DEPARTMENT OF MATHEMATICS

INTEGRAL OPERATORS OF STURM–LIOUVILLE TYPE

D. PORTER and D.S.G. STIRLING

Report 11/93

UNIVERSITY OF READING

INTEGRAL OPERATORS OF STURM–LIOUVILLE TYPE

by

D. PORTER and **D.S.G. STIRLING**

Department of Mathematics
P O Box 220
University of Reading
READING RG6 2AX
United Kingdom

Abstract

This paper extends the class of integral equations whose solutions can be generated from a finite number of particular cases to include those of Sturm–Liouville type, including the case where the associated operators are not self–adjoint. An explicit expression for the resolvent operator is generated from two particular solutions, in a form amenable to the use of approximation techniques. A usable estimate of the norm of the inverse operator is obtained even in cases where approximate solutions have to be used.

1. Introduction

This paper is concerned with the integral operator K on $L_2(0,1)$ defined by

$$(K\phi)(x) = \int_0^1 \chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))\phi(t)dt \tag{1.1}$$

and with other operators closely related to K. We assume that χ_1 and χ_2 are given members of $L_2(0,1)$; the bar denotes complex conjugation.

Interest in K, and the description we have assigned to it and its near relatives, stems from its connection with the Sturm–Liouville boundary value problem for second order ordinary differential equations. For suppose that ϕ is a twice differentiable function on (0,1) satisfying

$$(p\phi')' - q\phi + \lambda r\phi = h \tag{1.2}$$

in (0,1), together with

$$a_0\phi'(0) + b_0\phi(0) = c_0\,, \quad a_1\phi'(1) + b_1\phi(1) = c_1\,. \tag{1.3}$$

Then under certain conditions it can be shown (see, for example, Porter and Stirling (**3**)) that ϕ also satisfies the integral equation

$$\phi(x) = f(x) + \lambda\int_0^1 \chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))r(t)\phi(t)dt \quad (0 \le x \le 1), \tag{1.4}$$

in which the functions f, χ_1 and χ_2 can be determined.

The equation (1.4) can be investigated *via* the equation

$$\phi = f + \lambda KR\phi \tag{1.5}$$

in $L_2(0,1)$, R being the operator defined by

$$R\phi = r\phi \qquad (\phi \in L_2(0,1)). \tag{1.6}$$

Spectral and approximation methods applied to (1.5) provide information about the solution of (1.4) and therefore about the solution of the boundary value problem.

It is common in differential equation theory to assume that the given functions p, q, r and h are real–valued, that the parameter $\lambda \in \mathbb{R}$ and that a_i, b_i and c_i are real numbers for $i = 0,1$. These conditions imply that f, χ_1 and χ_2 are real–valued and that the operators K and R are self–adjoint.

In principle, the Sturm–Liouville problem consisting of (1.2) and (1.3) can, of course, be solved in terms of two linearly independent solutions of the homogeneous version of (1.2), using standard differential equation theory. In practice, this procedure is often not available and approximation techniques are required. Nevertheless, the question naturally arises whether (1.4) admits a construction similar to the boundary value problem, in the sense that its solution is determined by just two functions, defined in a particular way. Such properties have recently been identified for other integral operators by Porter and Stirling (4), who described the solutions of the associated operator equations as being finitely–generated.

Although we have used the Sturm–Liouville problem to motivate the operator K and to suggest that the solutions of (1.5) could be finitely–generated, our results are obtained without reference to differential equations. We assume only that χ_1, χ_2, and f are members of $L_2(0,1)$, and that $r \in L_\infty(0, 1)$; these functions are not necessarily real–valued. In consequence K need not be self–adjoint. However, its kernel is what is sometimes described as "complex symmetric"; that is, it satisfies $k(x,t) = k(t,x)$, but is not necessarily real–valued. To identify the corresponding property of K we need to use the conjugate–linear operator J, where

$$J\phi = \bar{\phi} \qquad (\phi \in L_2(0,1)) . \tag{1.7}$$

Then we see that the adjoint, K^*, of K is given by

$$K^* = JKJ \tag{1.8}$$

and also that

$$R^* = JRJ . \tag{1.9}$$

The class of operators considered by Porter and Stirling (4) includes those generated by difference kernels, which have received attention from previous authors; references may be found in Porter (2), which the later work generalises. The operator K defined by (1.1) is not included in previous work, although there is an isolated connection because the difference kernel $\exp(\alpha|x-t|)$ $(\alpha \in \mathbb{C})$ is also of the form $\chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))$ with $\chi_1(x) = \exp(-\alpha x)$ and $\chi_2(x) = \exp(\bar{\alpha} x)$.

