NUMERICAL SIMULATION OF HOT FLUID

INJECTION INTO POROUS MEDIA.

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CONTENTS

ABSTRACT

This report develops a mathematical model for the injection of a hot fluid into a porous media, with applications to heavy oil recovery. The resulting differential equations are then solved numerically by a finite element method.

ACKNOWLEDGEMENTS

§1 INTRODUCTION

The first stage in the extraction of light oils (oils with a low
viscosity) is by Primary Recovery, a method which simply involves sinking
a well into the reservoir and letting the fuid flow out under the natural
pressure of its surrounds (see[4] & [5]). However this method usually
removes less than half of the initial oil in place, and therefore secondary
recovery techniques are used to remove the remaining oil, for example water
flooding (see [4]). There also exist heavy oil fields, in which the oil is
at such a high viscosity that recovery by primary or secondary methods is
unfeasible. A number of different techniques exist for lowering the
viscosity of the oil and thus allowing extracting, for example in-situ
combustion (see[15]), injection of carbon dioxide, or steam injection
(see [10] & [15]), there being an increased interest in the numerical simulation
of these methods (see[1],[2],[3],[6] & [7]).

Steam injection, as its name suggests, involves the injection of steam
under high pressure into the reservoir where it acts as a heating agent.
After a specified period of time the steam is switched off and the melted
oil flows out due to the pressures in the reservoir. It is know that the
steam forms a front which propagates through the reservoir and the period
for which steam is injected is dependent on the position of the front.
Therefore accurate tracking of the front is essential, but existing numerical
methods tend to give unsatisfactory oscillatory behaviour in the solution.

In this report we consider both modelling and numerical techniques
for the simpler problem of injecting hot water into a reservoir of cold
water. We have analysed this problem analytically and used a numerical method
which appears to give good resolution without oscillations.

A mathematical model for the simpler problem is developed in section
two of the report, with the resulting algebraic and differential equations
being non-dimensionalised in section three. Analytic solutions are sought

for the non-dimensional equations in section four. In section five a
Finite Element technique (with fixed nodes) is used to obtain numerical
solutions for the equations, the results of which can be seen in section
six, with the conclusions being contained in section seven.

§2 MATHEMATICAL MODEL

As a first step we shall consider the injection of hot water into a porous medium containing cold water, the temperature difference being of the order of 50^0C.

We consider a cylindrical reservoir of radius L   and height h , with a well bore of radius R   dropping vertically through the centre of the reservoir (see fig. 1). The reservoir is considered homogeneous in all rock properties and isotropic with respect to permeability.

Fig. 1
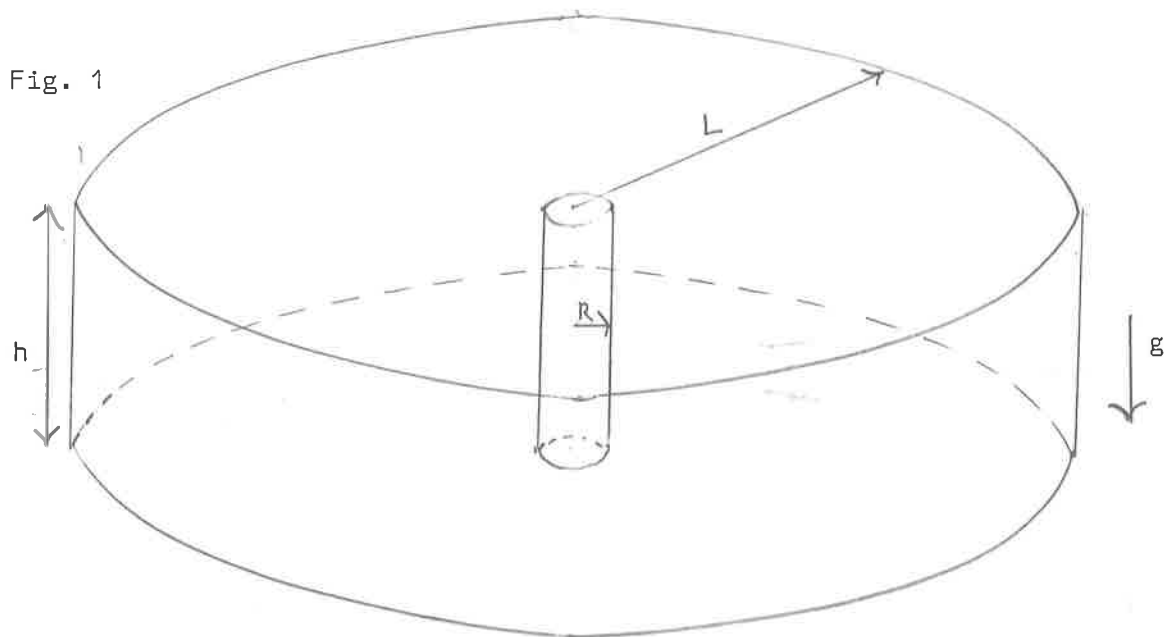
Fluid is injected from the well bore wall into the reservoir. The boundary condition at this point assumes the pressure is radially symmetric with a given pressure distribution vertically within the well bore. For the purpose of this report we shall assume a uniform pressure distribution within the well bore.

The strata lies horizontally with impermeable boundaries, the permeability throughout the rest of the well being homogeneous and isotropic. Therefore if we ignore the effects due to gravity the resulting flow may be assumed to be purely horizontal. As the reservoir is assumed to be radially symmetric with respect to all the rock properties, the flow of a fluid through

the reservoir will be radially symmetric.

§ 2.1   INITIAL AND BOUNDARY CONDITIONS

Initially the reservoir is at constant pressure, density and temperature, P_{int} , ρ_{int} and T_{int} , respectively. At the outer boundary of the reservoir (r = L) there exists an insulated no-flow boundary, ie. no heat or fluid flows across it. Also a no-flow condition is imposed on the top and bottom of the reservoir, but this is automatically satisfied, as we are considering only horizontal flow. For time greater than zero, water is injected at a constant mass rate q at temperature T_{inj} at the well bore (r = R) (see fig.2).

Fig. 2.
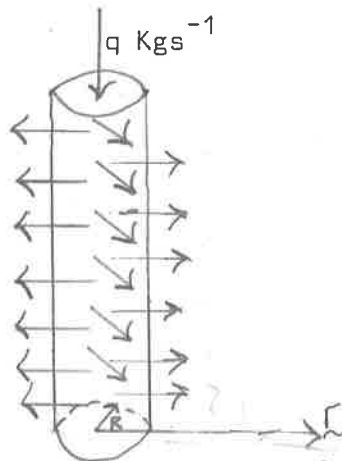
Flow out of the well bore is only in the radial direction. Therefore, as the surface area of the well bore is 2π Rh,

$$\text{flow} = \frac{q}{2\pi Rh} \; \hat{\underset{\sim}{r}} \; Kgm^{-2} \, s^{-1}$$

(mass/unit area/ unit time)

§ 2.2   GOVERNING EQUATIONS

As we are dealing with the flow of a heated liqiud (the heat not being sufficient enough to change the phase of the fluid) through a porous medium, the unknowns in this problem are (I) the temperature of the liquid,(II) the pressure of the liquid, (III) the density of the liquid and (IV) the velocity of the liquid. The other properties, e.g. viscosity of the liquid, density of the rock etc., are taken as constants (see § 2.3.1). Therefore we require four equations for the four unknowns. These are (a) Darcy's law, which relates

the velocity of a fluid in a porous medium to its pressure gradient, (b) conservation of mass, (c) conservation of energy and (d) an equation of state : for this the definition of isothermal compressibility was chosen.

§ 2.2.1   DARCY'S LAW

In 1856 Darcy (see [4]) derived an empirical law for the single phase flow of a fluid through a porous medium. Later King Hubbert (see [8]) obtained the same law from the Navier-Stokes equation of motion for a viscous fluid. The law may be stated in the form.

$$\underset{\sim}{V} = -\frac{K}{\mu} (\nabla p - \rho g \nabla d) \qquad (1)$$

where

$\underset{\sim}{V}$ - superficial velocity (volume/unit area/unit time)

μ - viscosity

p - pressure

ρ - density

g - gravity

d - depth

K - absolute permeability tensor (a function of the rock).

In this report we are ignoring the effects due to gravity and assuming a radially symmetric flow, so (1) simplifies to

$$\underset{\sim}{V} = -\frac{K}{\mu} \underset{\sim}{\nabla} p \qquad (2)$$

where ∇ now takes the form $\underset{\sim}{\hat{r}} \frac{\partial}{\partial r}$

$\underset{\sim}{\hat{r}}$ being the unit vector in the radial direction.

