A further look at Newmark Houbolt etc. time stepping formulae.

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SUMMARY

This paper looks at some well-known methods for the numerical integration of the vibration equation from the point of view of various groupings of the independent equations involved. New forms are given of the general three-parameter method introduced by Zienkiewicz. [1].

1. The original Newmark method [2] for the integration of the vibration equation

$$M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} = f \qquad (1.1)$$

takes

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t\underline{v}_n + \frac{1}{2}\Delta t^2 \left[(1 - 2\beta_N)\underline{a}_n + 2\beta_N\underline{a}_{n+1}\right] \qquad (1.2)$$

and

$$\underline{v}_{n+1} = \underline{v}_n + \Delta t \left[(1 - \gamma_N)\,\underline{a}_n + \gamma_N\underline{a}_{n+1}\right] \qquad (1.3)$$

where the subscript N is used to distinguish the Newmark parameters. There are then two more equations like (1.2) and (1.3) relating $n - 1$ and n time levels.

Taking these four equations together with

$$M\underline{a}_j + C\underline{v}_j + K\underline{x}_j = \underline{f}_j \qquad (1.4)$$

for $j = n - 1$, n , $n + 1$

gives altogether seven independent equations from which Zienkiewicz [1] quotes Leleux and Chaix [3] as pointing out we can eliminate the six quantities $\underline{a}_{n-1}$, $\underline{a}_n$, $\underline{a}_{n+1}$, $\underline{v}_{n-1}$, $\underline{v}_n$, $\underline{v}_{n+1}$ to arrive at a recurrence relation in $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$:

$$\begin{aligned}
&[M + \gamma_N\Delta tC + \beta_N\Delta t^2K]\,\underline{x}_{n+1} \\
&\quad + [-2M + (1 - 2\gamma_N)\,\Delta tC + (\tfrac{1}{2} + \gamma_N - 2\beta_N)\,\Delta t^2K]\,\underline{x}_n \\
&\quad + [M + (\gamma_N - 1)\,\Delta tC + (\tfrac{1}{2} - \gamma_N + \beta_N)\,\Delta t^2K]\,\underline{x}_{n-1} \\
&\qquad = \Delta t^2[(\tfrac{1}{2} - \gamma_N + \beta_N)\,\underline{f}_{n-1} + (\tfrac{1}{2} + \gamma_N - 2\beta_N)\underline{f}_n + \beta_N\underline{f}_{n+1}]
\end{aligned} \qquad (1.5)$$

From equation (1.5), knowing two previous values $\underline{x}_{n-1}$, $\underline{x}_n$ we can obtain $\underline{x}_{n+1}$ and thus step forward for values of the displacements only, but this does not conveniently give values of the velocities.

As an alternative we can use equations (1.2) and (1.3) to obtain $\underline{a}_n$ and $\underline{a}_{n+1}$ and then substitute in equation (1.4) used now only at time levels n, n+1 to produce two equations in $\underline{x}_n$, $\underline{x}_{n+1}$, $\underline{v}_n$, $\underline{v}_{n+1}$ which constitute a one-step method to obtain the vector $\underline{Y}_{n+1} = [\underline{x}_{n+1}, \Delta t v_{n+1}]^T$. These can be arranged in various ways but one arrangement which is simple to apply is as follows:

$$M\left(\gamma_N \underline{x}_{n+1} - \beta_N \Delta t \underline{v}_{n+1}\right) = \left[\gamma_N M + \left(\beta_N - \frac{\gamma_N}{2}\right)\Delta t^2 K\right]\underline{x}_n$$

$$+ \left[\left(\gamma_N - \beta_N\right) M + \left(\beta_N - \frac{\gamma_N}{2}\right)\right]\Delta t \underline{v}_n$$

$$+ \left(\frac{\gamma_N}{2} - \beta_N\right)\Delta t^2 f_n \qquad (1.6)$$

From equation 1.6 the vector $\gamma_N \underline{x}_{n+1} - \beta_N \Delta t \underline{v}_{n+1} = \underline{u}_{n+1}$, say, can be obtained by simply inverting the matrix M.