The operator defined by

$$(\hat{K}\phi)(x) = -\tfrac{1}{2}\int_0^1 \operatorname{sgn}(x-t)\{\chi_1(x)\bar{\chi}_2(t) - \chi_1(t)\bar{\chi}_2(x)\}\phi(t)dt \quad (\phi \in L_2(0,1)) \tag{1.10}$$

is a rank–two perturbation of K, since

$$\hat{K}\phi = K\phi - \tfrac{1}{2}(\phi, \bar{\chi}_1)\bar{\chi}_2 - \tfrac{1}{2}(\phi, \chi_2)\chi_1 . \tag{1.11}$$

A particular example of the kernel generating $\hat{K}$ is $-\frac{1}{2}\sin(\alpha|x-t|)$ $(\alpha \in \mathbb{C})$, but in general $\hat{K}$ is not included in existing work and we consider it here. The equation $\phi = f + \lambda\hat{K}R\phi$ is also related to a Sturm–Liouville problem with mixed boundary conditions.

A further operator closely related to K, which will prove useful later, is the Volterra operator V defined by

$$(V\phi)(x) = \int_0^x \{\chi_1(x)\bar{\chi}_2(t) - \chi_1(t)\bar{\chi}_2(x)\}\phi(t)dt \quad (\phi \in L_2(0,1)) , \tag{1.12}$$

for

$$K\phi = -V\phi + (\phi, \chi_2)\chi_1 = -JV^*J\phi + (\phi, \bar{\chi}_1)\bar{\chi}_2 . \tag{1.13}$$

The integral equation (1.4) is typical of many which arise in applications in that, unless it is a constant, the function r destroys the complex symmetry of the kernel. This is often a major stumbling block in practical terms. If r is real–valued and non-negative in [0,1], we can recast (1.5) as $\psi = Qf + \lambda QKQ\psi$, where $\psi = Q\phi = \sqrt{r}\phi$, which restores the symmetry in the sense that $(QKQ)^* = J(QKQ)J$. This device, if it is available, allows theoretical progress to be made but can complicate calculations. Here we require no restriction on r and, indeed, we can accommodate a further generalisation in this direction by extending our theory to the equation

$$\phi = f + \lambda SKR\phi \tag{1.14}$$

in $L_2(0,1)$, where

$$S\phi = s\phi \tag{1.15}$$

and $s \in L_\infty(0,1)$.

The present contribution therefore extends the class of operators for which the solution of the associated second–kind equation is known to be finitely–generated. Even in the special cases where K and $\hat{K}$ are generated by difference kernels, we expand existing theory by virtue of including the operators R and S.

In Section 2 we establish the main results, showing how the inverse of $I - \lambda SKR$ can be determined by the solutions of two particular versions of (1.14) or by the solutions of two Volterra equations. The theory allows approximations to $(I - \lambda SKR)^{-1}$ to be constructed, which give explicit bounds for the norm of that operator, as we demonstrate in Section 3. The main points in the corresponding theory of the operator $I - \lambda S\hat{K}R$ are sketched in Section 4.

2. The Inverse of $I - \lambda SKR$

It turns out to be simpler to invert the operator $I - \lambda KR$ and deduce the inverse of $I - \lambda SKR$ from this, so we shall proceed in this way.

