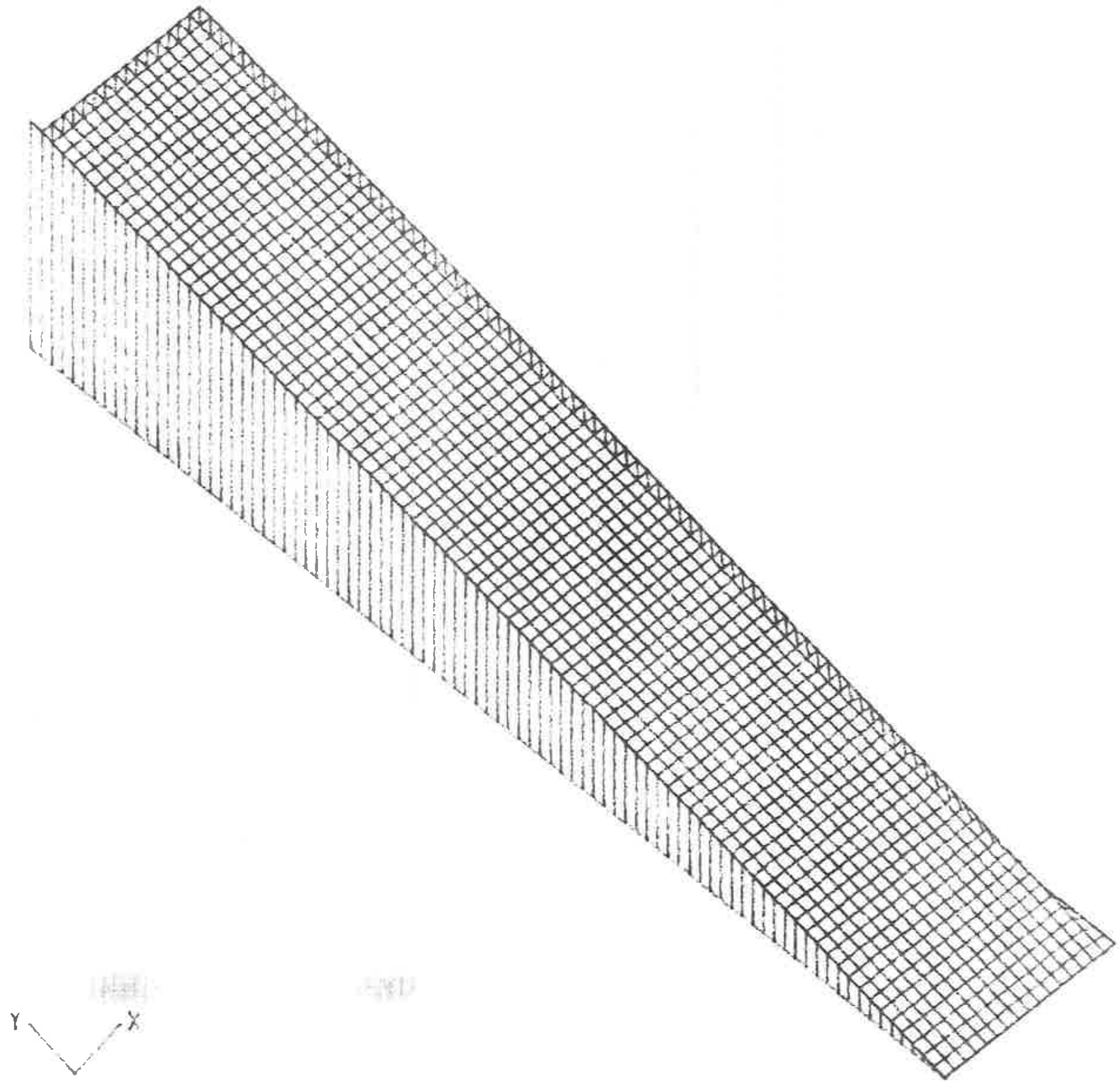

FUNCTION U
AT TIME= 0.4726E 03 SECS
ITIME= 47
MFE SOLUTION
S-W EQUATIONS
113 MOVING NODES
UMIN=-0.61736E-03 M/S
UMAX= 0.90582E-02 M/S
X
Y

Fig. 5.6



FUNCTION V

AT TIME= 0.4726E 03 SECS

ITIME= 47

MFE SOLUTION

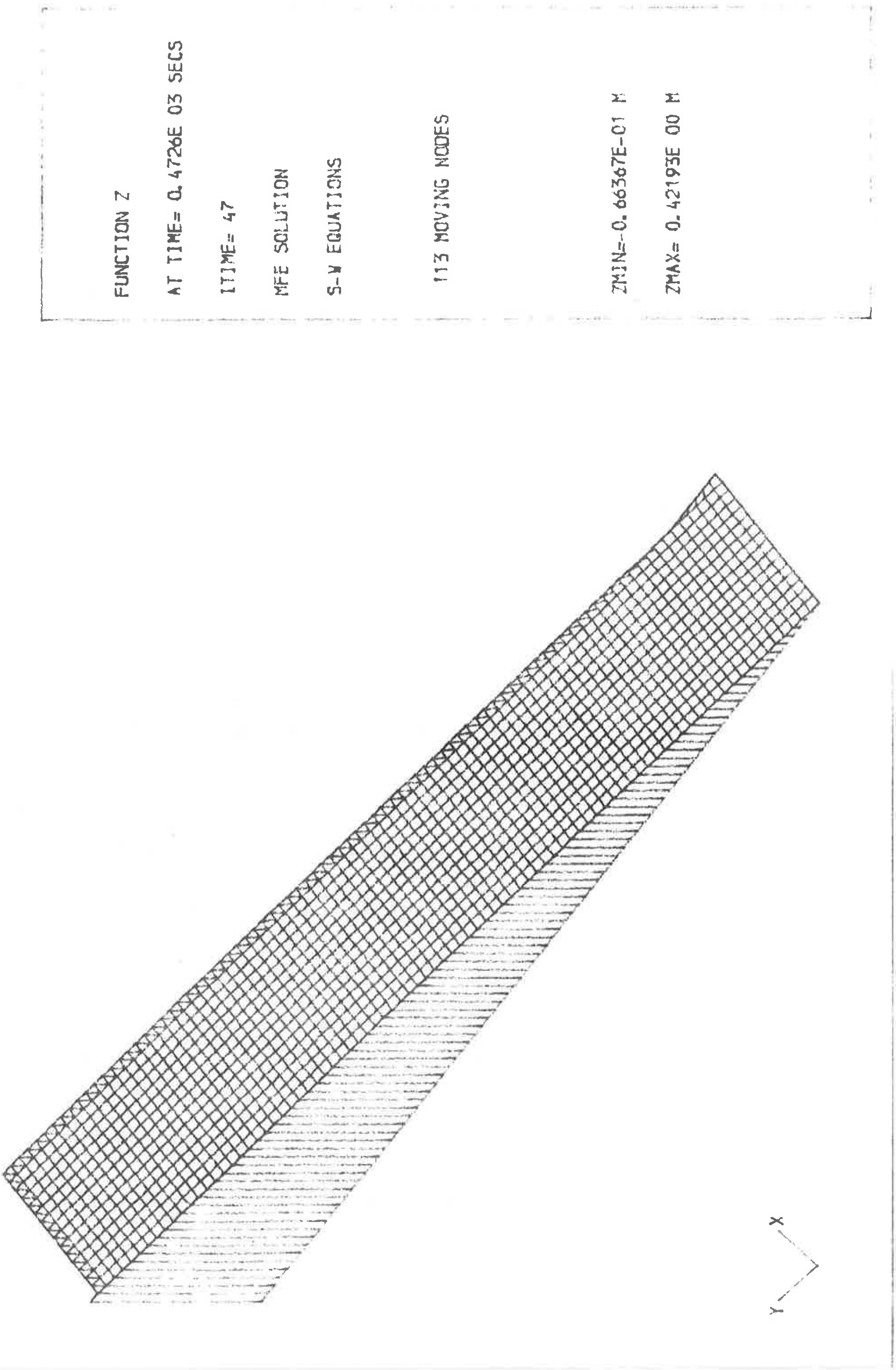
S-V EQUATIONS

113 MOVING NODES

VMIN=-0.46492E-01 M/S

VMAX= 0.29380E 00 M/S

Fig. 5.7



FUNCTION Z
AT TIME= 0.4726E 03 SECS
ITIME= 47
MFE SOLUTION
S-W EQUATIONS
113 MOVING NODES
ZMIN=-0.66367E-01 M
ZMAX= 0.42193E 00 M
X
Y