§ 2.2.2   CONSERVATION OF MASS

If we consider an arbitrary volume R with boundary ∂R contained

in the porous region, then conservation of mass yields

$$
\left\{ \begin{array}{l} \text{rate of change of} \\ \text{mass in } R \end{array} \right\} = - \left\{ \begin{array}{l} \text{flow of mass} \\ \text{out of } R \end{array} \right\} + \left\{ \begin{array}{l} \text{mass created in } R \\ \text{per unit volume} \end{array} \right\}
$$

i.e.
$$
\frac{\partial}{\partial t} \int_R \phi \rho_w \, dT = - \int_{\partial R} \rho_w \, \underset{\sim}{V}_w \cdot \hat{\underset{\sim}{dn}} + \int_R q \, dT \quad .
$$

where

$$
\phi - \text{porosity} = \frac{V_{\text{pores}}}{V_{\text{tot}}} = \frac{\text{Volume occupied by pores}}{\text{total volume}}
$$

$\hat{\underset{\sim}{dn}}$ - outward unit normal to R

q - mass injected per volume, which is zéro in the present problem

as fluid is injected or extracted only at the edge of the region.

Therefore mass injected enters only as a boundary condition.

From the divergence theorem

$$
\int_{\partial R} \rho_w \, \underset{\sim}{V}_w \cdot \underset{\sim}{dn} = \int_R \underset{\sim}{\nabla} \cdot \rho_w \, \underset{\sim}{V}_w \, dT
$$

or

$$
\int_R \left\{ \frac{\partial}{\partial t} (\phi \rho_w) + \underset{\sim}{\nabla} \cdot (\rho_w \, \underset{\sim}{V}_w) \right\} dT = 0 \quad ,
$$

and as we are considering an arbitrary region R;

$$
\frac{\partial}{\partial t} (\phi \, \rho_w) + \underset{\sim}{\nabla} \cdot (\rho_w \, \underset{\sim}{V}_w) = 0 \, , \tag{3}
$$

where the operator $\underset{\sim}{\nabla} \cdot \underset{\sim}{E}$ is taken as

$$
\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \frac{1}{r} \frac{\partial}{\partial r} (r \, \hat{\underset{\sim}{r}} \cdot \underset{\sim}{E}) \quad .
$$

§ 2.2.3 CONSERVATION OF ENERGY

Again considering an arbitrary volume R conservation of energy

gives :
$$\begin{pmatrix} \text{rate of change of} \\ \\ \text{internal energy in R} \end{pmatrix} = - \begin{pmatrix} \text{energy conducted} \\ \text{out of R by the} \\ \text{water} \end{pmatrix} - \begin{pmatrix} \text{heat conducted} \\ \text{out of R by} \\ \text{the rock} \end{pmatrix}$$

$$+ \begin{pmatrix} \text{energy} \\ \text{created in R} \\ \text{per unit volume} \end{pmatrix}$$

(ignoring effects due to radiation, because of their small magnitude),

i.e.

$$\frac{\partial}{\partial t} \int_R (\phi \rho_w U_w + (1 - \phi) \rho_r U_r) \, d\tau = \int_R q_h \, d\tau - \int_{\partial R} \rho_w H_w \underset{\sim}{V}_w \cdot d\underset{\sim}{n} - \int_{\partial R} -k_\theta \underset{\sim}{\nabla} T \cdot d\underset{\sim}{n}$$

where

U_w - specific internal energy

H_w - specific enthalpy

k_θ - coefficient of thermal conductivity

q - heat injected into R, taken as zero in the present problem as heat
is injected at the edge of the reservoir: it is therefore again treated
as a boundary condition.

After applying the divergence theorem,

$$\int_R \left\{ \frac{\partial}{\partial t} (\phi \rho_w U_w + (1-\phi) \rho_r U_r) + \underset{\sim}{\nabla} \cdot (\rho_w H_w \underset{\sim}{V}_w) - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T \right\} d\tau = 0 \quad ,$$

but because R is an arbitrary volume, we obtain

$$\frac{\partial}{\partial t} (\phi \rho_w U_w + (1-\phi) \rho_r U_r) + \underset{\sim}{\nabla} \cdot (\rho_w H_w \underset{\sim}{V}_w) - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0 \qquad (4)$$

§ 2.2.4 ISOTHERMAL COMPRESSIBILITY

As we are only dealing with a temperature range of $\sim 50^\circ C$, whereas

the pressure range should be of the order 500psi, we can assume that the density of water is a function of the pressure only. Therefore, from the definition of isothermal compressibility

$$C_w = - \frac{1}{V} \frac{\partial V}{\partial P} \qquad (5)$$

where

C_w - coefficient of isothermal compressibility for water.

But

$$\rho = \frac{m}{V}$$

so

$$C_w = - \frac{\rho}{m} \frac{\partial (m/\rho)}{\partial p} = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \qquad .$$

Integration gives

$$C_w \, p = \log \rho + const \quad ,$$

but initially $\rho = \rho_{int}$ when $p = p_{int}$:

therefore

$$C_w \, p_{int} - \log \rho_{int} = const$$

which gives

$$C_w \, p - C_w \, p_{int} = \log \rho - \log \rho_{int}$$

$$C_w \, (p - p_{int}) = \log (\rho / \rho_{int})$$

$$\rho = \rho_{int} \exp \{ C_w \, (p - p_{int}) \} \qquad (6)$$

A consequence of using the definition of isothermal compressibility as the equation of state is that it decouples the equation for the conservation of mass (3) from the equation for the conservation of energy (4).

§ 2.3 PHYSICAL ASSUMPTIONS

2.3.1 ROCK PROPERTIES

Due to the homogeneity and isotropy of the rock, ϕ , K , k_θ and ρ_r

can be assumed to be constant throughout the reservoir, for all time.

2.3.2 THERMODYNAMIC PROPERTIES

Defining

$$U = \int^{T} C_V \, d\theta$$

$$H = \int^{T} C_p \, d\theta$$

where

C_V - specific heat at constant volume,

C_p - specific heat at constant pressure,

it can be shown that (see [9])

$$U = H - \frac{p}{\rho} \,. \tag{7}$$

If we assume C_p^w and C_p^r are constants and that $H_w = H_r = 0$ when

$T = T_{base}$ (a base temperature)

then

$$H_w = C_p^w (T-T_{base}) \tag{8}$$

$$H_r = C_p^r (T-T_{base}) \,. \tag{9}$$

Also if the pressure in the rock is taken to be the same as in the water,

then by (7)

$$U_w = H_w - \frac{p}{\rho_w} \tag{10}$$

and

$$U_r = H_r - \frac{p}{\rho_r} \,. \tag{11}$$

§ 2.3.3 <u>VISCOSITY</u>

From [10] we take the viscosity of water μ to be given by the approximate formula

$$\frac{1}{\mu} = 0.14 + \theta/30 + 9\times10^{-6} \cdot \theta^2 \ cp^{-1} \qquad (\theta - {}^0C)$$

(see fig 3).

Fig 3

VISCOSITY OF WATER AGAINST TEMP.
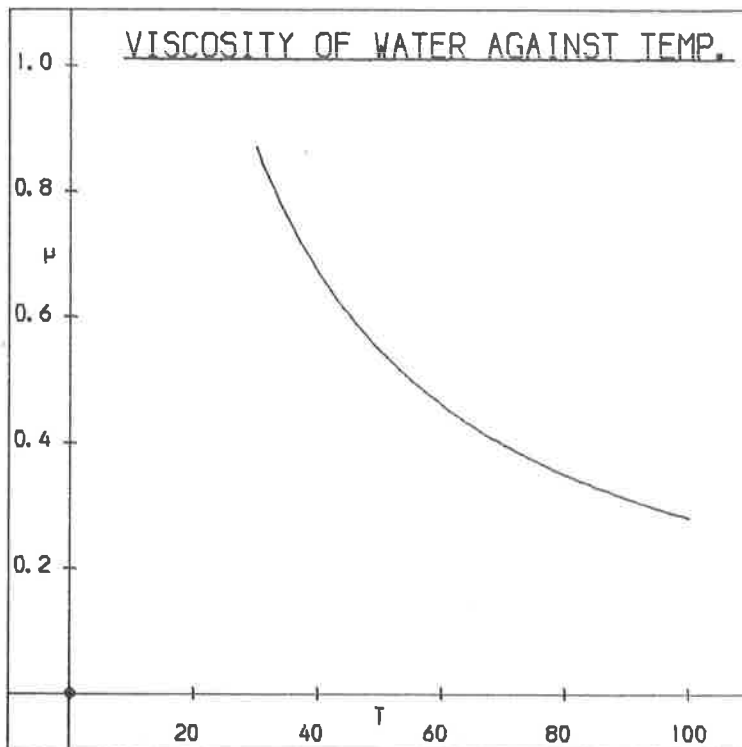
For the temperature range we are interested in it can be seen that μ varies by approximately 0.4 cp. but for the purpose of this report we will assume μ to be constant.