Then the second equation can be taken as

$$\Delta t^2 \gamma_N K \underline{x}_{n+1} + \left(M + \Delta t \gamma_N C\right)\Delta t \underline{v}_{n+1}$$

$$= -\Delta t^2 (1-\gamma_N) K \underline{x}_n + \left[M - \Delta t(1-\gamma_N)C\right]\Delta t \underline{v}_n$$

$$+ (1-\gamma_N)\Delta t^2 \underline{f}_n + \gamma_N \Delta t^2 \underline{f}_{n+1} \qquad (1.7)$$

If we now substitute for $\Delta t \underline{v}_{n+1}$ in terms of $\underline{x}_{n+1}$ and $\underline{u}$ we have to invert the matrix $\left(M + \Delta t \gamma_N C + \Delta t^2 \beta_N K\right)$ to obtain $\underline{x}_{n+1}$ and hence $\Delta t \underline{v}_{n+1}$.

If we want to compute $\underline{X}_n = [\underline{x}_n, \Delta t\underline{v}_n, \Delta t^2\underline{a}_n]$ then we can simply substitute for $\underline{x}_{n+1}, \underline{v}_{n+1}$ in equation 1.4, $j = n + 1$ to obtain

$$[M + \Delta t\gamma_N C + \Delta t^2\beta_N]\ \underline{a}_{n+1}$$

$$= -K\underline{x}_n - [C + \Delta t\ K]\underline{v}_n$$

$$- [\Delta t\ (1 - \gamma_N)C + \frac{1}{2}\Delta t^2(1-2\beta_N)K]\underline{a}_n$$

$$+ f_{n+1} \qquad 1.7a$$

This gives $\underline{a}_{n+1}$ and then $\underline{x}_{n+1}, \underline{v}_{n+1}$ are obtained by substituting for $\underline{a}_{n+1}$ into equations 1.2 and 1.3.

When $C = 0$ unconditional stability is achieved with $2\beta_N \geq \gamma_N \geq \frac{1}{2}$ for any of the above recurrence relations whether in $\underline{x}_n$, $\underline{Y}_n$ or $\underline{X}_n$.

Second Approach

If we start as in Zienkiewicz [1] from the supposition that

$$\ddot{\underline{x}} = \sum_{n-1}^{n+1} N_i(t)\underline{x}_i \tag{1.8}$$

where the $N_i(t)$ are quadratic in the time t, and substitute this into the Weighted Residual form of equation (1.1) :

$$\int_{-\Delta t}^{\Delta t} W\left[M\sum \ddot{N}_i\underline{x}_i + C\sum N_i\underline{x}_i + K\sum N_i\underline{x}_i\right]dt = \int_{-\Delta t}^{\Delta t} W\underline{f}dt \tag{1.9}$$

then we obtain immediately a recurrence relation between $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$:

$$\begin{aligned}
&\left[M + \frac{\Delta t}{2}(1 + 2\gamma_z)C + \frac{\Delta t^2}{2}(\gamma_z + \beta_z)K\right]\underline{x}_{n+1} \\
&+ \left[-2M - 2\gamma_z\Delta t\,C + \Delta t^2(1 - \beta_z)K\right]\underline{x}_n \\
&+ \left[M - \frac{\Delta t}{2}(1 - 2\gamma_z)C - \frac{\Delta t^2}{2}(\gamma_z - \beta_z)K\right]\underline{x}_{n-1} \\
&= \Delta t^2\left[-\frac{1}{2}(\gamma_z - \beta_z)\underline{f}_{n-1} + (1 - \beta_z)\underline{f}_n + \frac{1}{2}(\gamma_z + \beta_z)\underline{f}_{n+1}\right]
\end{aligned} \tag{1.10}$$

where

$$\left.\begin{aligned}
\gamma_z &= \frac{1}{p}\int_{-1}^{1} W\,\xi\,d\xi \\
\beta_z &= \frac{1}{p}\int_{-1}^{1} W\,\xi^2 d\xi \\
p &= \int_{-1}^{1} W\,d\xi
\end{aligned}\right\} \tag{1.11}$$

and $\xi = t/\Delta t$.