As before we presume K is defined by

$$(K\phi)(x) = \int_0^1 \chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))\phi(t)dt \qquad (0 \leq x \leq 1)$$

where χ_1 and χ_2 are given members of $L_2(0,1)$ and K is a compact operator from $L_2(0,1)$ to itself. Let r be a given function in $L_\infty(0,1)$, and define $(R\phi)(x) = r(x)\phi(x)$.

We make no further restrictions on χ_1 and χ_2 but notice in passing that there is a degenerate case where χ_1 and $\bar{\chi}_2$ are linearly dependent. Suppose $\bar{\chi}_2 = \bar{\mu}\chi_1$ so that K is the operator defined by $K\phi = (\phi, \mu\bar{\chi}_1)\chi_1$ which has rank at most 1. It is then trivial to check that $I - \lambda KR$ is invertible provided that $\lambda \neq 1/(r\chi_1, \mu\bar{\chi}_1)$. In the exceptional case $\lambda = 1/(r\chi_1, \mu\bar{\chi}_1)$ χ_1 is an eigenvector of KR and the equation $(I - \lambda KR)\phi = \chi_1$ has no solution, so in this degenerate case $I - \lambda KR$ is invertible exactly when $(I - \lambda KR)\phi_1 = \chi_1$, and hence also $(I - \lambda KR)\bar{\phi}_2 = \bar{\chi}_2$, are soluble.

Returning to the general case suppose that there are vectors $\phi_1, \phi_2 \in L_2(0,1)$ satisfying

$$(I - \lambda KR)\phi_1 = \chi_1, \quad (I - \lambda KR)\bar{\phi}_2 = \bar{\chi}_2 . \tag{2.1}$$

Then define the linear operators L_0 and W by

$$(L_0\phi)(x) = \int_0^1 \phi_1(\min(x,t))\bar{\phi}_2(\max(x,t))\phi(t)dt , \tag{2.2}$$

$$(W\phi)(x) = \int_0^x (\phi_1(x)\bar{\phi}_2(t) - \phi_1(t)\bar{\phi}_2(x))\phi(t)dt , \tag{2.3}$$

and let

$$C = 1 + \lambda(r\phi_1, \chi_2) = 1 + \lambda(r\chi_1, \phi_2) . \tag{2.4}$$

The equality of the two versions of C follows from the observation, using (1.8) and (1.9), that $(I - \bar{\lambda}K^*R^*)\phi_2 = \chi_2$ whence

$$\begin{aligned}(r\phi_1, \chi_2) &= (R\phi_1,(I - \bar{\lambda}K^*R^*)\phi_2) = ((I - \lambda RK)R\phi_1, \phi_2) \\ &= (R\chi_1, \phi_2) = (r\chi_1, \phi_2) .\end{aligned}$$

From the definition of K, ϕ_1 and ϕ_2 we have

$$\phi_1 = \chi_1 + \lambda KR\phi_1 , \quad \bar{\phi}_2 = \bar{\chi}_2 + \lambda KR\bar{\phi}_2 ,$$

whence, for almost all x in $(0,1)$,

$$A(x)\begin{pmatrix}\chi_1(x)\\ \\ \bar\chi_2(x)\end{pmatrix} = \begin{pmatrix}\phi_1(x)\\ \\ \bar\phi_2(x)\end{pmatrix},$$

where $A(x)$ is the matrix

$$A(x) = \begin{pmatrix} 1+\lambda\int_x^1 \bar\chi_2(t)r(t)\phi_1(t)dt & \lambda\int_0^x \chi_1(t)r(t)\phi_1(t)dt \\ \\ \lambda\int_x^1 \bar\chi_2(t)r(t)\bar\phi_2(t)dt & 1+\lambda\int_0^x \chi_1(t)r(t)\bar\phi_2(t)dt \end{pmatrix}.$$