§ 2.4 SIMPLIFICATION OF EQUATIONS

Substituting (10) and (11) into (4) gives

$$\frac{\partial}{\partial t} \left[\phi \rho_w \left(H_w - \frac{p}{\rho_w} \right) + (1 + \phi) \rho_r \left(H_r - \frac{p}{\rho_r} \right) \right] + \underset{\sim}{\nabla} \cdot (\rho_w H_w \underset{\sim}{V}_w) - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0$$

$$\frac{\partial}{\partial t} (\phi \rho_w H_w) + \frac{\partial}{\partial t} ((1-\phi) \rho_r H_r) - \frac{\partial p}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho_w H_w \underset{\sim}{V}_w) - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0$$

$$H_w \frac{\partial}{\partial t} (\phi \rho_w) + \phi \rho_w \frac{\partial H_w}{\partial t} + (1 - \phi) \rho_r \frac{\partial H_r}{\partial t} - \frac{\partial p}{\partial t}$$

$$+ H_w \underset{\sim}{\nabla} \cdot (\rho_w \underset{\sim}{V}_w) + \rho_w \underset{\sim}{V}_w \cdot \underset{\sim}{\nabla} H_w - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0$$

$$H_w \left\{ \frac{\partial}{\partial t} (\phi \rho_w) + \underset{\sim}{\nabla} \cdot (\rho_w V_w) \right\} + \phi \rho_w \frac{\partial H_w}{\partial t} + (1 - \phi) \rho_r \frac{\partial H_r}{\partial t} - \frac{\partial p}{\partial t}$$

$$+ \rho_w \underset{\sim}{V}_w \cdot \underset{\sim}{\nabla} H_w - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0 \ .$$

But from (3)

$$\frac{\partial}{\partial t} (\phi \rho_w) + \underset{\sim}{\nabla} \cdot (\rho_w \underset{\sim}{V}_w) = 0$$

so that equation (4) simplifies to

$$\phi \rho_w \frac{\partial H_w}{\partial t} + (1-\phi) \rho_r \frac{\partial H_r}{\partial t} - \frac{\partial p}{\partial t} + \rho_w \underset{\sim}{V}_w \cdot \underset{\sim}{\nabla} H_w - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0 \ .$$

Then substituting (8) and (9) in the above gives

$$(\phi \rho_w C_p^w + (1-\phi) \rho_r C_p^r) \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} + \rho_w \underset{\sim}{V}_w \cdot C_p^w \frac{\partial T}{\partial r} \hat{\underset{\sim}{r}} - \underset{\sim}{\nabla} \cdot k_\theta \underset{\sim}{\nabla} T = 0$$

and, finally substituting (2), namely

$$\underset{\sim}{V}_w = - \frac{K}{\mu} \frac{\partial p}{\partial r} \hat{\underset{\sim}{r}} \quad \text{into equation (3), and the above equation we obtain}$$

$$\frac{\partial}{\partial t} (\phi \rho_w) - \frac{1}{r} \frac{\partial}{\partial r} (\frac{r\rho_w K}{\mu} \frac{\partial p}{\partial r}) = 0 \qquad (12)$$

and

$$(\phi \rho_w C_p^w + (1-\phi) \rho_r C_p^r) \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} - \frac{\rho_w C_p^w K}{\mu} \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (rk_\theta \frac{\partial T}{\partial r}) = 0 \qquad (13)$$

along with (6)

$$\rho = \rho_{int} \, e^{C_w (p - p_{int})} \, , \qquad (6)$$

which gives us three equations for the unknowns p, T and ρ_w ,

§ 2.5 BOUNDARY CONDITONS

At $r = R$ (well bore radius) water is injected at a constant mass rate q at temperature T_{inj} .

The flow into the reservoir per unit area is

$$\rho_w \, \underset{\sim}{V}_w = \frac{q}{2\pi r h} \, \underset{\sim}{\hat{r}}$$

and, substituting for $\underset{\sim}{V}_w$, we obtain

$$- \frac{K\rho_w}{\mu} \frac{\partial p}{\partial r} = \frac{q}{2\pi R h}$$

which implies that

$$\left. \begin{array}{l} \dfrac{\partial p}{\partial r} = \dfrac{-q\mu}{2\pi R h K \rho_w} \\[3mm] \text{and } T = T_{inj} \, . \end{array} \right\} \quad \text{at } r = R \qquad (14)$$

At $r = L$ (outer boundary) no heat or fluid flows across the boundary, therefore

$$\left. \begin{array}{l} \rho_w \, \underset{\sim}{V}_w = 0 \quad \Rightarrow \quad \dfrac{\partial p}{\partial r} = 0 \\[3mm] \text{and } \dfrac{\partial T}{\partial r} = 0 \end{array} \right\} \quad \text{at } r = L \qquad (15)$$

Summarising,(12), (13) and (6) form our set of three equations,with

boundary conditions (14) and (15) and initial conditions

$$\rho = \rho_{int} \quad , \quad T = T_{int} \quad \text{and} \quad p = P_{int} .$$

§ 3 NON-DIMENSIONAL FORM

We now non-dimensionalise the equations of §2 so as to try and make the

variation in the variables lie in the range 0(1), giving us a better idea of

the size of the coefficients of the differential equations and thus leading

to simplification of the system of equations. Also equation (6) will be assumed

to be a linear function of pressure (see table 1).

Non-dimensional variables are represented by an overbar.

Let P_{max} , ρ_{max} , and t_{max} be the maximum pressure,density,and time

obtained in this problem.

TABLE 1

Relationship between variable and non-dimensionalesed variable	Range of non-dimensionalised variable
$r = L \tilde{r}$	$\tilde{r} \in [R/_L , 1]$
$T = T_{int} + (T_{inj} - T_{int}) \tilde{T}$	$\tilde{T} \in [0 , 1]$
$p = P_{int} + \hat{p} \tilde{p}$	$\tilde{p} \in [0 , (p_{max} - P_{int})/\hat{p}]$
$t = \tau \tilde{t}$	$\tilde{t} \in [0 , t_{max}/\tau]$
$\rho_w = \rho_{int} + b \tilde{p}$	$\rho_w \in [0 , (p_{max} - P_{int})/b \tilde{p}_{max}]$

Let
$$\delta = R/L$$

$$T_{Diff} = T_{inj} - T_{int} ,$$

where $\hat{p}$ is chosen such that $\dfrac{\partial \tilde{p}}{\partial \tilde{r}} = -1$ at $\tilde{r} = \delta$. Then

$$\frac{\partial p}{\partial r} = \frac{\hat{p}}{L}\frac{\partial \tilde{p}}{\partial \tilde{r}} = \frac{-q\,\mu}{2\pi Rh\,\rho_w K} \qquad \text{at} \quad r = R\,, \qquad (r=\delta) \quad \text{by} \; (14)\,,$$

and hence

$$\hat{p} = \frac{q\,\mu\,L}{2\pi Rh\rho_w K}\;. \tag{16}$$

τ is chosen so as to match the most dominant terms in the energy equation.

If we place $p = p_{int} + \hat{p}\,\tilde{p}$ into (6),

$$\rho = \rho_{int}\,e^{c_w\hat{p}\tilde{p}}$$

Then expanding $e^{c_w\hat{p}\tilde{p}}$ in the form

$$\rho = \rho_{int}\left(1 + c_w\hat{p}\tilde{p} + \frac{(c_w\hat{p}\tilde{p})^2}{2!} + \dots\right)$$

where

$$c_w\hat{p}\tilde{p} \sim 10^{-3}$$

$$\rho_{int} \sim 10^{3}$$

we have

$$\rho \sim \rho_{int} + b\,\tilde{p}$$

where

$$b = \rho_{int}\,c_w\hat{p} \sim 2 \quad .$$

Substituting the expressions in table 1 into equations (12) and (13) and the boundary conditions (14) and (15) gives

$$\frac{1}{\tau}\frac{\partial}{\partial \tilde{t}}(\phi(\rho_{int} + b\tilde{p})) - \frac{1}{L^2}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\,\frac{k}{\mu}\,(\rho_{int} + b\tilde{p})\,\frac{\partial}{\partial \tilde{r}}(p_{int} + \hat{p}\tilde{p})\right) = 0$$

and

$$\left(\phi(\rho_{int} + b\,\tilde{p})\, C_p^W + (1-\phi)\rho_r\, C_p^r\right) \frac{1}{\tau} \frac{\partial}{\partial \tilde{t}} (T_{int} + T_{Diff}\,\tilde{T}) - \frac{1}{\tau} \frac{\partial}{\partial \tilde{t}} (p_{int} + \hat{p}\,\tilde{p})$$

$$- \frac{(\rho_{int} + b\tilde{p})\, C_p^W\, K}{\mu\, L^2} \frac{\partial}{\partial \tilde{r}} (p_{int} + \hat{p}\tilde{p}) \frac{\partial}{\partial \tilde{r}} (T_{int} + T_{Diff}\,\tilde{T}) \frac{-1}{L^2 \tilde{r}} \frac{\partial}{\partial \tilde{r}}$$

$$(\tilde{r}k_\theta \frac{\partial}{\partial \tilde{r}} (T_{int} + T_{Diff}\,\tilde{T})) = 0$$

with

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = -1 \ \text{ at }\ \tilde{r} = \delta \ ,\ \frac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \ \text{ at }\ \tilde{r} = 1 \ ,\ \tilde{p} = 0 \ \text{ at }\ \tilde{t} = 0$$

and

$$\tilde{T} = 1 \ \text{ at }\ \tilde{r} = \delta,\ \tilde{t} > 0 \ ,\ \frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \ \text{ at }\ \tilde{r} = 1 \ ,\ \tilde{T} = 0 \ \text{ at }\ \tilde{t} = 0 \ .$$