The right hand side of equation 1.10 is obtained by assuming

$$\underline{f} = \sum_{n-1}^{n+1} N_i(t)\underline{f}_i \quad \text{i.e. the same interpolation as for } \underline{x} .$$

The recurrence relation (1.10) is exactly the same as that in equation (1.5) if we take

$$\left.\begin{aligned} \gamma_N &= \frac{1}{2} + \gamma_z \\ 2\beta_N &= \gamma_z + \beta_z \end{aligned}\right\} \qquad (1.12)$$

We must note that the Newmark equations (1.2) and (1.3) with independent β_N, γ_N are not consistent with the assumption of a polynomial variation in time.

Suppose we now start from equation (1.8) and (1.10) and form the pair of equations which constitute the one-step recurrence relation in the vector $\left[\underline{x}_n, \Delta t\underline{v}_n\right]^T = \underline{Y}_n$.

Equation (1.8) can be written as

$$\underline{x} = \frac{-t}{2\Delta t}\left(1-\frac{t}{\Delta t}\right)\underline{x}_{n-1} + \left(1 - \frac{t^2}{\Delta t^2}\right)\underline{x}_n + \frac{t}{2\Delta t}\left(1 + \frac{t}{\Delta t}\right)\underline{x}_{n+1} . \qquad (1.13)$$

This implies that

$$\dot{\underline{x}} = -\frac{1}{2\Delta t}\left(1 - \frac{2t}{\Delta t}\right)\underline{x}_{n-1} - \frac{2t}{\Delta t^2}\underline{x}_n + \frac{1}{2\Delta t}\left(1 + \frac{2t}{\Delta t}\right)\underline{x}_{n+1} \qquad (1.14)$$

and hence

$$\underline{v}_n = \frac{\underline{x}_{n+1} - \underline{x}_{n-1}}{2\Delta t} \qquad (1.15)$$

$$\underline{v}_{n+1} = \frac{3\underline{x}_{n+1} - 4\underline{x}_n + \underline{x}_{n-1}}{2\Delta t} . \qquad (1.16)$$

To obtain the pair of equations in $\underline{x}_{n+1}$, $\underline{v}_{n+1}$, $\underline{x}_n$, $\underline{v}_n$ we can eliminate $\underline{x}_{n-1}$ from the three equations (1.10), (1.15), (1.16), giving

$$\left[2M + 2\gamma_z \Delta t C + \Delta t^2 \beta_z K\right] \underline{x}_{n+1}$$

$$= \left[2M + 2\gamma_z \Delta t C - \Delta t^2 (1 - \beta_z) K\right] \underline{x}_n$$

$$+ \left[2M - \Delta t (1 - 2\gamma_z) C - \Delta t^2 (\gamma_z - \beta_z) K\right] \Delta t \underline{v}_n + \Delta t^2 \underline{F} \qquad (1.17)$$

(where $\Delta t^2 \underline{F}$ is written for the right hand side of equation (1.10))

and

$$2\underline{x}_{n+1} - \Delta t \underline{v}_{n+1} = 2\underline{x}_n + \Delta t \underline{v}_n \quad . \qquad (1.18)$$

Equation (1.17) can be used to generate $\underline{x}_{n+1}$ and then equation (1.18) to obtain $\Delta t \underline{v}_{n+1}$. This algorithm for obtaining $\left[\underline{x}_n, \ \Delta t \underline{v}_n\right]^T$ is much simpler than the method represented by the pair of equations (1.6), but it cannot be regarded as being equivalent to Newmark. With the damping matrix $C = 0$ we can again analyse the stability by taking $\underline{Y}_{n+1} = \lambda \underline{Y}_n$. We have unconditional stability for

$$\beta_z > \gamma_z \geq \frac{1}{2} \quad .$$

Now we can express the hypothesis that $\underline{x}$ is a quadratic in time in terms of three independent parameters.

(i) If we choose $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{x}_{n-1}$ to be these three parameters then the W.R. (Weighted Residual) equation gives a two-step recurrence in the displacements only which is convenient if we only need the displacements.