Fig. 5.8

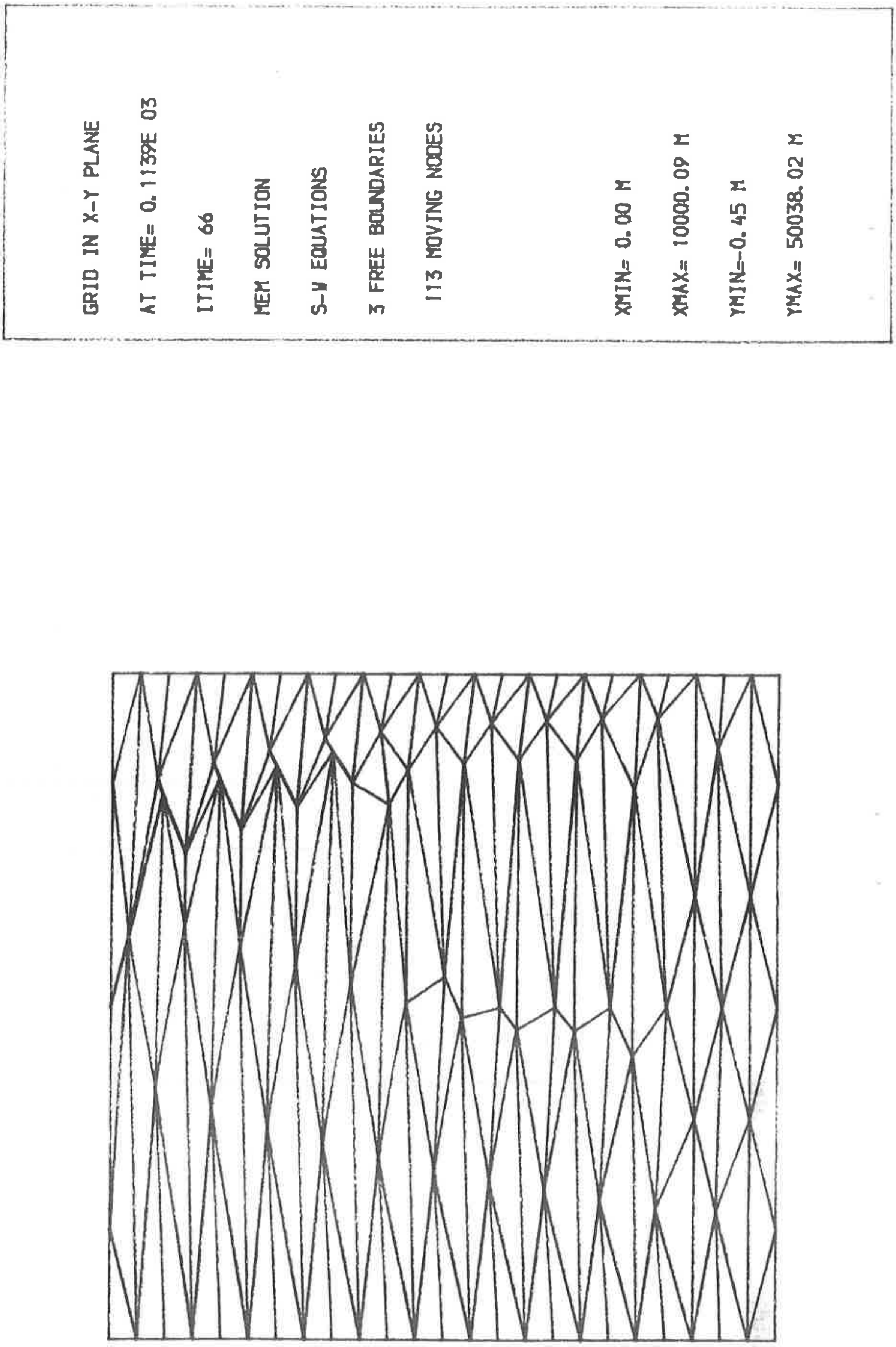
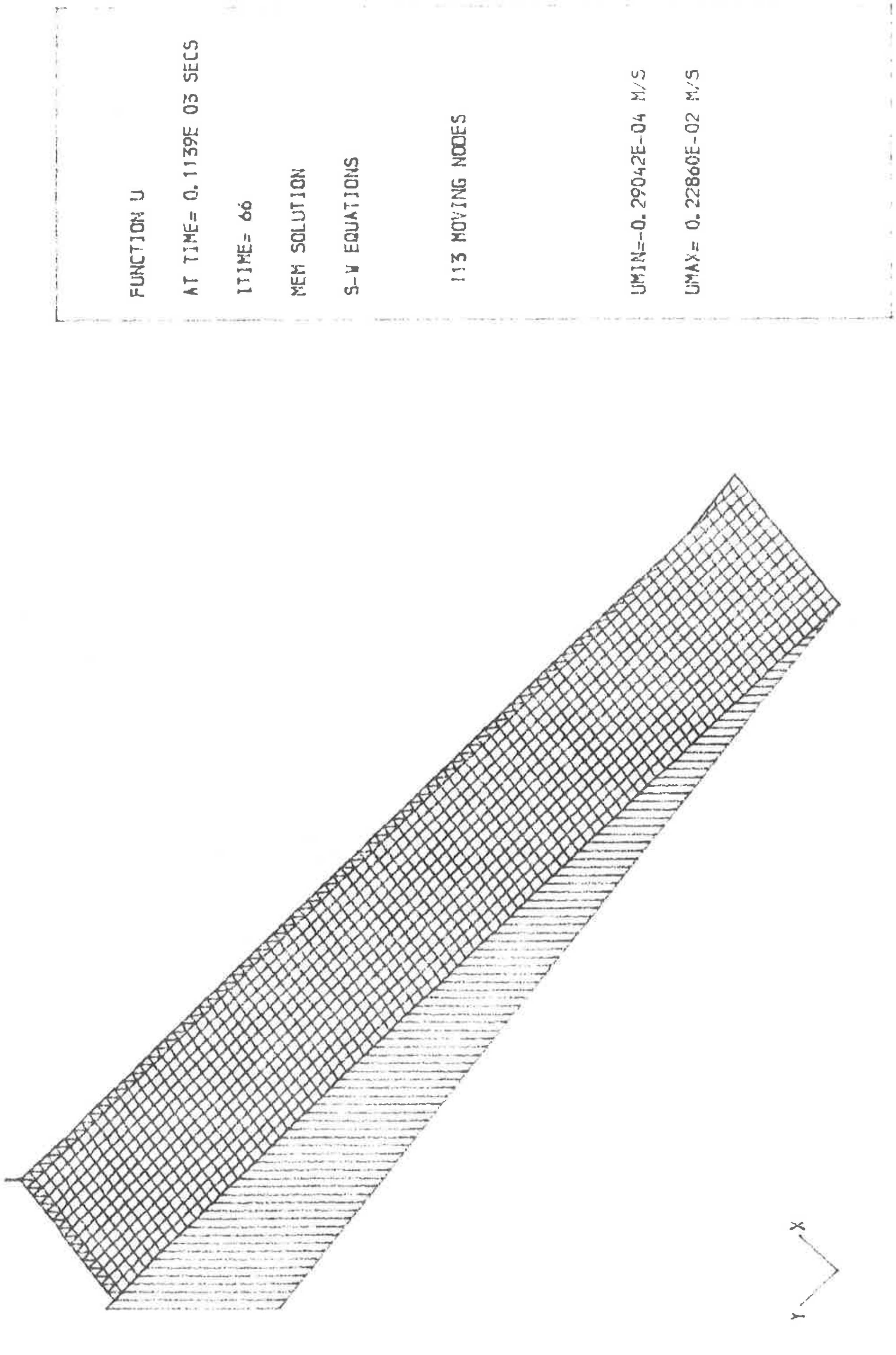
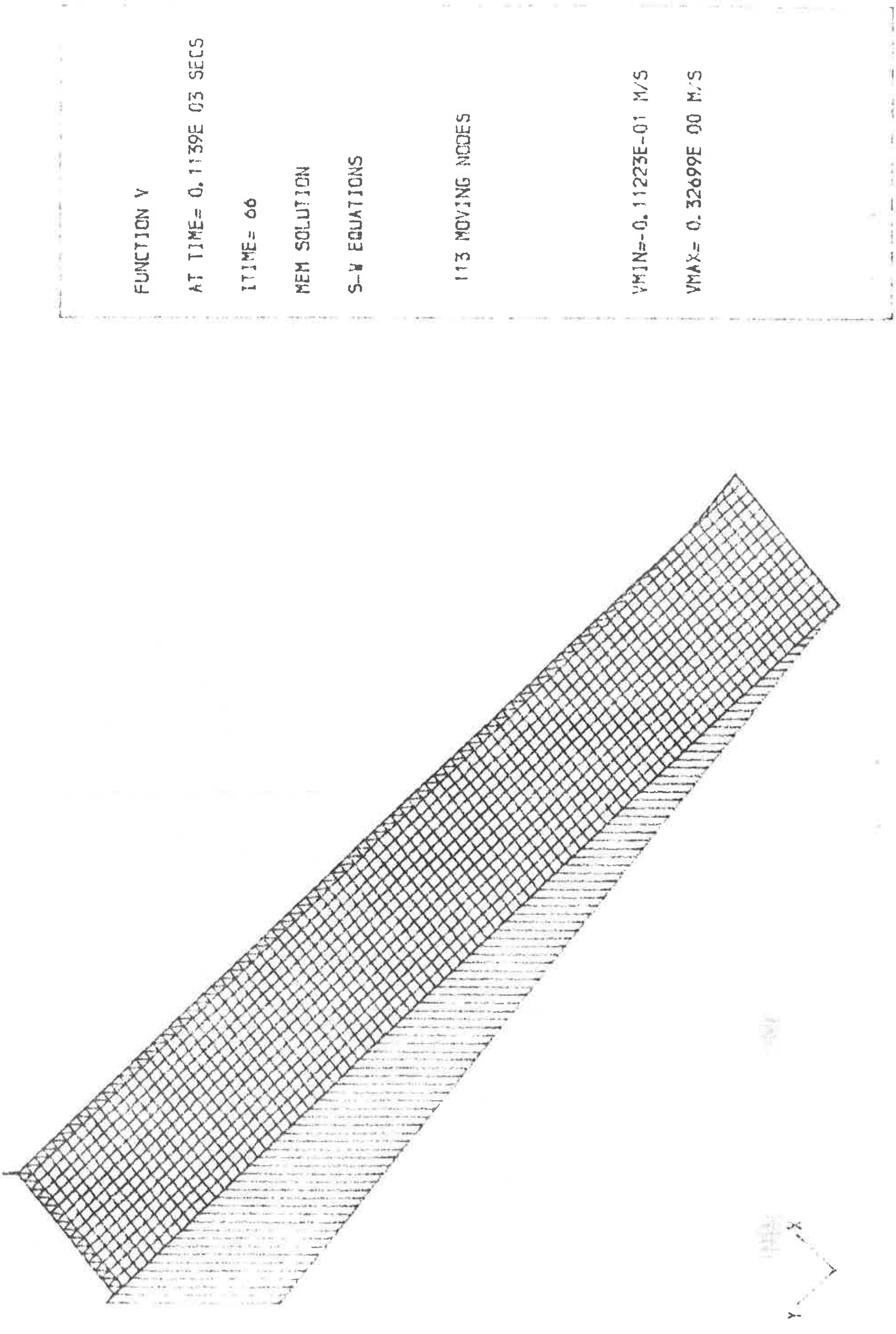

GRID IN X-Y PLANE
AT TIME= 0.11139E 03
ITIME= 66
MEM SOLUTION
S-W EQUATIONS
3 FREE BOUNDARIES
113 MOVING NODES
XMIN= 0.00 M
XMAX= 10000.09 M
YMIN=-0.45 M
YMAX= 50038.02 M

Fig. 5.9



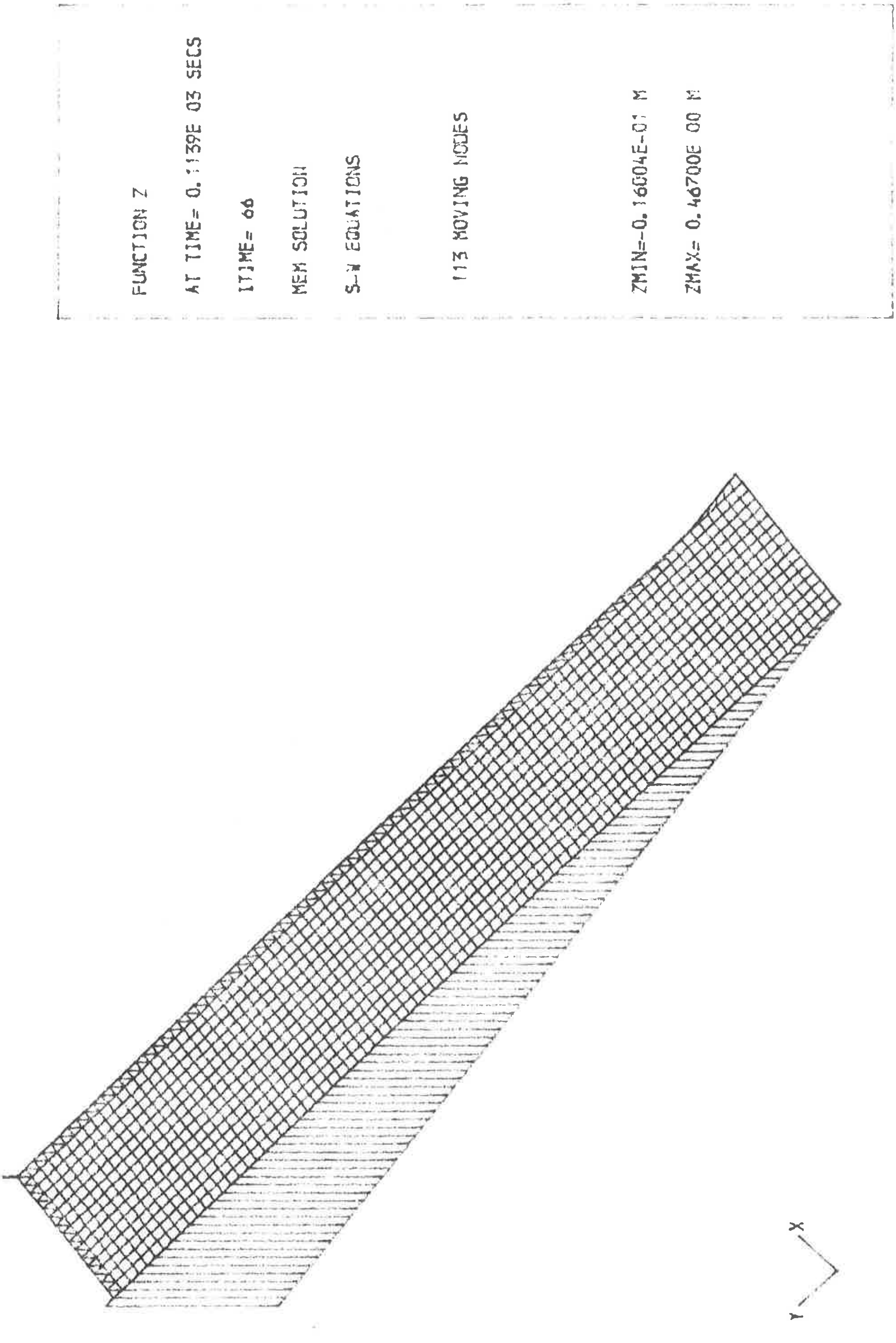
FUNCTION U
AT TIME= 0.1139E 03 SECS
ITIME= 66
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
UMIN=-0.29042E-04 M/S
UMAX= 0.22860E-02 M/S
X
Y