This simplifies to

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} - \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \left\{ \frac{K\, \rho_{int}\, \hat{p}\tau}{\mu\phi bL^2} + \frac{Kb\hat{p}\tau\, \tilde{p}}{\mu\phi bL^2} \right\} \frac{\partial \tilde{p}}{\partial \tilde{r}} \right) = 0 \tag{17}$$

and

$$(\phi\, \rho_{int}\, C_p^W + (1-\phi)\, \rho_r\, C_p^r) \frac{T_{Diff}}{\tau} \frac{\partial \tilde{T}}{\partial \tilde{t}} + \frac{\phi b\, C_p^W\, T_{Diff}}{\tau} \tilde{p} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{\hat{p}}{\tau} \frac{\partial \tilde{p}}{\partial \tilde{t}}$$

$$- \frac{\rho_{int}\, C_p^W\, K\, \hat{p}\, T_{Diff}}{\mu L^2} \frac{\partial \tilde{p}}{\partial \tilde{r}} \frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{b\, C_p^W\, K\, \hat{p}\, T_{Diff}}{\mu L^2} \tilde{p} \frac{\partial \tilde{p}}{\partial \tilde{r}} \frac{\partial \tilde{t}}{\partial \tilde{t}} - \frac{k_\theta\, T_{Diff}}{L^2} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} (\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}}) = 0$$

$$\tag{18}$$

From the data set (see [11] given in appendix A, the two dominant terms of the energy equation are

$$(\phi \, \rho_{int} \, C_p^W + (1-\phi) \, \rho_r \, C_p^r) \, \frac{T_{Diff}}{\tau} \qquad = f_1 \;\; , \; \text{say}$$

and

$$\frac{\rho_{int} \, C_p^W \, K \, \hat{p} \, T_{Diff}}{\mu \, L^2} \qquad = h_1 \;\; , \; \text{say} \;\; .$$

Thus τ is chosen such that $h_1 = f_1$

$$\tau = \hat{\tau} f_1 \;\; = \;\; \frac{(\phi \, \rho_{int} \, C_p^W + (1-\phi) \, \rho_r \, C_p^r) \, T_{Diff} \, \mu L^2}{\rho_{int} \, C_p^W \, K \, \hat{p} \, T_{Diff}} \;\; .$$

Now substitute τ into (17) and (18) and then divide equation (18) by the coefficient of

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} = f_1$$

to give the two non-dimensional differential equations which follow.

We have

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} - \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \, (\tilde{r} \, (\alpha + \beta \, \tilde{p}) \, \frac{\partial \tilde{p}}{\partial \tilde{r}}) = 0 \qquad\qquad (19)$$

with

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = -1 \;\; \text{at} \;\; \tilde{r} = \delta$$

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \;\; \text{at} \;\; \tilde{r} = 1$$

$$\tilde{p} = 0 \quad \text{at} \quad \tilde{t} = 0$$

and

$$(1+\lambda\tilde{p}) \frac{\partial \tilde{T}}{\partial \tilde{t}} - (1+w\tilde{p}) \frac{\partial \tilde{p}}{\partial \tilde{r}} \frac{\partial \tilde{T}}{\partial \tilde{r}} - \frac{\nu}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}} \right) = \eta \frac{\partial \tilde{p}}{\partial \tilde{r}} \qquad (20)$$

with

$$\tilde{T} = T_{inj} \quad \text{at} \quad \tilde{r} = \delta, \tilde{t} > 0,$$

$$\frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \quad \text{at} \quad \tilde{r} = 1,$$

$$\tilde{T} = 0 \quad \text{at} \quad \tilde{t} = 0 \quad .$$

Again from the data set of appendix A the coefficients may be calculated as

$$\alpha = 1687.80$$

$$\beta = 3.643$$

$$\lambda = 5.92488 \times 10^{-4}$$

$$\omega = 2.15843 \times 10^{-3}$$

$$\nu = 1.38088 \times 10^{-4}$$

$$\eta = 3.0395 \times 10^{-2} \quad .$$

§ 4 ANALYTIC SOLUTIONS

In this section we shall simplify the equations of §3 and seek some analytic solutions.

As β is small compared to α in equation (19) , we can as a first approximation place $\beta = 0$ in (19) giving
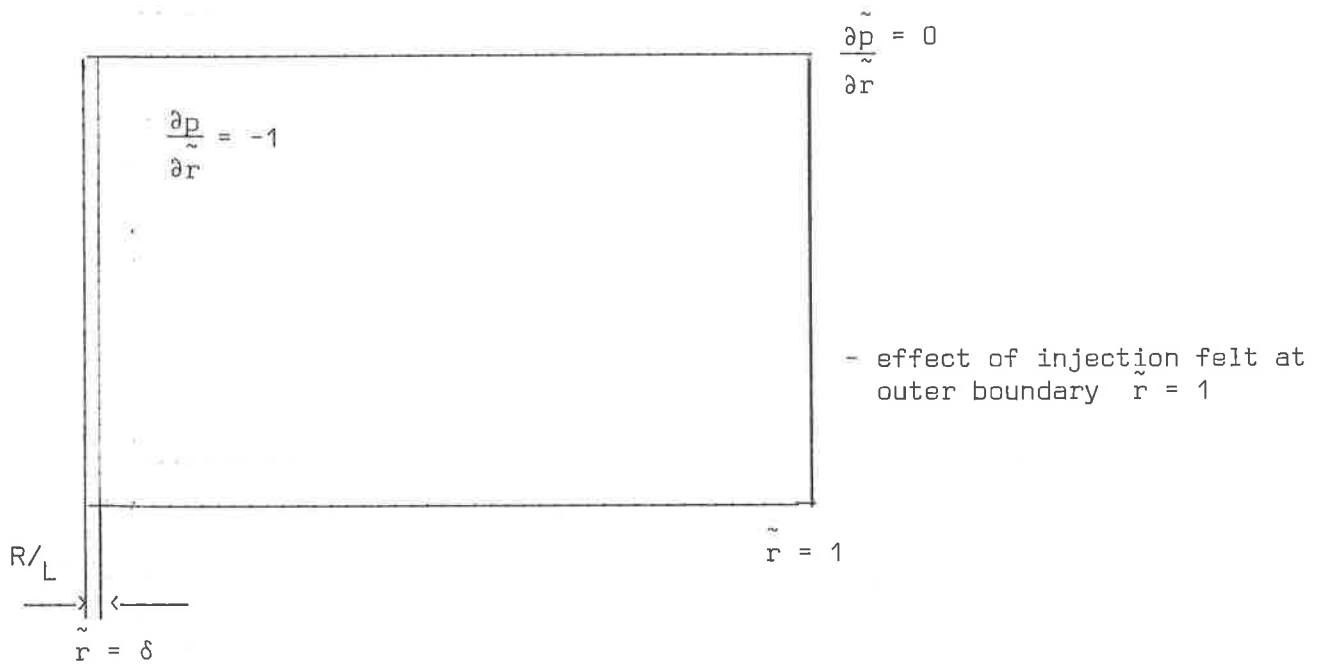
$$\frac{\partial \tilde{p}}{\partial \tilde{t}} - \frac{\alpha}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \tilde{p}}{\partial \tilde{r}} \right) = 0 \qquad (21)$$

with

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \quad \text{at} \quad \tilde{r} = 1 \; ,$$

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = 1 \quad \text{at} \quad \tilde{r} = \delta \; .$$

Fig 4

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = 0$$

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = -1$$

- effect of injection felt at outer boundary $\tilde{r} = 1$

$\tilde{r} = 1$

R/L

$\tilde{r} = \delta$

After the injection has been taken place for a sufficient period of time – such that the effect of the outer boundary $(r = 1)$ has been felt (see fig 4) – a semi-steady state condition occurs, that is to say if the injection rate is constant, then the reservoir pressure will increase in such a way that

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} \qquad \text{constant for all } r \text{ and } t \text{ (see [4]).}$$

From (5) we have

$$C_w = -\frac{1}{V} \frac{\partial V}{\partial p} \quad \text{and} \quad C_w V \frac{dp}{dt} = -\frac{dv}{dt} = \text{Volume rate of injected fluid.}$$

But the volume rate of injected fluid $= \dfrac{q}{\rho_W}$, so that

$$\frac{dp}{dt} = \frac{q}{C_W V \rho_W} \quad , \qquad\qquad V = \pi \, (1-R)^2 \, \phi h \ ,$$