(ii) If we need to evaluate velocities as well as displacements we could choose to express the quadratic variation in terms of $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{v}_n$

i.e. $$\underline{x} = \underline{x}_n + \underline{v}_n t + \frac{t^2}{\Delta t^2}(\underline{x}_{n+1} - \underline{x}_n - \Delta t v_n) \quad . \tag{1.19}$$

Substituting from equation 1.19 into the W.R. equation 1.9 gives

$$2M(\underline{x}_{n+1} - \underline{x}_n - \Delta t v_n) + \Delta t C\left[\Delta t \underline{v}_n + 2\gamma_z(\underline{x}_{n+1} - \underline{x}_n - \Delta t \underline{v}_n)\right]$$
$$+ \Delta t^2 K\left[x_n + \gamma_z \Delta t \underline{v}_n + \beta_z(x_{n+1} - x_n - \Delta t v_n)\right] = \Delta t^2 \underline{F}$$

i.e.

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \underline{v}_n + \left[2M + 2\gamma_z \Delta t C + \beta_z \Delta t^2 K\right]^{-1}\left[-\Delta t^2 K \underline{x}_n - \Delta t^2 (C - \gamma_z \Delta t K)\underline{v}_n + \Delta t^2 \underline{F}\right] \tag{1.20}$$

where γ_z, β_z are the same as in equation 1.11.

Equation (1.20) can be used to obtain $\underline{x}_{n+1}$ and then equation (1.19) differentiated gives

$$\Delta t \underline{v}_{n+1} = 2(\underline{x}_{n+1} - \underline{x}_n) - \Delta t \underline{v}_n \tag{1.21}$$

which is of course the same as equation (1.18).

Equation (1.20) is the same as equation (1.17) provided the vector $\underset{\sim}{F}$ from the forcing function is obtained in the same way.

Similarly we could make the W.R. equation produce a relation between $\underset{\sim}{v}_{n+1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{v}_n$ to obtain $\underset{\sim}{v}_{n+1}$, then afterwards get $\underset{\sim}{x}_{n+1}$ from equation (1.21).

To summarise this section we note that (a) In the original Newmark approach [2] there are seven independent equations in the nine quantities $\underset{\sim}{x}_{n-1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{x}_{n+1}$, $\underset{\sim}{v}_{n-1}$, $\underset{\sim}{v}_n$, $\underset{\sim}{v}_{n+1}$, $\underset{\sim}{a}_{n-1}$, $\underset{\sim}{a}_n$, $\underset{\sim}{a}_{n+1}$. We can (i) eliminate six of these quantities from the seven equations to obtain a single recurrence relation in $\underset{\sim}{x}_{n-1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{x}_{n+1}$, (ii) we can use the two equations (1.2) and (1.3) together with equation (1.4) at time levels n , $n+1$ only i.e. four independent equations now in $\underset{\sim}{x}_n$, $\underset{\sim}{x}_{n+1}$, $\underset{\sim}{v}_n$, $\underset{\sim}{v}_{n+1}$, $\underset{\sim}{a}_n$, $\underset{\sim}{a}_{n+1}$ and eliminate $\underset{\sim}{a}_n$, $\underset{\sim}{a}_{n+1}$ to produce two equations (1.6), (1.7) which constitute the recurrence relation in $\underset{\sim}{Y}_n$, (iii) we take equations 1.7a, 1.2, 1.3 to form a recurrence in $\underset{\sim}{X}_n$.

(b) In the Zienkiewicz approach [1]

(i) The recurrence relation (1.10) in $\underset{\sim}{x}_{n-1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{x}_{n+1}$ is obtained immediately by substituting the Lagrange form of the quadratic variation in time assumption (1.8) into the W.R. equation (1.9).