Fig. 5.10



FUNCTION V
AT TIME= 0.1139E 03 SECS
ITIME= 66
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
VMIN=-0.11223E-01 M/S
VMAX= 0.32699E 00 M/S

Fig. 5.11



FUNCTION Z
AT TIME= 0.1139E 03 SECS
ITIME= 66
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
ZMIN=-0.16604E-01 M
ZMAX= 0.46700E 00 M
X
Y

Fig. 5.12

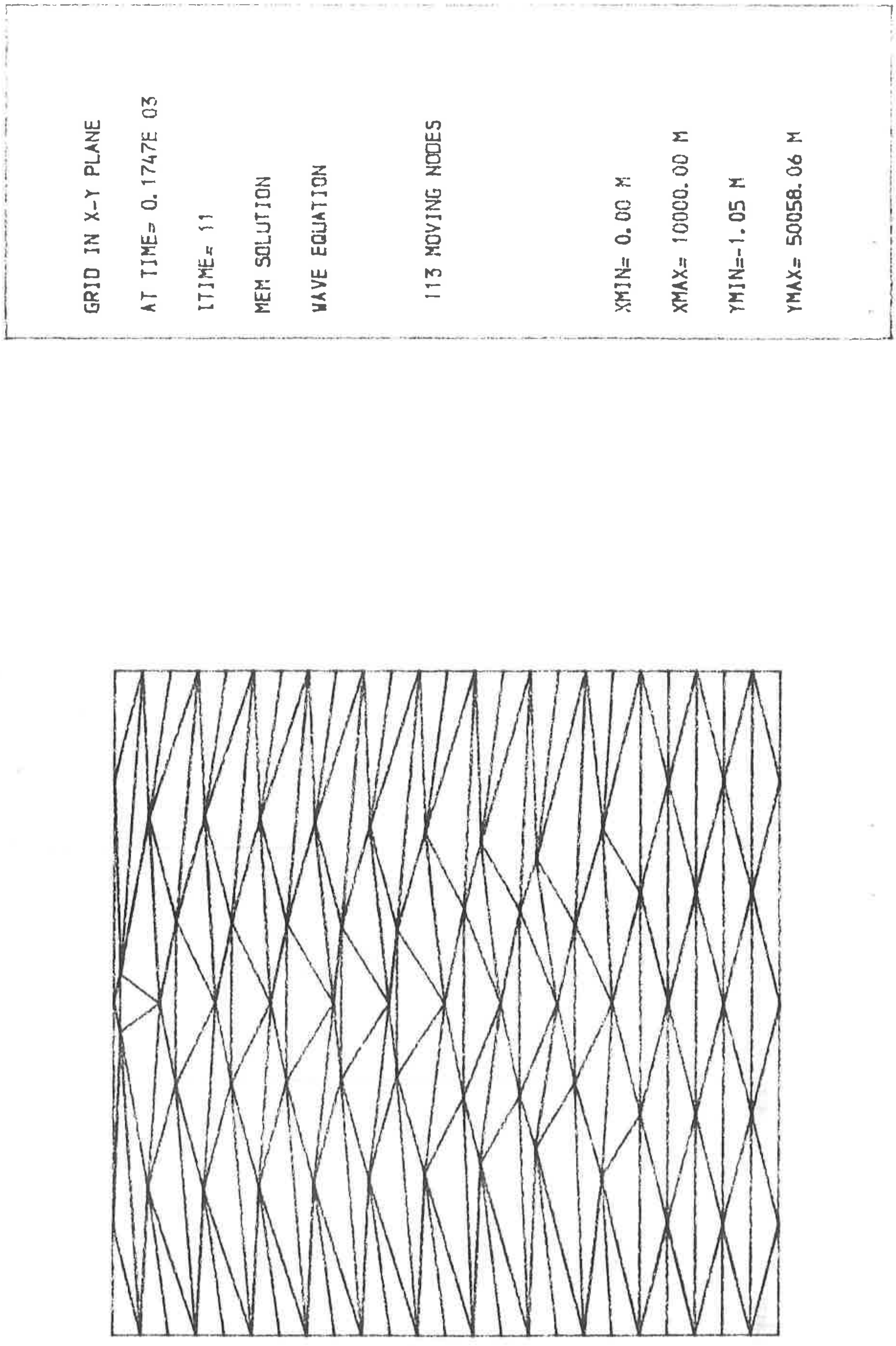

GRID IN X-Y PLANE
AT TIME= 0.1747E 03
ITIME= 11
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
XMIN= 0.00 M
XMAX= 10000.00 M
YMIN=-1.05 M
YMAX= 50058.06 M

Fig. 5.13

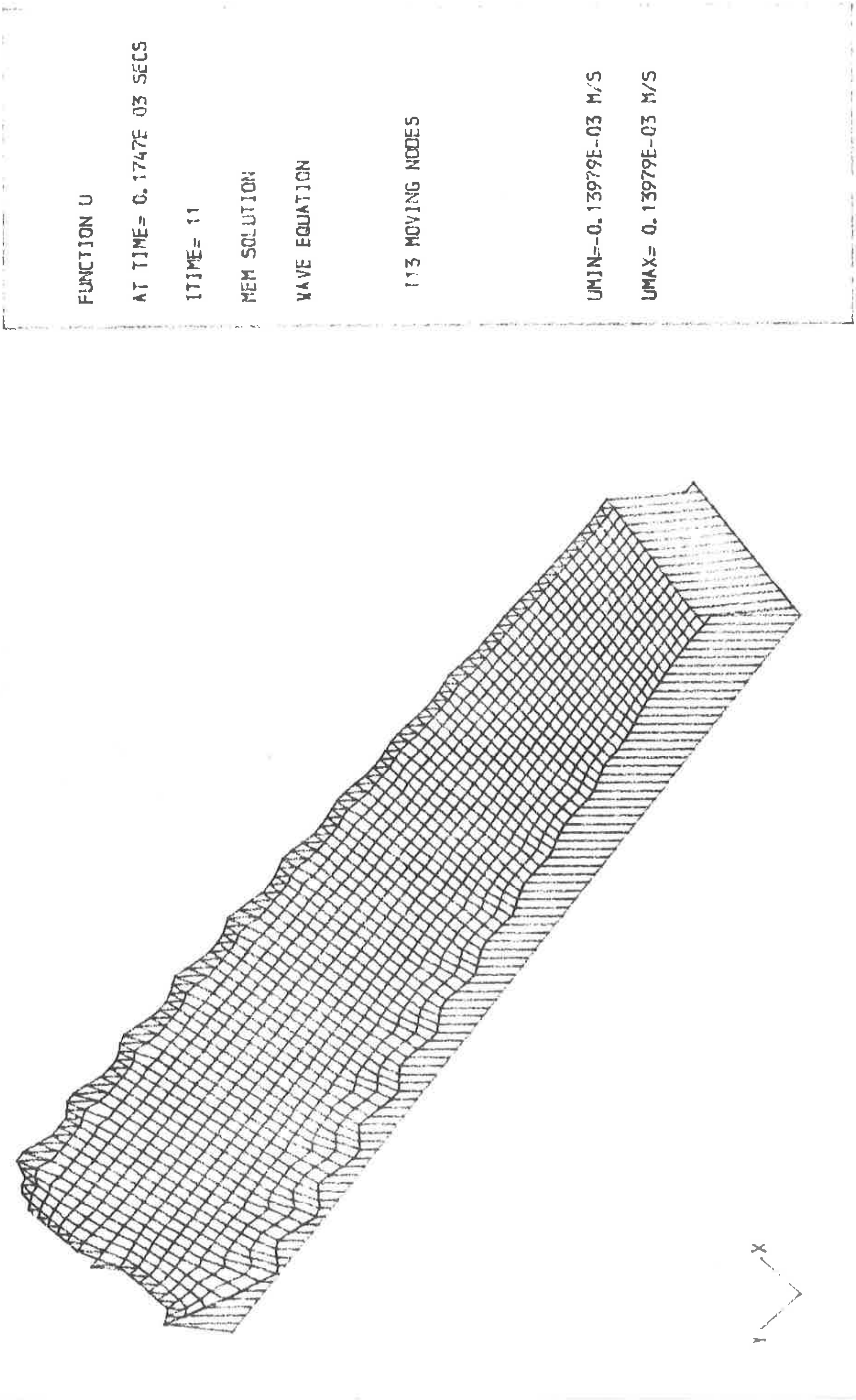

FUNCTION U
AT TIME= 0.1747E 03 SECS
ITIME= 11
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
UMIN=-0.13979E-03 M/S
UMAX= 0.13979E-03 M/S
X
Y