Hence

$$\frac{dp}{dt} = \frac{q}{C_W \, \pi (L-R)^2 \, h\rho\phi} = \frac{\hat{p}}{T} \, \frac{\partial \tilde{p}}{\partial \tilde{t}}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} = \frac{q \, T}{\hat{p} \, C_W \, \pi (L-R)^2 \, h\rho\phi} = \Psi(\sim 0.901) \quad . \tag{22}$$

Then (19) becomes

$$\frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \, \frac{\partial \tilde{p}}{\partial \tilde{r}} \right) = \frac{\tilde{r}\Psi}{\alpha} = \tilde{r}\sigma \qquad \text{where} \quad \sigma = \frac{\Psi}{\alpha} \sim 2\delta$$

$$\tilde{r} \, \frac{\partial \tilde{p}}{\partial \tilde{r}} = \frac{\tilde{r}^2}{2} \sigma + C \quad (C = C(\tilde{t}));$$

when $\quad \tilde{r} = 1 \quad \dfrac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \quad \forall \tilde{t}$

which implies that

$$C = -\frac{\sigma}{2} \sim \delta \quad .$$

Letting $\quad \sigma = 2\delta$

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = \delta\tilde{r} - \frac{\delta}{\tilde{r}}$$

$$\left\{ \text{note that when } \tilde{r} = \delta, \frac{\partial \tilde{p}}{\partial \tilde{r}} = -1 + \delta^2 \sim -1 \right\}$$

$$\tilde{p} = \frac{\delta}{2} \, \tilde{r}^2 - \delta \log r + C_o \qquad C_o = C_o(\tilde{t}) \quad .$$

Then differentiating with respect to t,

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} = \frac{\partial C_0}{\partial \tilde{t}} = \Psi \qquad C_0 = \Psi \tilde{t} + const$$

so that

$$\tilde{p} = \frac{\delta}{2} \tilde{r}^2 - \delta \log \tilde{r} + \Psi \tilde{t} + const., \qquad (23)$$

which gives an equation for the non-dimensional pressure in the reservoir.

To solve equation (20) we require $\frac{\partial \tilde{p}}{\partial r}$ which from (23) is

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = \delta \tilde{r} - \frac{\delta}{\tilde{r}}$$

Fig. 5

delta/r & delta.r against r.
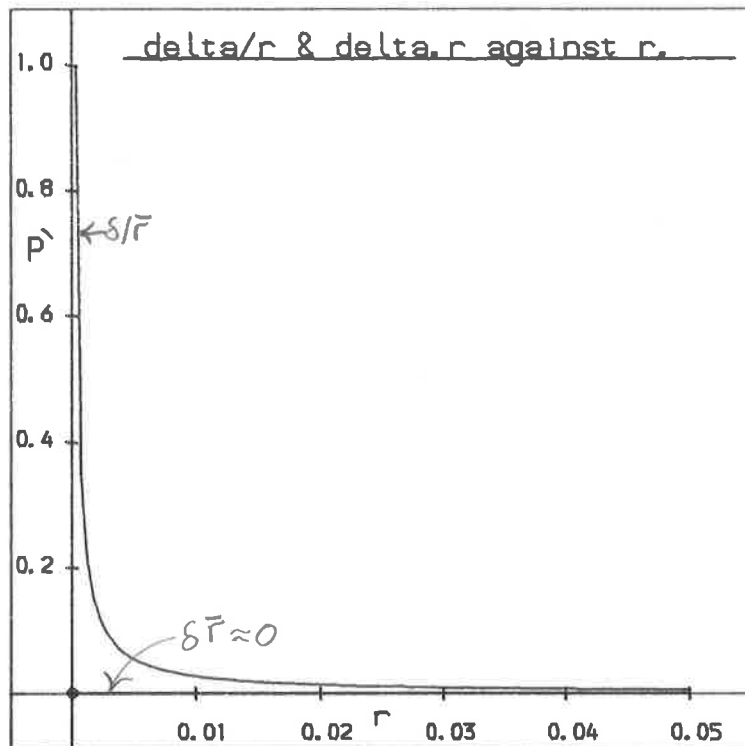
From Fig. 5 it can be seen that for $\tilde{r} < 0.05$, $\delta \tilde{r} \sim 0$, therefore

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} \sim \frac{\delta}{\tilde{r}} \qquad (24)$$

Substituting into equation (20) gives

$$(1+\lambda\tilde{p}) \frac{\partial\tilde{T}}{\partial\tilde{t}} + (1+w\tilde{p}) \frac{\delta}{\tilde{r}} \frac{\partial\tilde{T}}{\partial\tilde{r}} - \frac{\nu}{\tilde{r}} \frac{\partial}{\partial\tilde{r}} (\tilde{r} \frac{\partial\tilde{T}}{\partial\tilde{r}}) = \eta \frac{\partial\tilde{p}}{\partial\tilde{r}} \quad .$$

Next if we assume that λ , w and η are zero
we have

$$\frac{\partial\tilde{T}}{\partial\tilde{t}} + \frac{\delta}{\tilde{r}} \frac{\partial\tilde{T}}{\partial\tilde{r}} - \frac{\nu}{\tilde{r}} \frac{\partial}{\partial\tilde{r}} (\tilde{r} \frac{\partial\tilde{T}}{\partial\tilde{r}}) = 0 \qquad (25)$$

with boundary conditions

$$\tilde{T} = 0 \quad \text{at} \quad \tilde{t} = 0 \quad \forall \quad \tilde{r}$$

$$\tilde{T} = 1 \quad \text{at} \quad \tilde{r} = \delta \quad \tilde{t} > 0$$

$$\frac{\partial\tilde{T}}{\partial\tilde{r}} = 0 \quad \text{at} \quad \tilde{r} = 1 \quad \forall \quad \tilde{t} \quad .$$

If we assume that $\tilde{T} = 1$ at $\tilde{r} = 0$ for $\tilde{t} > 0$

and instead of $\frac{\partial\tilde{T}}{\partial\tilde{r}} = 0$ at $\tilde{r} = 1$ we use $\tilde{T} = 0$ at $\tilde{r} = \infty \quad \forall \quad \tilde{t}$

we can find a similarity solution to (25) .

Let $\tilde{T}(\tilde{r},\tilde{t}) = \tilde{t}^{\alpha} y(x)$, $\quad x = \frac{\tilde{r}}{\tilde{t}^{\beta}}$ $\qquad (26)$

$$\frac{\partial\tilde{T}}{\partial\tilde{t}} = \alpha\tilde{t}^{-\alpha-1} y + \tilde{t}^{\alpha} y_x X_{\tilde{t}}$$

$$X_{\tilde{t}} = -\beta\tilde{r}\tilde{t}^{-\beta-1} = -\frac{\beta\tilde{r}\tilde{t}^{-1}}{\tilde{t}^{\beta}} = -\beta x\tilde{t}^{-1}$$

so

$$\frac{\partial\tilde{T}}{\partial\tilde{t}} = \tilde{t}^{\alpha-1} (\alpha y - \beta x y_x) \qquad (27)$$

$$\frac{\partial \tilde{T}}{\partial \tilde{r}} = \tilde{t}^{\alpha} y_x \tilde{X}_r$$

$$\tilde{X}_r = \tilde{t}^{-\beta}$$

$$\frac{\partial \tilde{T}}{\partial \tilde{r}} = \tilde{t}^{\alpha-\beta} y_x \tag{28}$$

$$\frac{\partial}{\partial \tilde{r}} (\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}}) = \tilde{t}^{\alpha-\beta} \frac{\partial}{\partial \tilde{r}} (\tilde{r} y_x) = \tilde{t}^{\alpha-\beta} (y_x + \tilde{r} y_{xx} \tilde{X}_r)$$

$$= \tilde{t}^{\alpha-\beta} (y_x + \tilde{r} y_{xx} \tilde{t}^{-\beta})$$

$$= \tilde{t}^{\alpha-\beta} y_x + \tilde{t}^{\alpha-2\beta} \tilde{r} y_{xx} . \tag{29}$$

Substituting (27), (28) and (29) into (25) gives

$$\tilde{t}^{\alpha-1} (\alpha y - \beta x y_x) + \frac{\delta}{\tilde{r}} \tilde{t}^{\alpha-\beta} y_x - \frac{\nu}{\tilde{r}} t^{\alpha-\beta} y_x - \nu \tilde{t}^{\alpha-2\beta} y_{xx} = 0 .$$

Next substituting (26) $x = \frac{\tilde{r}}{\tilde{t}^{\beta}}$ into the above gives

$$t^{\alpha-1} (\alpha y - \beta x y_x) + \frac{\delta t^{\alpha-2\beta}}{x} y_x - \frac{\nu}{x} t^{\alpha-2\beta} - \nu t^{\alpha-2\beta} y_{xx} = 0$$

or

$$\tilde{t}^{\alpha-1} (\alpha y - \beta x y_x) + \tilde{t}^{\alpha-2\beta}((\delta-\nu) \frac{y_x}{x} - \nu y_{xx}) = 0 .$$