(ii) It is pointed out here that the quadratic variation in time can be expressed in other ways each in terms of <u>three</u> quantities e.g. $\underset{\sim}{v}_{n+1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{v}_n$ or $\underset{\sim}{x}_{n+1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{v}_n$ and the W.R. equation will then give a relation between these <u>three</u> quantities in each case. The equation which states the quadratic variation can then be used to give a relation between <u>four</u> quantities, thus giving $\underset{\sim}{x}_{n+1}$ in terms of $\underset{\sim}{v}_{n+1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{v}_n$ or $\underset{\sim}{v}_{n+1}$

in terms of $\underset{\sim}{x}_{n+1}$, $\underset{\sim}{x}_n$, $\underset{\sim}{v}_n$. We then have a one-step recurrence in $\underset{\sim}{Y}_n$, no longer equivalent to Newmark.

(iii) There does not appear to be a useful one-step recurrence in $\underset{\sim}{X}_n$ from this approach.

2. Higher Order Formulae

In reference [1] Zienkiewicz obtained a very general higher-order formula by assuming that the displacement $\underline{x}$ is cubic in time and writing

$$\underline{x} = \sum_{j=n-2}^{n+1} N_j(t)\, \underline{x}_j \tag{2.1}$$

i.e. a Lagrangian interpolation expressing $\underline{x}$ in terms of the four quantities $\underline{x}_{n-2}$, $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$. He used this to obtain a four term recurrence relation in the displacements via the Weighted Residual Method. This recurrence relation is in terms of three parameters α, β, γ. It has been analysed by Wood [4] and shown to include the Houbolt [5], Wilson -θ [6] and Farhoomand [7] methods; it also includes the Bossak [8] and Hilber-Hughes-Taylor [9], when these are second order methods. The correspondence with Hilber-Hughes-Taylor (using subscripts H, Z to distinguish Hilber-Hughes-Taylor and Zienkiewicz parameters respectively) is that when they take

$$\gamma_H = \frac{1}{2} - \alpha_H \tag{2.2}$$

we have

$$\left.\begin{aligned} \gamma_Z &= 2 \\ \alpha_Z &= 8 + 12\beta_H + 6\alpha_H\beta_H - 6\alpha_H^{2} \\ \beta_Z &= 4 - 2\alpha_H^{2} + 2\beta_H \end{aligned}\right\} \tag{2.3}$$

[Hilber, Hughes, Taylor use the undamped form of equation (1) with C = o]

We now look at the Zienkiewicz approach from the point of view of the number and type of independent equations involved. There is the recurrence relation in $\underline{x}_{n-2}$, $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$, given in reference [1]. Equation 2.1 can also be differentiated to obtain

$$\dot{\underline{x}} = \sum_{j=n-2}^{n+1} \dot{N}_j(t)\, \underline{x}_j \tag{2.4}$$

Substituting the appropriate values for t then gives three equations for $\underline{v}_{n+1}$, $\underline{v}_n$, $\underline{v}_{n-1}$ in terms of $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{x}_{n-1}$, $\underline{x}_{n-2}$. From these four independent equations we can eliminate $\underline{x}_{n-2}$ to give three equations relating now $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{x}_{n-1}$, $\underline{v}_{n+1}$, v_n, v_{n-1}. If we aim to obtain a two-step recurrence in $\underline{Y}_n = \left[\underline{x}_n, \ \Delta t \underline{v}_n\right]^T$ we can arrange the equations so that one gives $\underline{x}_{n+1}$ directly and the other gives $\underline{v}_{n+1}$.

Equation 2.4 can be differentiated again and used to give equations for $\underline{a}_n$, $\underline{a}_{n+1}$ in terms of $\underline{x}_{n-2}$, $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$ i.e. we now have five equations involving $\underline{x}_{n-2}$, $\underline{x}_{n-1}$, $\underline{x}_n$, $\underline{x}_{n+1}$, $\underline{v}_n$, $\underline{v}_{n+1}$, $\underline{a}_n$, $\underline{a}_{n+1}$. We can eliminate $\underline{x}_{n-2}$, $\underline{x}_{n-1}$ from these five equations to obtain three equations in $\underline{x}_n$, $\underline{x}_{n+1}$, $\underline{v}_n$, $\underline{v}_{n+1}$, $\underline{a}_n$, $\underline{a}_{n+1}$ which constitute the recurrence

$$\underline{X}_{n+1} = A\underline{X}_n$$

Now we can express the cubic variation of $\underline{x}$ in time in various ways each involving four discrete values of displacement, velocity, acceleration, chosen as convenient. The W.R. equation then gives a relation between these four quantities. The statement of cubic variation gives further relations between four or five quantities:

(i) We can express the cubic variation as above in equation (2.4) using $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{x}_{n-1}$, $\underline{x}_{n-2}$ giving a three step recurrence in displacements as in [1].