Fig. 5.14

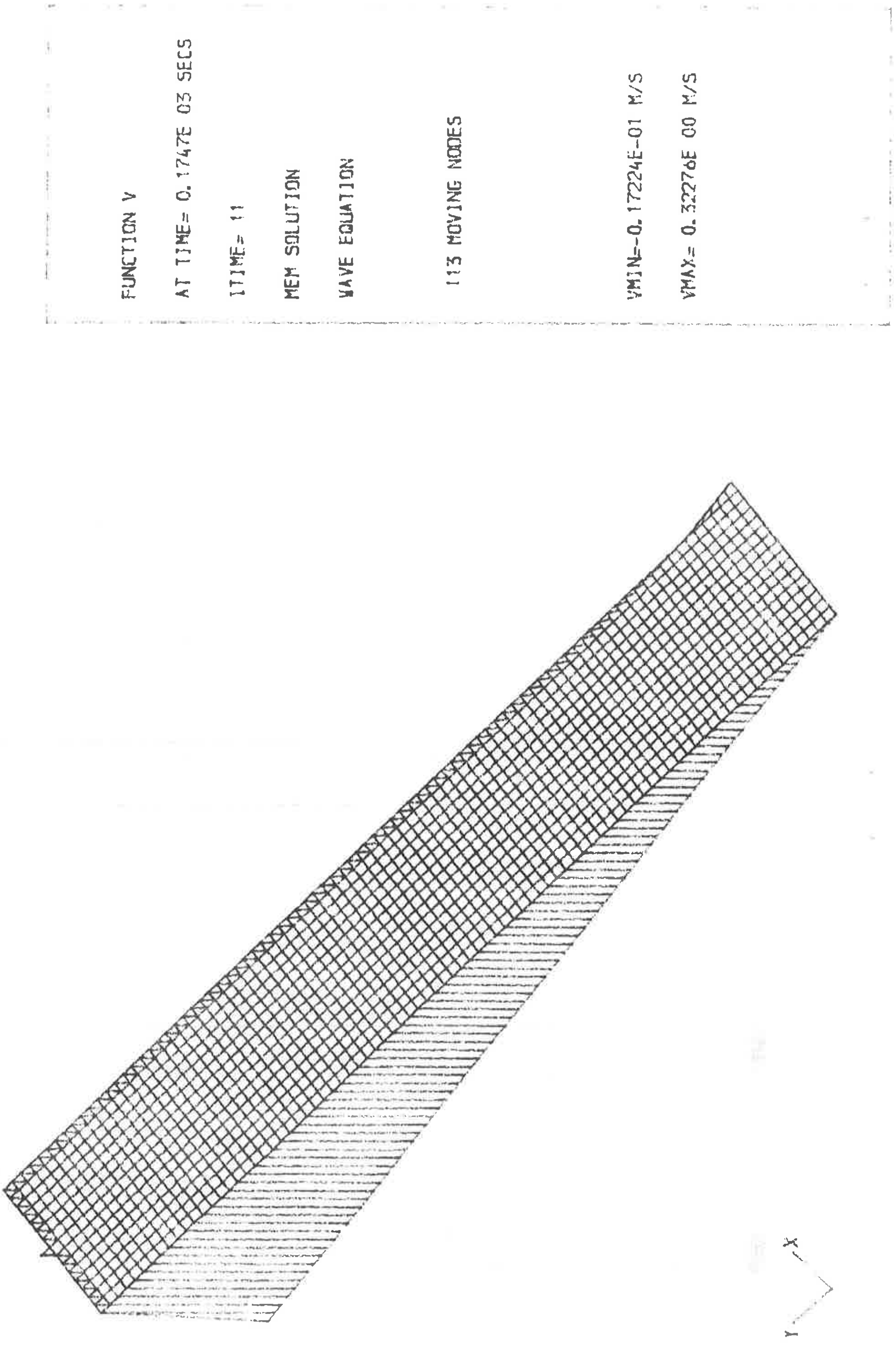

FUNCTION V
AT TIME= 0.1747E 03 SECS
ITIME= 11
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
VMIN=-0.17224E-01 M/S
VMAX= 0.32276E 00 M/S
X
Y

Fig. 5.15

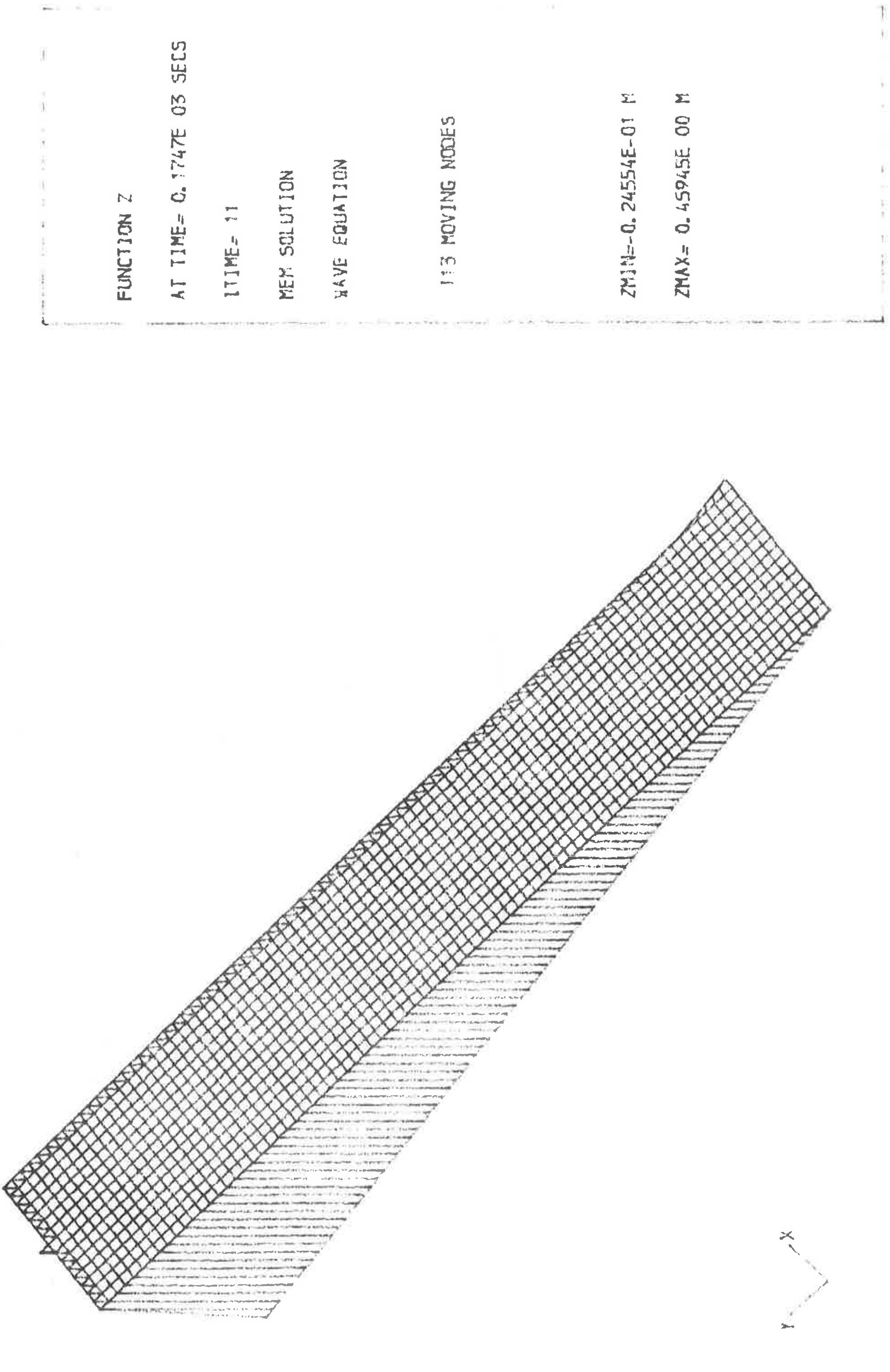

FUNCTION Z
AT TIME= 0.1747E 03 SECS
ITIME= 11
MEM SOLUTION
WAVE EQUATION
MOVING NODES
ZMIN=-0.245554E-01 M
ZMAX= 0.45945E 00 M
X
Y

Fig. 5.17

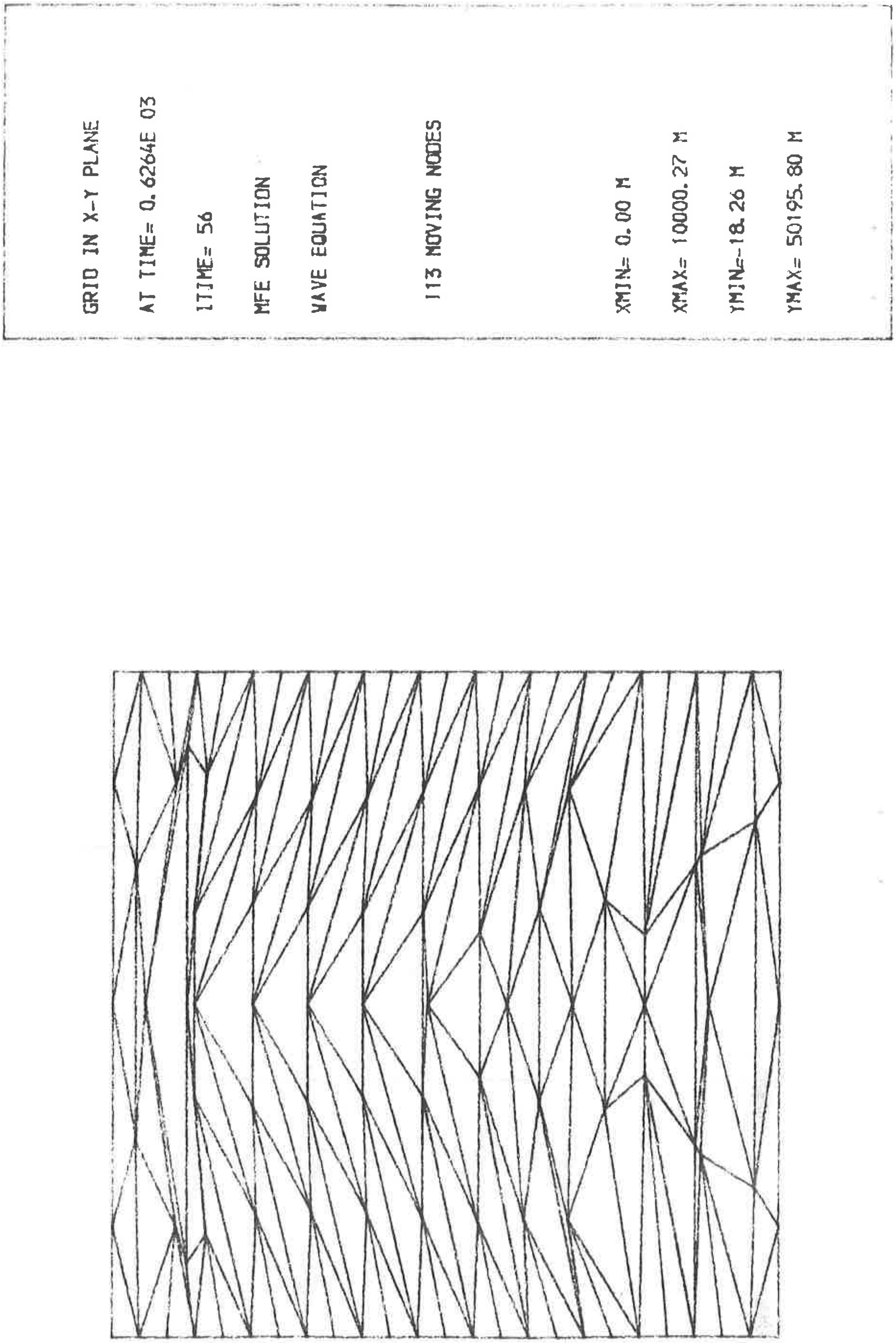

GRID IN X-Y PLANE
AT TIME= 0.6264E 03
ITIME= 56
MFE SOLUTION
WAVE EQUATION
113 MOVING NODES
XMIN= 0.00 M
XMAX= 10000.27 M
YMIN=-18.26 M
YMAX= 50195.80 M