For a solution to exist

$$\tilde{t}^{\alpha-1} = \tilde{t}^{\alpha-2\beta} \quad \text{which implies that}$$

$$\alpha-1 = \alpha-2\beta \quad \text{or} \quad \beta = \tfrac{1}{2} .$$

which gives

$$(\alpha y - \tfrac{1}{2} x y_x) + \left((\delta - \nu) \frac{y_x}{x} - \nu y_{xx} \right) = 0 \quad . \tag{30}$$

The new boundary conditions are

(a) $\tilde{T} = 1$ at $\tilde{r} = 0$, $\tilde{t} > 0$ $\Rightarrow$ $1 = \tilde{t}^{\alpha} y(0)$ $\forall$ $\tilde{t} > 0$

(b) $\tilde{T} = 0$ at $\tilde{t} = 0$ $\forall \tilde{r}$ $\Rightarrow$ $0 = \lim_{t \to 0} \tilde{t}^{\alpha} y(x)$ $\forall \tilde{r}$ $x \to \infty$

(c) $\tilde{T} = 0$ as $\tilde{r} \to \infty$ $\forall t$ $0 \to \tilde{t}^{\alpha} y(x)$ $x \to \infty$

From (a) $\alpha = 0$ and $y(0) = 1$

and from (c) $y(\infty) = 0$

which gives from (26)

$$\tilde{T} = y(x) \quad \text{where} \quad x = \frac{\tilde{r}}{\tilde{t}^{\frac{1}{2}}} \quad .$$

Substituting α into (30) gives us an ODE for y ,

namely

$$-\tfrac{1}{2} x \, y_x + (\delta - \nu) \frac{y_x}{x} - \nu y_{xx} = 0 \quad ,$$

$$y_{xx} + \frac{1}{2\nu} x y_x - \frac{(\delta - \nu)}{\nu x} y_x = 0 \quad .$$

Letting $\iota = \dfrac{1}{2\nu}$ $\kappa = \dfrac{(\delta - \nu)}{\nu}$

where from the data set [11]

$\iota = 3620.8794$

$\kappa = 0.931136$

we obtain

$$y'' + \left(\iota x - \frac{\kappa}{x} \right) y' = 0 \tag{31}$$

Then letting $\quad z = y'$, $\qquad z' = y''$,

$$\frac{z'}{z} = \frac{(\kappa - \iota x)}{x}$$

$$\log z = \kappa \log x - \frac{1}{2} x^2 + \log c \quad (\text{c-const.})$$

or

$$\log y' = \log c \, x^{\kappa} - \frac{1}{2} x^2$$

$$y' = c \, x^{\kappa} \, e^{-\frac{\iota x^2}{2}}$$

giving

$$y = \int_{0}^{x} c \, x^{\kappa} \, e^{-\frac{\iota x^2}{2}} \, dx + d \quad (\text{d-const}) .$$

When $\quad x = 0$, $\quad y = 1$ $\quad$ we find that $\quad d = 1$

and when $\quad x = \infty$, $\quad y = 0$,

$$y = 1 + c \int_{0}^{\infty} x^{\kappa} \, e^{-\frac{\iota x^2}{2}} \, dx = 0$$

so that

$$c = - \frac{1}{\displaystyle\int_{0}^{\infty} x^{\kappa} \, e^{-\frac{\iota x^2}{2}} \, dx} .$$

Therefore

$$\tilde{T}(\tilde{r}, \tilde{t}) = y(x) = 1 - \frac{\displaystyle\int_{0}^{x} x^{\kappa} \, e^{-\frac{\iota x^2}{2}} \, dx}{\displaystyle\int_{0}^{\infty} x^{\kappa} \, e^{-\frac{\iota x^2}{2}} \, dx}$$

where $x = \dfrac{\tilde{r}}{\tilde{t}^{\frac{1}{2}}}$.

As $k = 0.9311$, the approximation $k = 1.00$ will lead to a simplification of the above equations, namely,

$$\int_0^x x\, e^{-\frac{\imath x^2}{2}}\, dx = \left[-\frac{1}{\imath} e^{-\frac{\imath x^2}{2}} \right]_0^x = \frac{1}{\imath} \{1 - e^{-\frac{\imath x^2}{2}} \}$$

Then $\displaystyle\lim_{x\to\infty} \int_0^x x e^{-\frac{\imath x^2}{2}}\, dx = \lim_{x\to\infty} \frac{1}{\imath} \{ 1 - e^{-\frac{\imath x^2}{2}} \} = \frac{1}{\imath}$

which gives

$$y = 1 - \dfrac{1/\imath\, \{1 - e^{-\frac{\imath x^2}{2}} \}}{1/\imath} = e^{-\frac{\imath x^2}{2}}$$

therefore

$$\tilde{T}(\tilde{r}, \tilde{t}) = y = e^{-\frac{\imath x^2}{2}} \qquad \text{with} \quad x = \tilde{r}/\tilde{t}^{\frac{1}{2}}$$

$$\tilde{T}(\tilde{r}, \tilde{t}) = e^{-\frac{\imath \tilde{r}^2}{2\tilde{t}}} \qquad\qquad \text{for small } \tilde{r} \tag{32}$$

which along with (23) gives us two analytic solutions for the presssure and temperature in the reservoir.

§ 5 NUMERICAL METHODS

We require a numerical method capable of solving the system of non-dimensional differential equations of § 3. Although the majority of oil reservoir simulators use finite difference techniques on a fixed rectangular grid (see [2.], [5], and [6]), we shall use a finite element method (see [13]) with fixed nodes on an irregular grid. This method was chosen for two reasons: firstly further simplification of equation

(19) yields a linear parabolic equation and it is known the finite element method gives good results for this type of problem (see [13]) and secondly it is anticipated that it may be necessary to use finite element techniques with moving nodes in future work.

§ 5.1 THE FINITE ELEMENT METHOD

As in this section of the report we shall always be dealing with the non-dimensional form of the equations, we shall ignore the overbars and assume each variable is in its non-dimensional form.

We seek semi-discrete approximations for the unknowns p and T , in the form

$$p(r , t) = \sum_{i=1}^{n+1} p_i(t) \, \phi_i(r) \tag{33}$$

$$T(r , t) = \sum_{i=1}^{n+1} T_i(t) \, \phi_i(r) \tag{34}$$

where p_i and T_i are coefficients and ϕ_i are linear basis functions (see fig (6)).

The region of solution is the line $r = \delta$ to $r = 1$ (using the symmetry of the problem), divided into n intervals (elements).
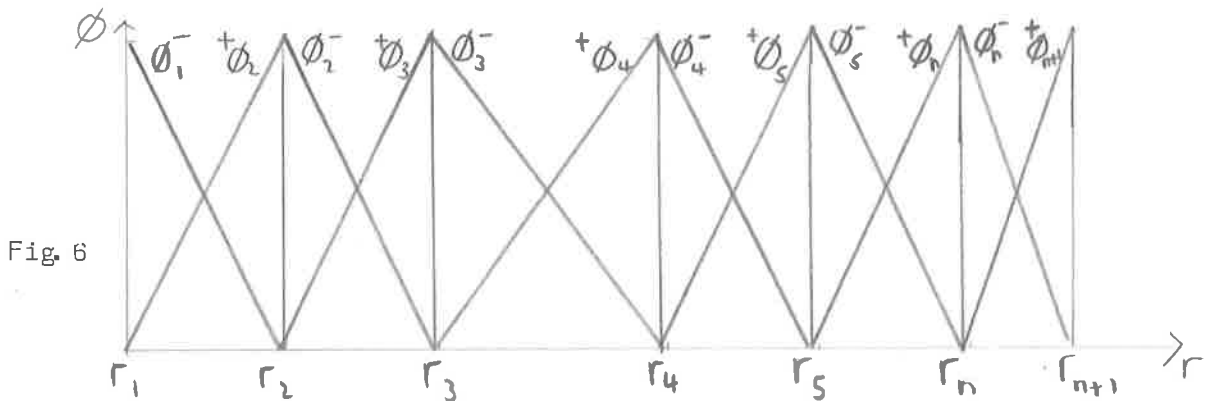
Fig. 6

From Fig 6 we note that

$$\phi_1 = \phi_1^-$$

$$\phi_{n+1} = \phi_{n+1}^+$$

and $\quad \phi_i = \phi_i^+ \quad \phi_i^- \quad i = 2 , \ldots , n .$

If we write equations (19) and (20) of §4 as

$$P_t - L(p) = 0 \tag{35}$$

and

$$(1+\lambda p) \, T_t + \eta P_t - L'(T , p) = 0 , \tag{36}$$

where $\quad L(p) = \dfrac{1}{r} \dfrac{\partial}{\partial r} \left(r(\alpha+\beta p) \dfrac{\partial p}{\partial r} \right)$

and

$$L'(T , P) = (1+w) \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} + \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

then ODE's for the 2(n+1) unknowns $P_1 \, P_2, \ldots , P_{n+1}$ and $T_1 , T_2 , \ldots ,$