(ii) We can express the cubic variation in terms of $\underline{x}_{n+1}$, $\underline{x}_n$, $\underline{v}_n$ and either $\underline{x}_{n-1}$ or $\underline{v}_{n-1}$. The W.R. equation then relates these four quantities and thus gives an equation for $\underline{x}_{n+1}$. The velocity $\underline{v}_{n+1}$ is then obtained from a five term equation.

For example, stating the cubic variation as

$$\underline{x} = \underline{x}_n + \underline{v}_n t + \frac{t^2}{2\Delta t^2}(\underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1}) + \frac{t^3}{2\Delta t^3}(\underline{x}_{n+1} - \underline{x}_{n-1} - 2\underline{v}_n\Delta t) \qquad (2.5)$$

we have

$$\dot{\underline{x}} = \underline{v}_n + \frac{t}{\Delta t^2}(\underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1}) + \frac{3t^2}{2\Delta t^3}(\underline{x}_{n+1} - \underline{x}_{n-1} - 2\underline{v}_n\Delta t) \qquad (2.6)$$

$$\ddot{\underline{x}} = \frac{1}{\Delta t^2}(\underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1}) + \frac{3t}{\Delta t^3}(\underline{x}_{n+1} - \underline{x}_{n-1} - 2\underline{v}_n\Delta t) \qquad (2.7)$$

The W.R. equation then gives

$$\frac{1}{\Delta t^2} M \left[\underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1} + 3\gamma_2(\underline{x}_{n+1} - \underline{x}_{n-1} - 2\underline{v}_n \Delta t)\right]$$

$$+ \frac{C}{\Delta t}\left[\Delta t \underline{v}_n + \gamma_2(\underline{x}_{n+1} - 2\underline{x}_n + \underline{x}_{n-1}) + \frac{3\beta_2}{2}2(\underline{x}_{n+1} - \underline{x}_{n-1} - 2\underline{v}_n \Delta t)\right]$$

$$+ K\left[\underline{x}_n + \gamma_2 \Delta t \underline{v}_n + \frac{\beta_2}{2}2(x_{n+1} - 2x_n + x_{n-1}) + \frac{\alpha_2}{2}2(x_{n+1} - x_{n-1} - 2v_n \Delta t)\right] = \underline{F} \qquad (2.8)$$

where $\underline{F}$ is the result of a suitable interpolation on the forcing function $\underline{f}$, and the parameters α_2, β_2, γ_2 are given by

$$\left.\begin{aligned} \alpha_2 &= \frac{1}{q\Delta t^3}\int_{-\Delta t}^{\Delta t} t^3 W dt \\ \beta_2 &= \frac{1}{q\Delta t^2}\int_{-\Delta t}^{\Delta t} t^2 W dt \\ \gamma_2 &= \frac{1}{q\Delta t}\int_{-\Delta t}^{\Delta t} t W dt \end{aligned}\right\} \qquad (2.9)$$

where $q = \int_{-\Delta t}^{\Delta t} W dt$

Equation (2.8) then gives $\underline{x}_{n+1}$, knowing $\underline{x}_n$, $\underline{x}_{n-1}$, $\underline{v}_n$, and $\underline{v}_{n+1}$ is given by substituting $t = \Delta t$ in equation (2.6):

i.e.

$$\underline{v}_{n+1} = \frac{5}{2\Delta t}\underline{x}_{n+1} - \frac{2}{\Delta t}\underline{x}_n - \frac{1}{2\Delta t}\underline{x}_{n-1} - 2\underline{v}_n \qquad (2.10)$$

The results represented by equations (2.8) and (2.10) are similar to those produced by the more elaborate procedure described above starting with the W.R. equation with the displacements only.