Fig. 5.17

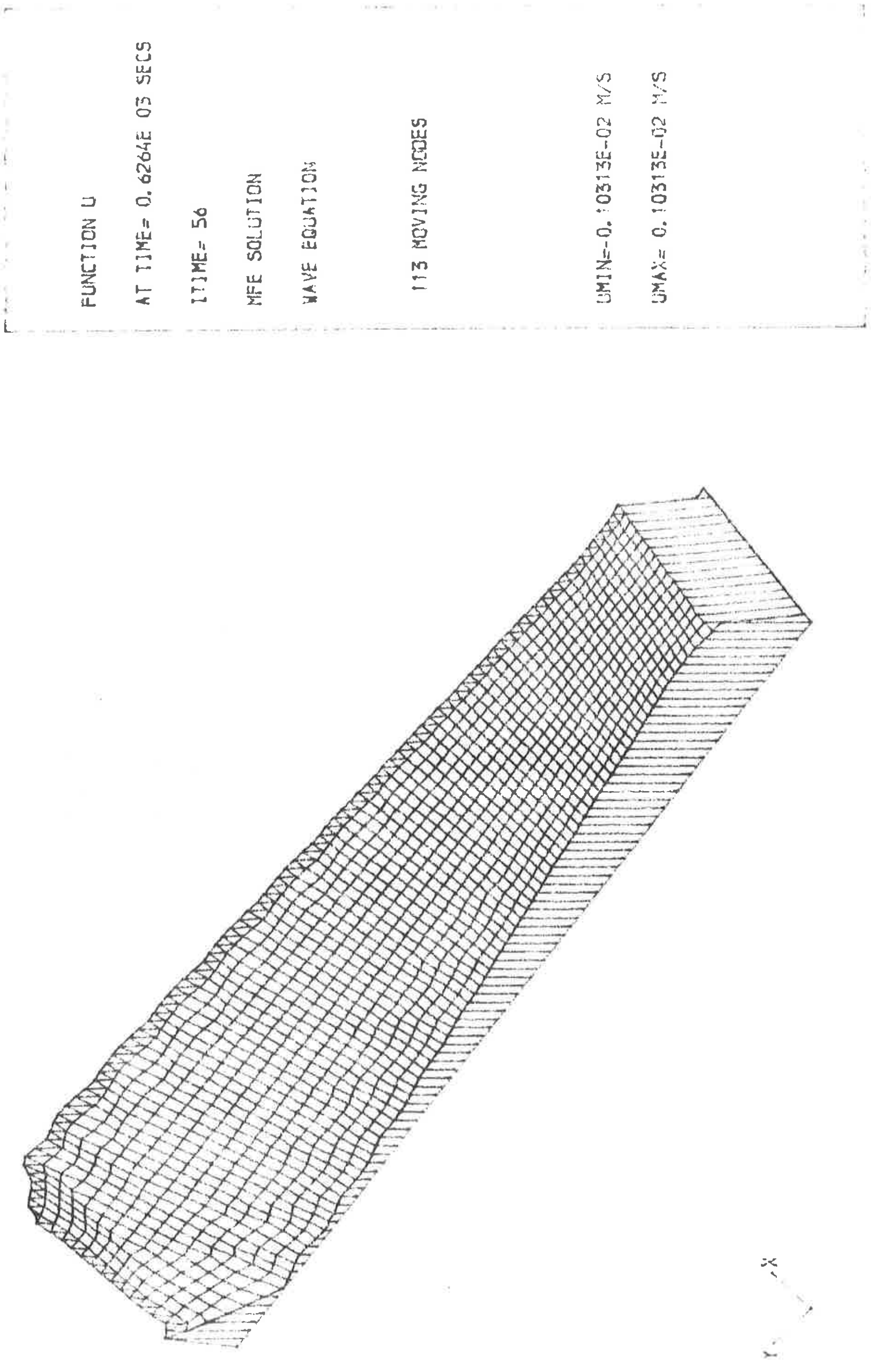

FUNCTION U
AT TIME= 0.6264E 03 SECS
ITIME= 56
MFE SOLUTION
WAVE EQUATION
113 MOVING NODES
UMIN=-0.10313E-02 M/S
UMAX= 0.10313E-02 M/S
X
Y

Fig. 5.18

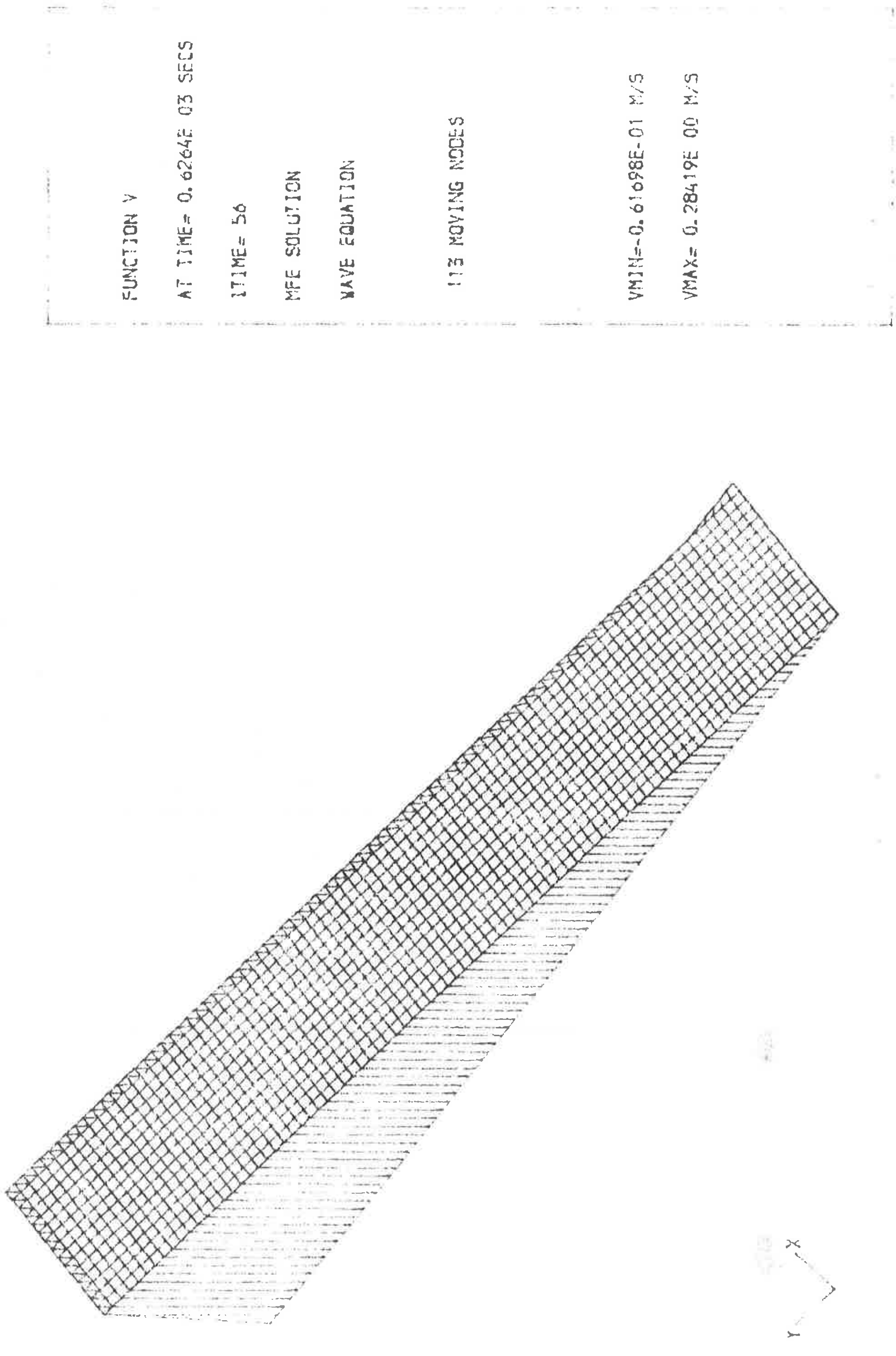
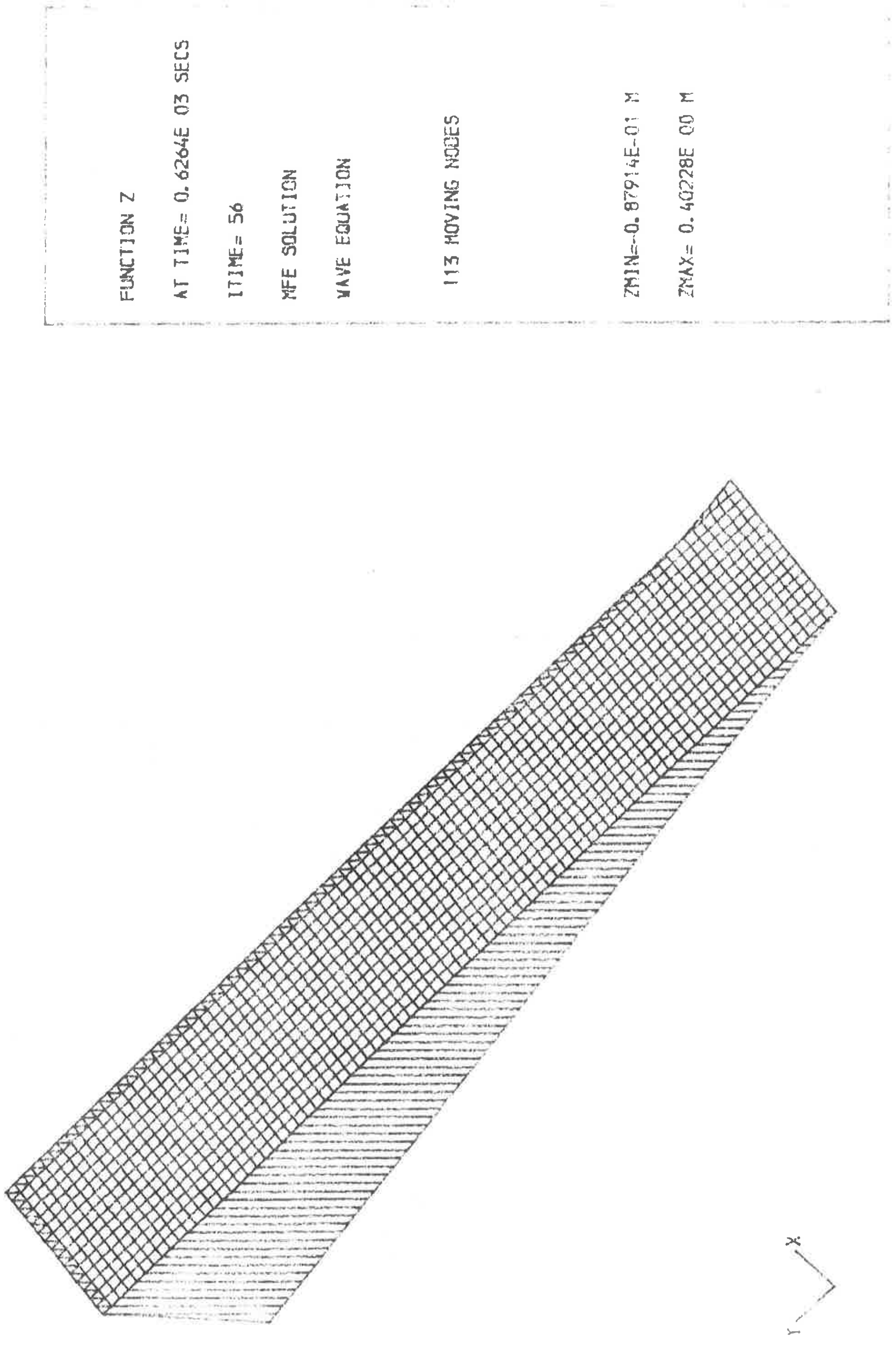

