DEPARTMENT OF MATHEMATICS

The Spectral Lagrange-Galerkin Method for the Atmospheric Transportation of Pollutants. Part II. Stability of the Non-Exact Integration.

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Numerical Analysis Report 4/89

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Introduction

In the previous report in this series, Priestley (1989), the Lagrange-Galerkin method was described for use with spectral methods.

Priestley (1986) and Morton, Priestley, Süli (1988) showed that, with the more conventional finite elements, the Lagrange-Galerkin method lost its unconditional stability when quadrature was applied to the integrations involved. Numerical integration is needed in all but the most trivial case.

Süli & Ware (1988) showed that if the global basis functions of the Fourier spectral method were used then not only is the exactly integrated scheme stable but so is the approximately integrated version, provided that the quadrature points are the Gaussian points for the polynomials - equally spaced points in this case - and that the number of quadrature points is greater than the number of modes in the spectral representation.

Süli (1989) has proved stability, with exact integration, for Chebyshev and Legendre polynomials. These proofs were for convection - diffusion problems but it is worth emphasising here that the diffusion was not required for its stabilizing properties but so that the characteristic could be assumed not to leave the domain. In similar circumstances here we will assume not that $\underline{a} = 0$ at the boundaries but that $\underline{a} \cdot \underline{n} = 0$ which of course in one dimension comes to the same thing but means that we can concern ourselves with pure advection.

2. Preliminaries

Let $\rho(x)$ be any non-negative (not identically zero) function on the interval [-1-1] .

We define orthonormal polynomials $Y_q(x)$ such that

$$\int_{-1}^{1} \rho(x)\, Y_q(x)\, Y_p(x)\, dx = \begin{cases} 0 & p \neq q \\ 1 & p = q \end{cases} .$$

The quadrature associated with these polynomials is then

$$\int_{-1}^{1} \rho(x)\, f(x) \simeq \sum_{k=1}^{L} A_k f(x_k) \tag{1}$$

where the A_k are the weights and the abscissae x_k, are the zeroes of $Y_L(x)$. The quadrature is exact for polynomial $f(x)$ of order $2L-1$ or less.

Following the notation of Süli & Ware (1988) we write the spectral Lagrange-Galerkin method as

$$\hat{U}_p^{n+1} = \sum_{q \leq N} \hat{A}(p,q)\, \hat{U}_q^n \qquad p \leq N \tag{2}$$

where

$$\hat{A}(p,q) = \int_{-1}^{1} \rho(x)\, Y_q(X(x,\Delta t;0)) \, . \, Y_p(x)dx \, .$$

Here $X(x,\Delta t;0)$ is the usual notation for a trajectory that at time $t = 0$ was at x and at time $t = \Delta t$ is at X .

The approximately integrated version of (2) can then be written as

$$\hat{U}_p^{n+1} = \sum_{q \leq N} \hat{A}_{pq}\, \hat{U}_q^n \qquad p \leq N \tag{3}$$

where

$$\hat{A}_{pq} = \sum_{k=1}^{L} A_k \; Y_q(x_k + a\Delta t) \; Y_p(x_k) \; .$$

3. Stability of the Spectral Lagrange Galerkin Method with Non-Exact Integration

The proof of stability is split into two bits. It is first given for a constant velocity a . For this part of the proof we assume that the domain, $[-1,1]$, and the orthonormal polynomials are periodic in order that the characteristics do not leave the domain.

For the variable velocity case no restriction needs to be placed upon the polynomials but we must assume that the velocity field is such that no trajectory may leave the domain and that any approximation to a trajectory also remains in the domain. In this case we only manage to prove the weaker time-stability.

In both cases it is required that the exactly integrated version of the scheme is stable and that the number of quadrature points, L , exceeds the number of modes N .