Equations (2.8) and (2.10) together constitute a two-step recurrence in $\underline{Y}_n = \left[\underline{x}_n, \ \Delta t\underline{v}_n\right]^T$

(iii) We can express the cubic variation in terms of the four discrete quantities $\underline{x}_n$, $\underline{v}_n$, $\underline{a}_n$, $\underline{a}_{n+1}$ as follows

$$\underline{x} = \underline{x}_n + t\underline{v}_n + \frac{1}{2}t^2\underline{a}_n + \frac{t^3}{6\Delta t}(\underline{a}_{n+1} - \underline{a}_n) \qquad (2.11)$$

Then substituting for $\underline{x}$, $\dot{\underline{x}}$, $\ddot{\underline{x}}$ in the W.R. equation we obtain immediately

$$\left[M\gamma_3 + \frac{\beta_3}{2}\Delta tC + \frac{\alpha_3}{6}\Delta t^2K\right]\underline{a}_{n+1} = -K\underline{x}_n - (C + \gamma_3\Delta tK)\underline{v}_n - \left[(1-\gamma_3)M + \left(\gamma_3 - \frac{\beta_3}{2}\right)\Delta tC + \frac{(3\beta_3 - \alpha_3)\Delta t^2K}{6}\right]\underline{a}_n + \Delta t^2\underline{F} \qquad (2.12)$$

where the forcing vector $\underline{F}$ is again obtained from some suitable interpolation and the parameters α_3, β_3, γ_3 are given by

$$\left.\begin{aligned}
\alpha_3 &= \frac{1}{r\Delta t^3}\int_0^{\Delta t} t^3 W dt \\
\beta_3 &= \frac{1}{r\Delta t^2}\int_0^{\Delta t} t^2 W dt \\
\gamma_3 &= \frac{1}{r\Delta t}\int_0^{\Delta t} t W dt \\
r &= \int_0^{\Delta t} W dt
\end{aligned}\right\} \tag{2.13}$$

Equation (2.12) together with the equations

$$\underline{x}_{n+1} - \frac{1}{6}\Delta t^2 \underline{a}_{n+1} = \underline{x}_n + \Delta t \underline{v}_n + \frac{1}{3}\Delta t^2 \underline{a}_n \tag{2.14}$$

and

$$\Delta t \underline{v}_{n+1} - \frac{1}{2}\Delta t^2 \underline{a}_{n+1} = \Delta t \underline{v}_n + \frac{1}{2}\Delta t^2 \underline{a}_n \tag{2.15}$$

(obtained from equation (2.11)), then constitute a one-step scheme for $\underline{X}_n = \left[\underline{x}_n,\ \Delta t \underline{v}_n,\ \Delta t^2 \underline{a}_n\right]^T$. Equation (2.12) has been constructed especially to give $\Delta t^2 \underline{a}_{n+1}$ in terms of the values at the previous step and then $\underline{x}_{n+1}$ and $\Delta t \underline{v}_{n+1}$ are obtained simply from equations (2.14) and (2.15).

In order to use the results on stability etc. in reference [4] we can match the coefficients in the stability polynomials resulting from the methods in (ii) and (iii) above by putting $\underline{Y}_{n+1} = \lambda \underline{Y}_n$, $\underline{X}_{n+1} = \lambda \underline{X}_n$ respectively used on the equation (1.1) with $C = 0$ by taking

$$\left.\begin{aligned} \gamma_2 &= \gamma_z - \frac{4}{3} \\ \beta_2 &= \beta_z - \frac{8\gamma_z}{3} + \frac{14}{9} \\ \alpha_2 &= \alpha_z - 4\beta_z + \frac{14\gamma_z}{3} - \frac{14}{9} \end{aligned}\right\} \qquad (2.16)$$

for the method (ii)

and

$$\left.\begin{aligned} \gamma_3 &= \gamma_z - 1 \\ \beta_3 &= \beta_z - 2\gamma_z + \frac{2}{3} \\ \alpha_3 &= \alpha_z - 3\beta_z + 2\gamma_z \end{aligned}\right\} \qquad (2.17)$$

for method (iii).