FUNCTION V
AT TIME= 0.6264E 03 SECS
ITIME= 56
MFE SOLUTION
WAVE EQUATION
113 MOVING NODES
VMIN=-0.61698E-01 M/S
VMAX= 0.28419E 00 M/S
X
Y

Fig. 5.19



FUNCTION Z
AT TIME= 0.6264E 03 SECS
ITIME= 56
MFE SOLUTION
WAVE EQUATION
113 MOVING NODES
ZMIN=-0.87914E-01 M
ZMAX= 0.40228E 00 M
X
Y

Fig. 5.20

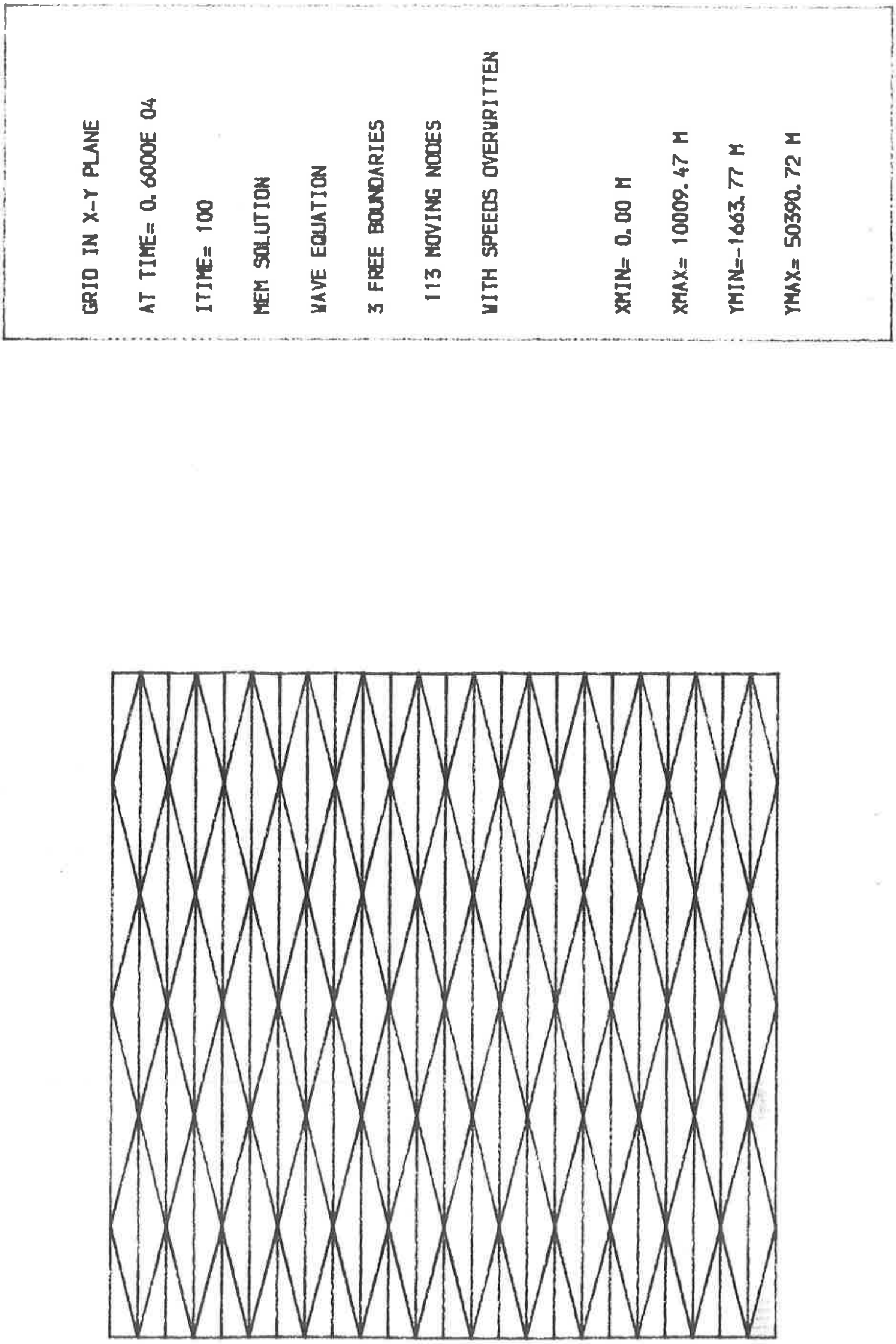

GRID IN X-Y PLANE
AT TIME= 0.6000E 04
ITIME= 100
MEM SOLUTION
WAVE EQUATION
3 FREE BOUNDARIES
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
XMIN= 0.00 M
XMAX= 10009.47 M
YMIN=-1663.77 M
YMAX= 50390.72 M

Fig. 5.21

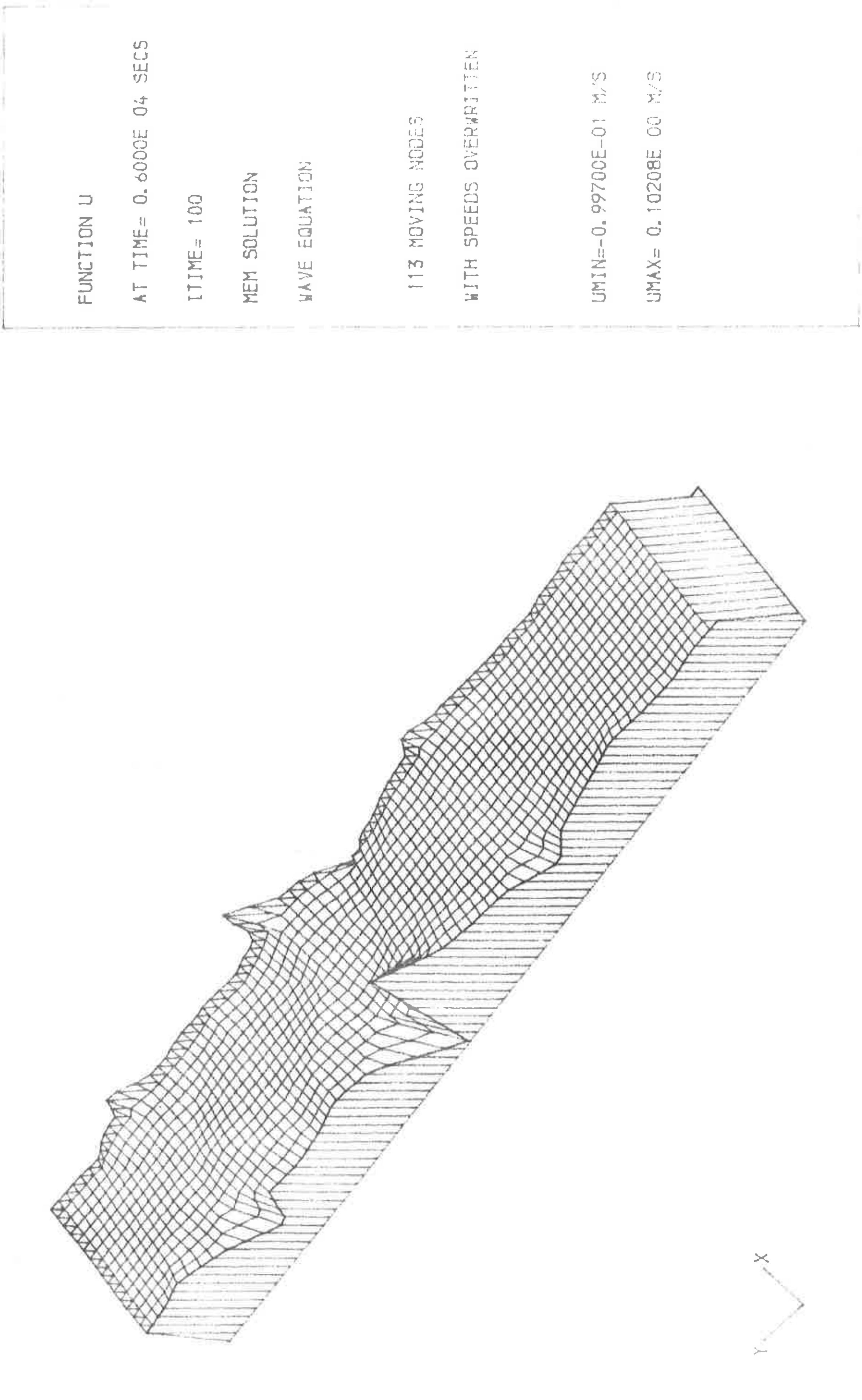
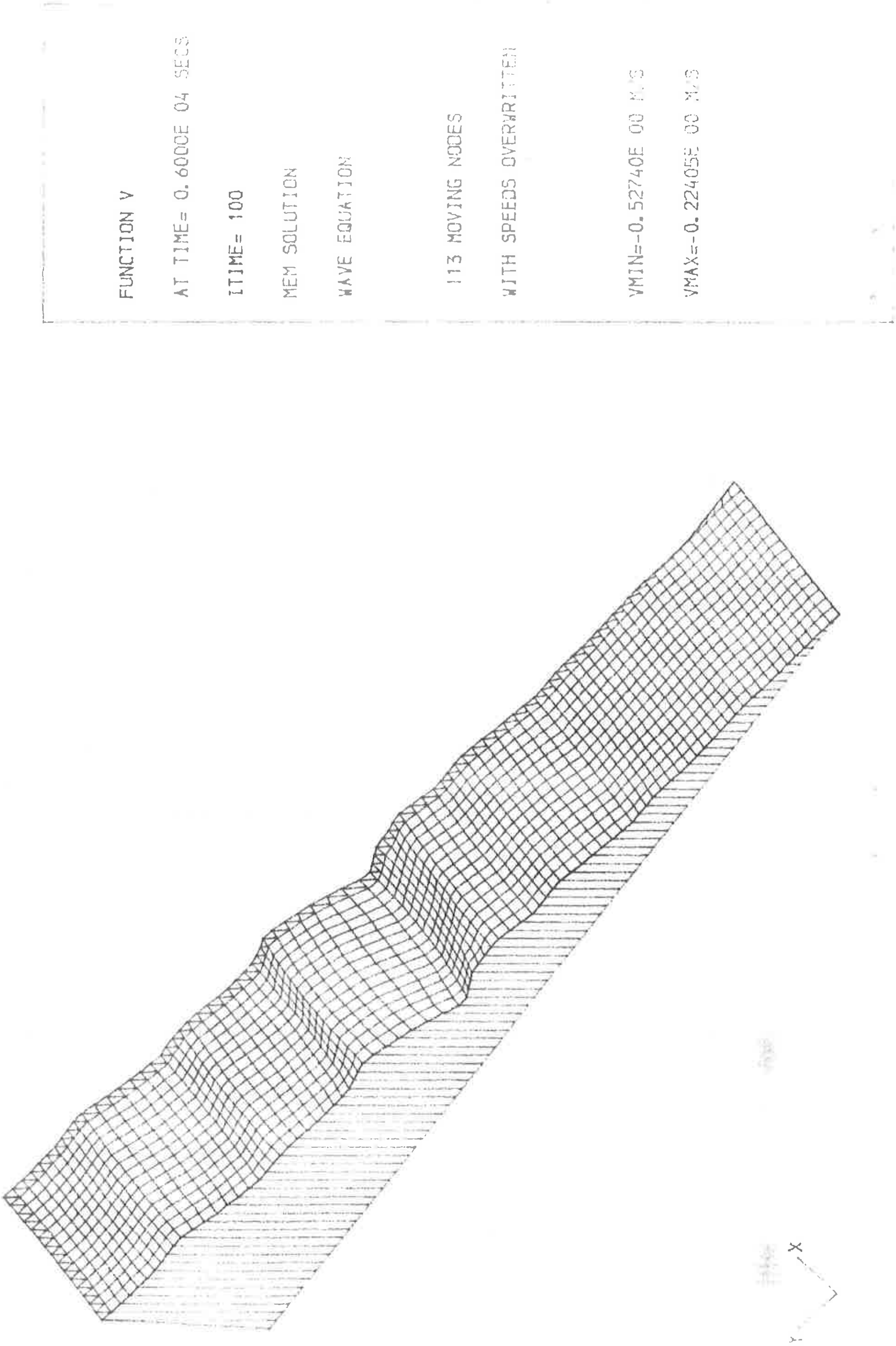