T_{n+1} can be obtained by taking a Galerkin weak form of (35) and (36)

using test functions $\phi_i \quad i = 1, \ldots , n+1$, to give

$$\langle \, P_t + L(p) \, , \quad \phi_i \, \rangle = 0 \tag{37}$$

$$\langle \, (1+\lambda p) \, T_t + \eta P_t + L'(T , p) \, , \, \phi_i \, \rangle = 0 \tag{38}$$

$$i = 1, \ldots , n+1$$

where

$$\langle a , b \rangle = \int_\tau a b \, d\tau .$$

In this case τ represents the volume of the reservoir,

with

$$\delta < r < 1$$

$$0 < z < h/L$$

$$0 < \theta < 2\pi$$

and $d\tau = rdrd\theta dz$, so that

$$\langle a , b \rangle = \int_o^{h/L} \int_o^{2\pi} \int_\delta^1 ab \ rdrd\theta dz$$

$$\langle a , b \rangle = \frac{2\pi h}{L} \int_\delta^1 ab \ rdr \tag{39}$$

Due to the lower triangular coupling of the equations equation (37) may be solved first, and the values obtained for the p_i's used in the solution of (38).

By considering the term $\alpha + \beta p$ in equation (19) and using the values obtained for α and β in §3 it can be seen that $(\alpha+\beta p) = 1687.80 \ (1+0.0022p)$. As it is thought that $p \in [0,1]$, it may be assumed for the purpose of this report that $\beta = 0$, therefore simplifying equation (19) to

$$\frac{\partial p}{\partial t} - \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) = 0 \quad ,$$

$$p = 0 \quad \text{at} \quad t = 0$$

$$\frac{\partial p}{\partial r} = -1 \quad \text{at} \quad r = \delta , t > 0$$

$$\frac{\partial p}{\partial r} = 0 \quad \text{at} \quad r = 1 , \forall t .$$

From the analytic solution of §5 it can be seen that

$$p = -\delta \log r + O(r^2) \quad , \quad \delta < r < 1 \quad ;$$

therefore as $p = -\delta \log r$ satisfies (40), it was decided to transform the variable p to $\tilde{p}$, where

$$p = \tilde{p} - \delta \log r \quad .$$

This has the effect of changing the initial and boundary conditions of equation (40) to

$$\tilde{p} = \delta \log r \text{ at } t = 0 \ ,$$

$$\frac{\partial \tilde{p}}{\partial r} = 0 \text{ at } r = \delta \; \forall \, t \ ,$$

$$\frac{\partial \tilde{p}}{\partial r} = \delta \text{ at } r = 1 \; t > 0 \quad .$$

Having non-zero initial data allows us to equi-distribute the nodes (see [14]) with respect to $f = \delta \log r$. This has the effect of placing the majority of the nodes near $r = \delta$, where it is expected that the greatest disturbance will occur. The new boundary conditions have the effect of introducing a small flux into the region at $r = 1$ with a no-flow condition at $r = \delta$, rather than a large flux at $r = \delta$ and a no-flow condition at $r = 1$.

Taking the weak form of (40) with test functions ϕ_i leads to a set of $n+1$ linear ODE's

$$M \, \tilde{\underset{\sim}{P}} = \alpha \, K \, \tilde{\underset{\sim}{P}} + \underset{\sim}{L} \tag{41}$$

where $\tilde{\underset{\sim}{P}} = (\tilde{p}_1 , \tilde{p}_2 , \cdots , \tilde{p}_{n+1})^T$,

M being a mass matrix , K a stiffness matrix and L a load vector obtained from the Neuman boundary conditions. From the nature of the inner product (39), both K and M are symmetric.

A simple one step θ time-stepping scheme is employed to reduce (41) to $n+1$ linear eqations

$$M \left(\frac{\underset{\sim}{\tilde{p}}^{k+1} - \underset{\sim}{\tilde{p}}^{k}}{\Delta t} \right) = \theta \alpha K \underset{\sim}{\tilde{p}}^{k+1} + (1-\theta) \alpha K \underset{\sim}{\tilde{p}}^{k} + \underset{\sim}{L} \quad . \tag{42}$$

$\underset{\sim}{\tilde{p}}^{k+1}$ can then be obtained by applying an LU solver to the above

with

$$p_i^{k+1} = \tilde{p}_i^{k+1} - \delta \log r_i \quad .$$

With $\underset{\sim}{p}$ known, (20) can be solved. This was done first with $\lambda = w = \eta = 0$

and later with $\eta \neq 0$.

After taking the weak form of (20) with test functions θ_i , we obtain the set of

η linear ODE's (as $T_1 = 1$ for $t > 0$)

$$M \underset{\sim}{T} = \tilde{K} \underset{\sim}{T} + \nu K \underset{\sim}{T} + \underset{\sim}{\tilde{L}} \quad , \tag{43}$$

where M and K are the same mass and stiffness matrixes, as derived in

equation (41), but K is a different stiffness matrix obtained from

$$\langle \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} , \phi_i \rangle \quad \forall_i \quad ,$$

so that $\tilde{K} = \tilde{K}(\underset{\sim}{p})$ and $\underset{\sim}{\tilde{L}}$ is a load vector, obtained from the Neuman

boundary conditions which in the present case is always zero.

Note that $\tilde{\underset{\sim}{K}}$ is unsymmetric.

Again using a one step θ time stepping scheme we can reduce (43) to n

linear equations.

$$M \left(\frac{\underset{\sim}{T}^{k+1} - \underset{\sim}{T}^{k}}{\Delta t} \right) = \theta_1 \tilde{K}^{k+1} \underset{\sim}{T}^{k+1} + (1-\theta_1) \tilde{K}^{k} \underset{\sim}{T}^{k} + \theta_2 \nu K \underset{\sim}{T}^{k+1} + (1-\theta_2) \nu K \underset{\sim}{T}^{k} \tag{44}$$

and T^{k+1} may be obtained by applying an LU solver to the above.

If we take $\eta \neq 0$, then we have an extra term on the right hand side,

namely

$$\langle \, \eta \, \frac{\partial p}{\partial t} \, , \, \phi_i \, \rangle \quad , $$

but if we only consider times after which p has reached semi-steady state,
then by the analysis of §5 ,

$$\frac{\partial p}{\partial t} \underset{\sim}{} \text{constant}$$

and this can be calculated from the definition of isothermal compressibility
as

$$\frac{\partial p}{\partial t} = \Psi \quad .$$

Therefore taking $\eta \neq 0$ changes (43) into

$$M \, \underset{\sim}{T} = K \, \underset{\sim}{T} + \nu \, K \, \underset{\sim}{T} + \underset{\sim}{\tilde{L}} + \underset{\sim}{\hat{L}} \tag{45}$$

where $\underset{\sim}{\hat{L}} = \langle \, \eta \, \Psi \, , \, \phi_i \, \rangle \quad , \quad i = 1 , \, \ldots \, , \, n+1$

Again a θ time stepping scheme was used to reduce (45) to n linear equations

$$M \, \frac{(T^{k+1} - T^k)}{\Delta t} = \theta \, \underset{\sim}{\tilde{K}}^{k+1} \, \underset{\sim}{T}^{k+1} + (1-\theta) \, \underset{\sim}{\tilde{K}}^{k} \, \underset{\sim}{T}^{k} + \theta_2 \, \nu K T^{k+1} + (1-\theta_2) \, \nu K T^{k} + \underset{\sim}{\hat{L}}$$

$$(\underset{\sim}{\tilde{L}} = 0)$$

and a LU solver was used to calculate $\underset{\sim}{\tilde{T}}^{k+1}$

§ 5.2 RELATIVE NORMS

If we have two piecewise linear functions f and g defined on the
interval $[a , b]$, we often require a measure of the relative difference
between them. If f and g are defined on the same grid, then a relative
pointwise norm L_r , may be defined as

$$L_r^2 = \frac{\displaystyle\sum_{i=1}^{n+1} f_i - g_i{}^2}{\displaystyle\sum_{i=1}^{n+1} \left(\frac{f_i + g_i}{2}\right)^2}$$

there being n intervals and f_i and g_i being the values of f and g at node i , respectively. However if f and g are defined on different grids with possibly different numbers of nodes a more general relative L_2 norm L_r must be used to measure the relative difference between f and g , derived from

$$L_r^2 = \frac{\displaystyle\int_a^b (f-g)^2 \, dr}{\displaystyle\int_a^b \left(\frac{f+g}{2}\right)^2 dr}$$

with appropriate quadrature.

§ 6 NUMERICAL RESULTS

§ 6.1 Pressure

Up to a time of $t \sim 0.0001$ transient flow occurs. This is due to the fact that the effects of the outer boundary of the reservoir have not yet affected the flow and can be seen in fig (7) where a fully implicit scheme ($\theta=1$) is used to solve equation (42) (as for the rest of the results).