From reference [4] we have that the conditions for unconditional stability and positive damping simplify into

$$a > 0 \ , \quad b > 0 \ , \quad 3ab > c > 0 \qquad (2.18)$$

where for method (ii)

$$\begin{aligned} a &= \gamma_2 - \frac{1}{6} \ , \quad b = \beta_2 - \frac{1}{3}\gamma_2 - \frac{1}{2} \ , \\ c &= \alpha_2 - \frac{1}{2}\beta_2 - \gamma_2 + \frac{1}{4} \ . \end{aligned} \qquad (2.19)$$

Hence the Houbolt method which corresponds to $\alpha_z = 27$, $\beta_z = 9$, $\gamma_z = 3$ corresponds also to method (ii) used with

$$\alpha_2 = \frac{31}{9} \ , \quad \beta_2 = \frac{23}{9} \ , \quad \gamma_2 = \frac{5}{3} \ .$$

These values give $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = \frac{3}{4}$ as obtained in reference [4] in connection with the recurrence in the displacements only.

For method (iii) we have

$$a = \gamma_3 - \frac{1}{2} , \quad b = \beta_3 - \gamma_3 - \frac{1}{6} ,$$

$$c = \alpha_3 - \frac{3\beta_3}{2} + \frac{1}{4} .$$

Hence the Houbolt method which corresponds to method (iii) used with $\alpha_3 = 6$, $\beta_3 = \frac{11}{3}$, $\gamma_3 = 2$ gives the same values of a, b, c again.

The other methods described in reference [4] can be similarly linked up with methods (ii) and (iii) in this paper.

To summarise this section we note that the cubic variation of the displacement in time can be expressed in various ways conveniently to give

(i) a four-term three-step recurrence in displacements alone,

(ii) a three-term two-step recurrence in displacements and velocities

(iii) a two-term one-step recurrence in displacements, velocities and accelerations.

If the starting values exactly correspond and the forcing terms are interpolated in the same way the results will be the same because of the linearity of the relations.

Adams [10] has demonstrated the equivalence of working with displacements only or with the vector $\underline{X}_n = [\underline{x}_n , \Delta t \underline{v}_n , \Delta t^2 \underline{a}_n]$ in a comparison of the Hilber, Hughes, Taylor [9] and Bossak [8] methods.

REFERENCES

[1] O. C. Zienkiewicz. A new look at Newmark, Houbolt and other time stepping formulae. A Weighted Residual approach. University of Wales, Swansea Civil Engineering Report C/R/273/76.

[2] N. M. Newmark. A method of computation for structural dynamics. Proc. ASCE 85 EM3 67-94 1959.

[3] M. Chaix and F. Leleux. Cetim, Senlis, France (private communication to O. C. Zienkiewicz).

[4] W. L. Wood. On the Zienkiewicz four-time-level scheme for the numerical integration of vibration problems. Int. J. Num. Meth. Eng. 11, 1519-1528 (1977).

[5] J. C. Houbolt. A recurrence matrix solution for the dynamic response of elastic aircraft. J. Aero. Sci. 17 540-550 (1950).

[6] E. L. Wilson. A computer program for the dynamic stress analysis of underground structures. University of California, Berkeley, SEL Rept. 68-1 (1968).

[7] I. Farhoomand. Non-linear dynamic stress analysis of two-dimensional solids. Ph.D. dissertation. University of California, Berkeley 1970.

[8] W. L. Wood, M. Bossak and O. C. Zienkiewicz. An alpha modification of Newmark's method. Int. J. Num.Meth. Eng. 15. 1519-1528 1980.

[9] H. H. Hilber, T. J. R. Hughes and R. L. Taylor. Improved numerical dissipation for time integration algorithms in structural dynamics. Earthquake Eng. and Struct. Dyn. 5 283-292 (1977).

[10] D. D. Adams. Investigation of the properties of the Bossak-Newmark method for the numerical integra tion of structural dynamics equations. M.Sc. dissertation. Department of Mathematics, University of Reading, 1982.