FUNCTION U
AT TIME= 0.6000E 04 SECS
ITIME= 100
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
UMIN=-0.99700E-01 M/S
UMAX= 0.10208E 00 M/S
X
Y

Fig. 5.22



FUNCTION V
AT TIME= 0.6000E 04 SECS
ITIME= 100
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
VMIN=-0.52740E 00 M/S
VMAX=-0.22405E 00 M/S
X
Y

Fig. 5.23

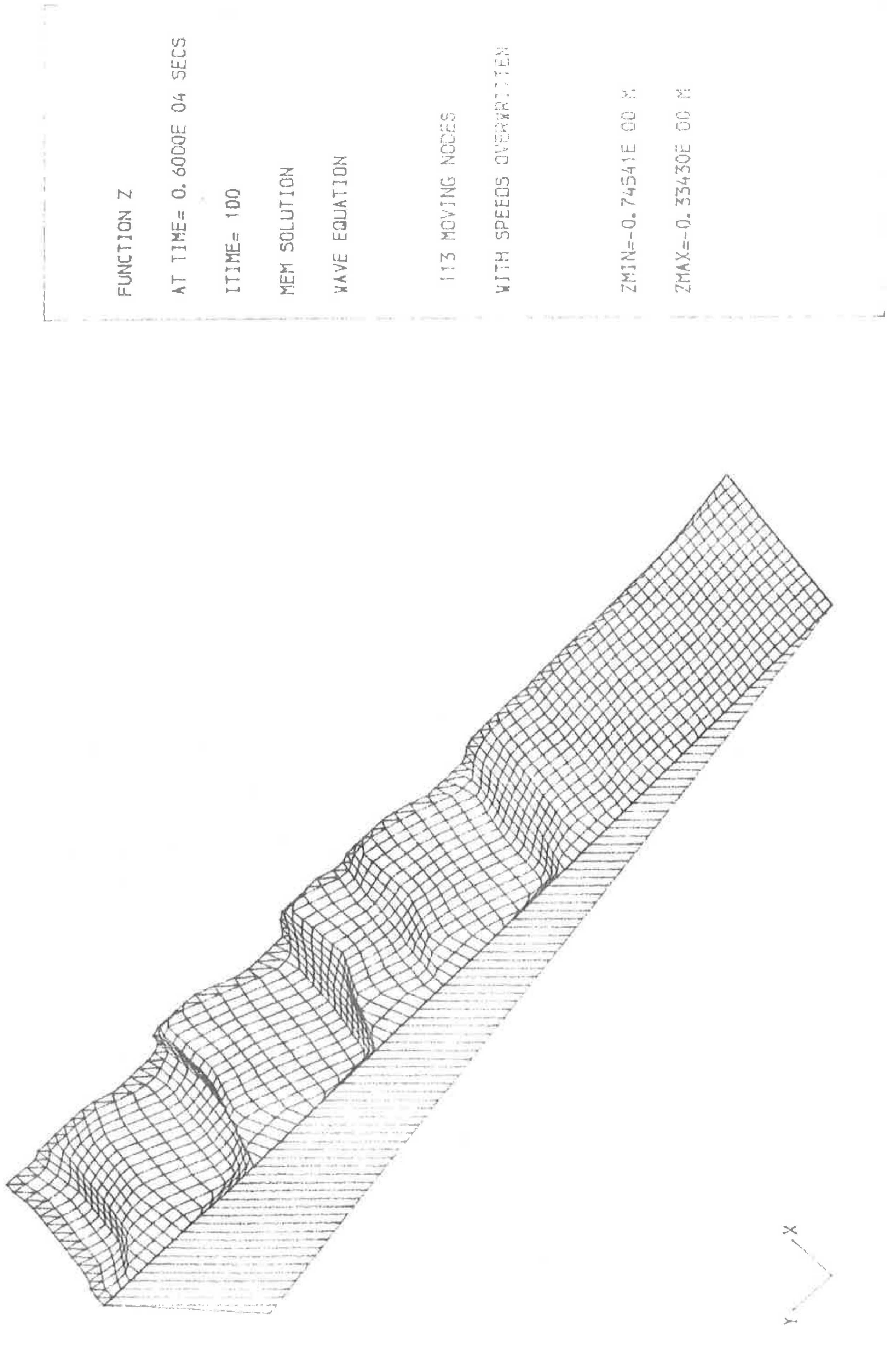

FUNCTION Z
AT TIME= 0.6000E 04 SECS
ITIME= 100
MEM SOLUTION
WAVE EQUATION
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
ZMIN=-0.74541E 00 M
ZMAX=-0.33430E 00 M
X
Y

Fig. 5.24

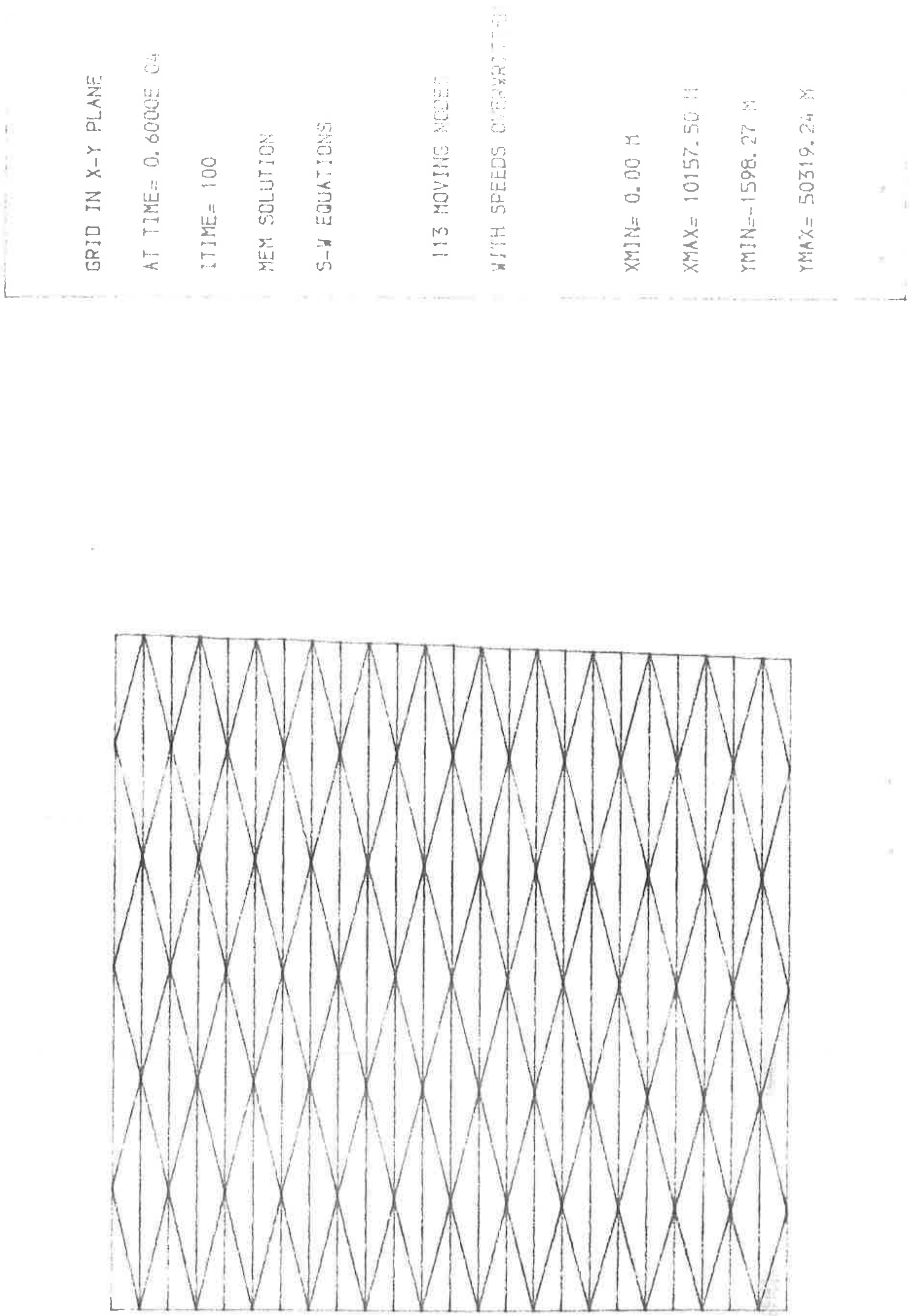

GRID IN X-Y PLANE
AT TIME= 0.6000E 04
ITIME= 100
MEM SOLUTION
S-W EQUATIONS
113 MOVING
WITH SPEEDS
XMIN= 0.00 M
XMAX= 10157.50 M
YMIN=-1598.27 M
YMAX= 50319.24 M