Figures (8) and (9) show the pressure distribution for times up to $t=0.01$, the result for fig (9) having twice as many elements as the result for fig (8). A relative L_2 norm for the difference between them gave $L_r \neq 9.527 \times 10^{-4}$, while a relative pointwise norm between fig (9) and the analytic solution at

this time fig (10) gave $L_r = 3.2642 \times 10^{-3}$. Figures (11) and (12)

show the numerical and analytic solution up to $t = 1.00$, the relative

pointwise norm between these results being $L_r = 9.5792 \times 10^{-4}$

§ 6.2 Temperature (η=0)

Placing $\eta = 0$ has the physical effect of putting the internal energy
equal to the enthalpy.

The results shown in figures (13) and (14) show the temperature up to
time $t = 1.00$ (using implicit time stepping) with the results in figure (4)
having twice the number of nodes as the results in figure (13). A relative
L_2 norm for the difference between the two solutions gave $L_r = 2.4318 \times 10^{-3}$,
while a relative pointwise error between fig (14) and the approximate analytic
solution in fig (15) gave $L_r = 1.2328 \times 10^{-2}$. Figure (16) shows the
temperature up to t = 1.00 but with $\theta_1 = 0$ (the convection term calculated
using explicit time stepping), a pointwise norm between this and the analytic
solution giving $L_r = 1.2652 \times 10^{-2}$

§ 6.3 Temperature(η ≠ 0.)

Having $\eta \neq 0$ should allow for heating by compressibility, thus
increasing the temperature through out the whole reservoir, not only near
the well bore as in the case $\eta = 0$. This phenomenon can clearly be seen
in figures (17) and (18), the results of figure (17) being calculated using
a fully implicit time stepping scheme $(\theta_1 = \theta_2 = 1)$, whereas the result
in figure (18) uses a time stepping scheme with $\theta = 1$ (explicit on the
convection term), a relative L_2 norm for the difference between the two
results giving $L_r = 3.7417 \times 10^{-3}$.

§ 6.4 Convective flow

It was next decided to check the robustness of the numerical method by letting $\nu \to 0$, the flow being purely convective when $\nu = 0$. Physically, this is equivalent to lowering the thermal conductivity of the rock until the heat transfer is only by convection in the water.

Figures (19) and (20) show that ν can be reduced by factors of 10 and 20 respectively, and the method will still reproduce the solution with 100 elements. However if ν is reduced by a factor of 100, figure (21) shows that oscillations occur using 100 elements, but if 200 elements are used a smooth solution is again obtained (fig (22)) similarly, if ν is reduced by a factor of 200, a solution using 100 elements leads to oscillations (fig (23)), but using 200 elements gives an oscillation - free solution. Finally taking $\nu = 0$ and using 100 nodes, we obtained very bad oscillations (fig (25)),but again using 200 elements they are almost totally eliminated.

FIGURE 7

Y-axis   Pressure
X-axis   Distance
No of elements = 50
Equi-Distributed
dt = 0.0000100
Up to t = 0.000100
δ = 0.000266667
α = 1687.79968
θ = 1.0000

Non-Dim. Pressure in Reservoir

FIGURE 8

Y-axis   Pressure
X-axis   Distance
No of elements = 50
Equi-Distributed
dt = 0.0010000
Up to t = 0.010000
δ = 0.000266667
α = 1687.79968
θ = 1.0000

Non-Dim. Pressure in Reservoir

FIGURE 9

Y-axis   Pressure
X-axis   Distance
No of elements = 100
Equi-Distributed
dt = 0.0010000
Up to t = 0.010000
δ = 0.000266667
α = 1687.79968
θ = 1.0000

Non-Dim. Pressure in Reservoir

FIGURE 10

Y-axis   Pressure
X-axis   Distance
No of elements = 100
Equi-Distributed
dt = 0.0020000
Up to t = 0.010000
δ = 0.000266667
α = 1687.79968

Non-Dim. Pressure in Reservoir
Analytic Solution

FIGURE 11

Y-axis Pressure
X-axis Distance
No of elements = 50
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
a = 1687.79968
θ = 1.0000

Non-Dim. Pressure in Reservoir

FIGURE 12

Y-axis Pressure
X-axis Distance
No of elements = 50
Equi-Distributed
dt = 0.2000000
Up to t = 1.000000
δ = 0.000266667
a = 1687.79968

Non-Dim. Pressure in Reservoir
Analytic Solution

FIGURE 13

Y-axis Temperature
X-axis Distance
No of elements = 50
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000138088
η = 0.000000000
θ_1 = 1.0000
θ_2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 14

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000138088
η = 0.000000000
θ_1 = 1.0000
θ_2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 15

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.200000
Up to t = 1.000000
δ = 0.000266667
v = 0.000138088
η = 0.000000000

Non-Dim. Temperature in Reservoir
Similarity Solution

FIGURE 17

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
v = 0.000138088
η = 0.030394996
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 16

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
v = 0.000138088
η = 0.000000000
θ1 = 0.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 18

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
v = 0.000138088
η = 0.030394996
θ1 = 0.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 19

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000013809
n = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 20

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000006904
n = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 21

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000001381
n = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 22

Y-axis Temperature
X-axis Distance
No of elements = 200
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000001381
n = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

（page）

FIGURE 23

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000000000
η = 0.000000690
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 24

Y-axis Temperature
X-axis Distance
No of elements = 200
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000000000
η = 0.000000690
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 25

Y-axis Temperature
X-axis Distance
No of elements = 100
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000000000
η = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

FIGURE 26

Y-axis Temperature
X-axis Distance
No of elements = 200
Equi-Distributed
dt = 0.0100000
Up to t = 1.000000
δ = 0.000266667
γ = 0.000000000
η = 0.000000000
θ1 = 1.0000
θ2 = 1.0000

Non-Dim. Temperature in Reservoir

§ 7 CONCLUSIONS

In this report we have discussed the modelling of hot fluid injection
into porous media, with analytic and numerical solutions being obtained
to the resulting differential equations. In section two we showed that the
problem could be written as a system of three equations, one algebraic and
two differential. In section three, the three equations were non-dimensionalised
with the algebraic equation being simplified to a linear function of pressure,
which when substituted into the two differential equations resulted in two
differential equations for the unknowns pressure and temperature. After
simplifying the equations it was possible to obtain analytic solutions for
the pressure and temperature, this being done in section 4, with numerical
methods being described in section 5. The results were displayed
in section 6.

The numerical method coped very well with the simplified pressure
equation (40), as expected, with good comparisons between the numerical and
analytic solutions. The method also gave good results for the temperature
equation (since most of the activity takes place near the well bore), the
nodal distribution used for the pressure equation proving very suitable for
the temperature problem.

When the flow became purely advective $(\nu \to 0)$ the method still coped
well, but at the cost of a large number of elements, mainly distributed
near the well bore. Therefore if the problem was run to a greater
time, such that the temperature front reached further into the well, it is
expected the results would break down due to the lack of nodes in the region.

Also if we considered a 2-D problem the method would become very expensive in terms of computer time. In order to have enough nodes to resolve the solution some kind of adaptive gridding would therefore appear to be necessary.

APPENDIX

For completeness, we include the data set (see [14]) used for this report.

Reservoir parameters

Outer radius $L = 300.0m$

Height $h = 30.0m$

Well bore radias $R = 0.08m$

Rock Properties

Porosity $\phi = 0.2$

Permeability $K = 1.97385 \times 10^{-3} m^2$

Density $\rho_r = 2.643 \times 10^3 Kgm^{-3}$

Thermal Conductivity $K_\theta = 1.8028 Jm^{-1} 5' K^{-1}$

Specific heat at

constant volume $C_V^r = 1.0 \times 10^3 J K_g^{-1} K^{-1}$

Water Properties

Thermal compressibility $C_w = 4.3511 \times 10^{-10} pascals^{-1}$

Specific heat at

constant volume $C_V^w = 4.0 \times 10^3 J Kg^{-1} K^{-1}$

Viscosity $\mu = 3.0 \times 10^{-4} Kgm^{-1} S^{-1}$

Stock Tank Conditions

Temperature $T_{stc} = 288.0K$

Density of Water $\rho_{stc} = 1.0 \times 10^3 Kgm^{-3}$

Atmospheric Pressure $p_{str} = 1.0135 \times 10^5 pascals$

Initial Conditions in Reservoir

Temperature $\qquad$ T_{int} = 310.0 K

Density of Water $\qquad$ ρ_{int} = 1.0 x 10^3 Kgm^{-3}

Pressure $\qquad$ P = 1.0341 x 10^7 pascals

Injection Conditions

Mass injection rate q = 0.16406 Kgs^{-1}

Heat injection rate q_h = 7.4040 x 10^5 JS^{-1}

Temperature $\qquad$ T_{inj} = 366.0K

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