From equation (3) we can write

$$\sum_{p \leq N} |\hat{U}_p^{n+1}|^2 = \sum_{p,q \leq N} |\hat{U}_q^n \; \overline{\hat{U}}_p^{n+1} \; \hat{A}\,(p,q)$$

$$+ \quad \sum_{p,q} \hat{U}_q^n \; \overline{\hat{U}}_p^{n+1} \quad (A_{pq} - A(p,q))$$

$$\equiv T_1 + T_2 \; .$$

Since the exactly integrated scheme is to be stable implies

$$T_1 \leq ||U^{n+1}|| \; . \; ||U^n||$$

and hence to prove stability we need to show that

$$||A_{pq} - A(p,q)|| = ||\int_{-1}^{1} \rho(x)\, Y_q(x+a\Delta t)\, Y_p(x)dx$$

$$-\sum_{k=1}^{L} A_k\, Y_q(x_k+a\Delta t)\, Y_p(x_k)|| \qquad (4)$$

is a quantity of order Δt .

Case 1 Periodic and constant velocity.

This is quite trivial because the Y_q's are still clearly polynomials on our domain and since $p + q \leq 2N < 2L - 1$ the quadrature will perform the integrations exactly i.e.,

$$||A_{pq} - A(p,q)|| = 0$$

and hence stability is proved for this case.

Case 2 Non-periodic and variable velocity.

Theorem

The Spectral Lagrange-Galerkin method is time-stable under the conditions given above provided that the velocity field is continuously differentiable and that $||a||_\infty$ and $||a||_{1,\infty}$ are bounded.

<u>Proof</u>

Consider

$$||A_{p,q} - A(p,q)|| = ||\int_{-1}^{1} \rho(x)\ Y_q(X(x,\Delta t;0))\ Y_p(x)\ dx$$

$$- \sum_{k=1}^{L} A_k\ Y_q(X(x_k,\Delta t;0))\ Y_p(x_k)||\ . \qquad (5)$$

We replace $X(x,\Delta t,0)$ by $x + a(x)\Delta t + 0(\Delta t^2)$ but ignore the Δt^2 terms on the understanding that $x + a(x)\Delta t$ is still a valid approximation to the trajectory. Equation (5) then becomes

$$||A_{pq} - A(p,q)|| = ||\int_{-1}^{1} \rho(x)\left\{Y_q(x) + a(x)\Delta t\ Y_p'(\eta)\right\} Y_p(x)dx$$

$$- \sum_{k=1}^{L} A_k\left\{Y_q(x_k) + a(x_k)\Delta t\ Y_p'(\xi_k)\right\} Y_p(x_k)||$$

where $\eta \in (x,x+a(x)\Delta t)$, $\xi_k \in (x_k,x_k+a(x_k)\Delta t)$.

Now

$$||A_{pq} - A(p,q)|| = \Delta t \left|\left| \int_{-1}^{1} \rho(x)\ a(x)\ Y_p'(\eta)\ Y_p(x) dx \right.\right.$$

$$\left.\left. - \sum_{k=1}^{L} A_k\ a(x_k)\ Y_p'(\xi_k)\ Y_p(x_k) \right|\right| + O(\Delta t^2)$$

$$\leq \Delta t\ ||a||_\infty \left[\left|\left| \int_{-1}^{1} \rho(x)\ |Y_q'(\eta)\ Y_p(x)| dx \right|\right| \right.$$

$$\left. + \left|\left| \sum_{k=1}^{L} A_k |Y_p'(\xi_k)\ Y_p(x_k)|\ \right|\right| \right] + O(\Delta t^2)$$

$$= \Delta t ||a||_\infty\ C(N) + O(\Delta t^2) = O(\Delta t)$$

and hence time-stability is proved for this case.

4. Conclusion

We have shown, under mild restrictions upon the velocity field, that if any orthonormal polynomial basis leads to a stable scheme in the exactly integrated case then provided that we use the associated quadrature with more points than modes then the approximately integrated scheme will also be stable. For the variable velocity case we had to assume that our spacial descretization is fixed, because the 'constants' depend on N , but for all practical purposes this time-stability result should be adequate.

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