Fig. 5.25

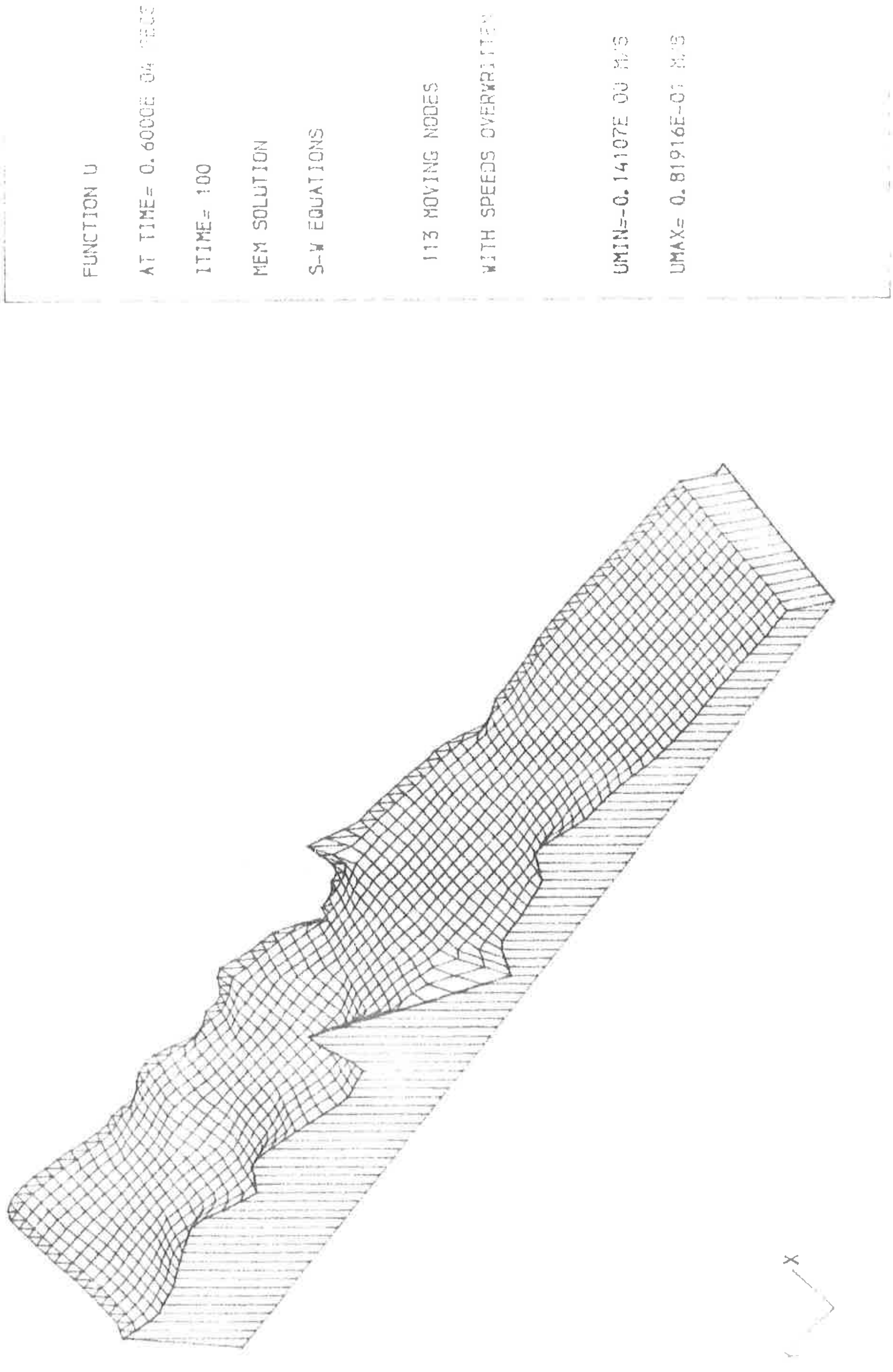

FUNCTION U
AT TIME= 0.6000E 04
ITIME= 100
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
UMIN=-0.14107E 00 M/S
UMAX= 0.81916E-01 M/S
X

Fig. 5.26

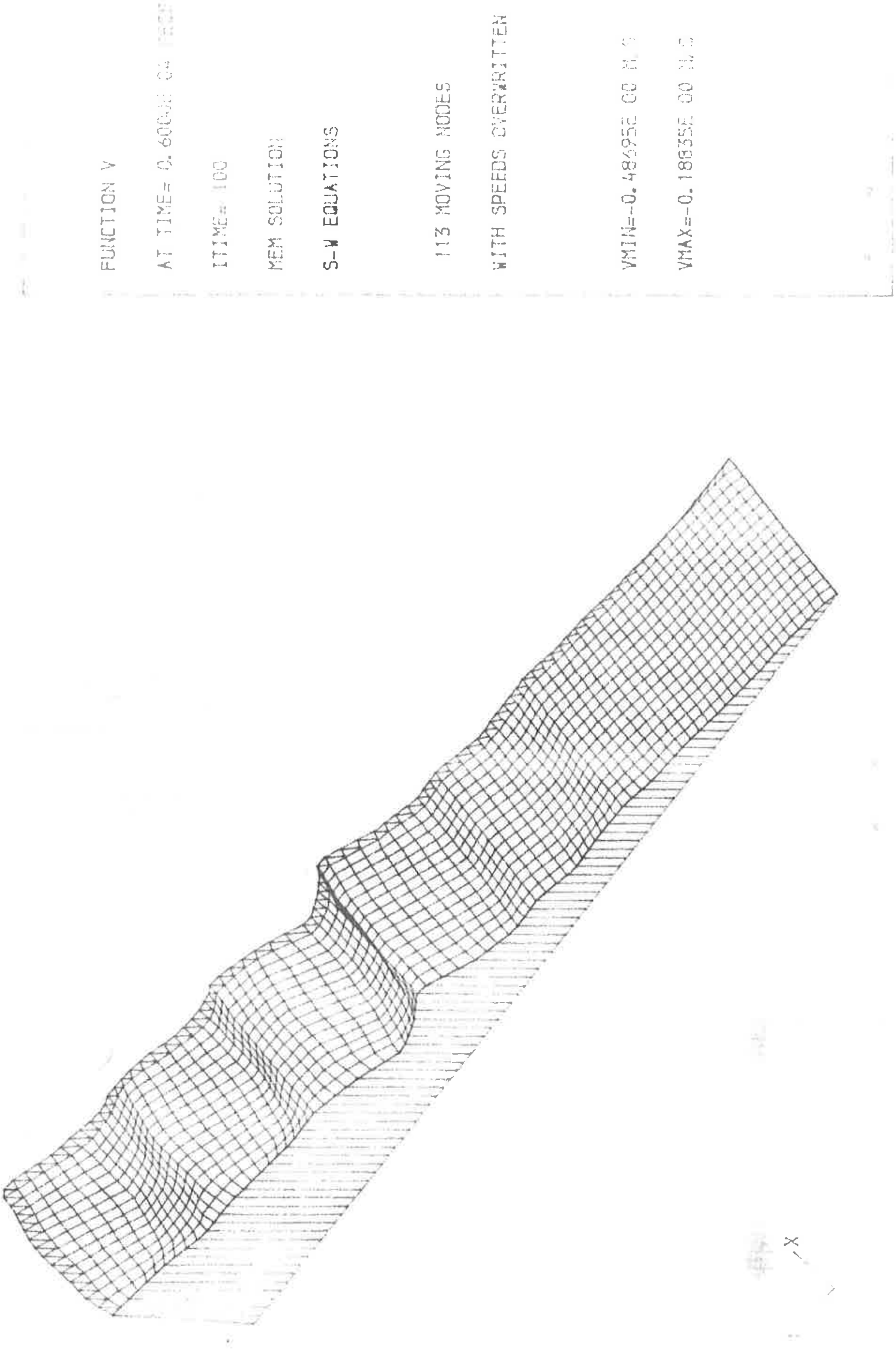
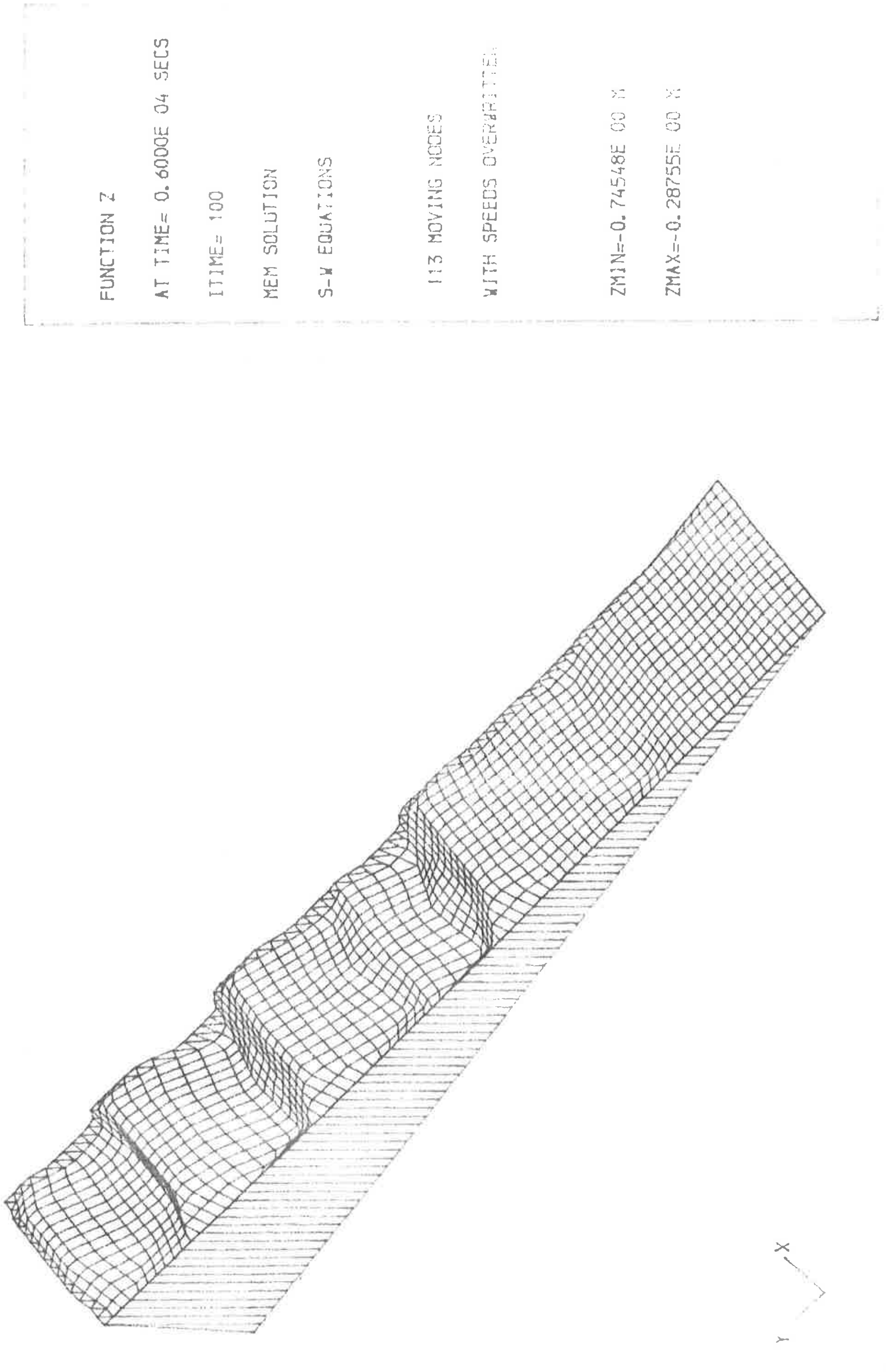

FUNCTION V
ITIME= 100
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
WITH SPEEDS OVERWRITTEN

Fig. 5.27


FUNCTION Z
AT TIME= 0.6000E 04 SECS
ITIME= 100
MEM SOLUTION
S-W EQUATIONS
113 MOVING NODES
WITH SPEEDS OVERWRITTEN
ZMIN=-0.74548E 00 M
ZMAX=-0.28755E 00 M
X
Y