Now the determinant of A is an absolutely continuous function of x and its derivative is equal almost everywhere to zero, and hence $\det A(x)$ is constant. [See, for example, Stromberg (**5**), Theorems 6.84 and 6.85.] The constant is $1+\lambda(r\phi_1, \chi_2)$, that is, C. Therefore, setting

$$B(x) = \begin{pmatrix} 1+\lambda\int_0^x \chi_1(t)r(t)\bar\phi_2(t)dt & -\lambda\int_0^x \chi_1(t)r(t)\phi_1(t)dt \\ \\ -\lambda\int_x^1 \bar\chi_2(t)r(t)\bar\phi_2(t)dt & 1+\lambda\int_x^1 \bar\chi_2(t)r(t)\phi_1(t)dt \end{pmatrix},$$

we have $A(x)B(x) = B(x)A(x) = CI$, giving

$$C\chi_1 = \phi_1 + \lambda WR\chi_1, \qquad \bar C\chi_2 = \phi_2 + \bar\lambda W^*R^*\chi_2, \tag{2.5}$$

the second equation arising from the conjugate of the second component in the equation $B(\phi_1, \bar\phi_2)^T = C(\chi_1, \bar\chi_2)^T$.

From (2.2) and (2.3), we have for all ϕ,

$$L_0\phi = -W\phi + (\phi, \phi_2)\phi_1 \tag{2.6}$$

whence $\qquad L_0^* \phi = -W^*\phi + (\phi, \phi_1)\phi_2 .$

Therefore, from (2.5) we have

$$C\chi_1 = C\phi_1 - \lambda L_0 R\chi_1, \quad C\bar{\chi}_2 = C\bar{\phi}_2 - \lambda L_0 R\bar{\chi}_2 , \tag{2.7}$$

the second equation arising through the equation $\bar{C}\chi_2 = \bar{C}\phi_2 - \bar{\lambda}L_0^* R^*\chi_2$.

The equations (2.7) are in a sense reciprocal to (2.1). We shall, however, find it more convenient to work with (2.5), and the use of V, as defined in (1.12), along with (1.13) yields from (2.1),

$$\phi_1 = C\chi_1 - \lambda VR\phi_1 , \quad \bar{\phi}_2 = \bar{\chi}_2 + \lambda(r\bar{\phi}_2, \chi_2)\chi_1 - \lambda VR\bar{\phi}_2 . \tag{2.8}$$

Also, the second equation in (2.5) yields (on expressing W^* in terms of W)

$$C\bar{\chi}_2 = C\bar{\phi}_2 - \lambda(r\bar{\phi}_2, \chi_2)\phi_1 + \lambda WR\bar{\chi}_2 . \tag{2.9}$$

This rather lengthy set of preliminaries allows us to prove:

Lemma 1 With the notation above

$$\lambda VRWR = \lambda WRVR = CVR - WR .$$

Proof: From the definitions of V and W we have, for all $\phi \in L_2(0,1)$,

$$(\lambda VRWR)\phi(x) = \int_0^x \alpha(x,t)r(t)\phi(t)dt$$

where

$$\alpha(x,t) = \lambda\int_t^x \{\chi_1(x)\bar{\chi}_2(s) - \chi_1(s)\bar{\chi}_2(x)\}r(s)\{\phi_1(s)\bar{\phi}_2(t) - \phi_1(t)\bar{\phi}_2(s)\}ds$$

$$= \lambda\int_0^x \{\chi_1(x)\bar{\chi}_2(s) - \chi_1(s)\bar{\chi}_2(x)\}r(s)\{\phi_1(s)\bar{\phi}_2(t) - \phi_1(t)\bar{\phi}_2(s)\}ds$$

$$+ \lambda\int_0^t \{\phi_1(t)\bar{\phi}_2(s) - \phi_1(s)\bar{\phi}_2(t)\}r(s)\{\chi_1(x)\bar{\chi}_2(s) - \chi_1(s)\bar{\chi}_2(x)\}ds$$

$$= \bar{\phi}_2(t)(\lambda VR\phi_1)(x) - \phi_1(t)(\lambda VR\bar{\phi}_2)(x)$$

$$+ \chi_1(x)(\lambda WR\bar{\chi}_2)(t) - \bar{\chi}_2(x)(\lambda WR\chi_1)(t)$$

$$= C(\chi_1(x)\bar{\chi}_2(t) - \chi_1(t)\bar{\chi}_2(x)) + (\phi_1(t)\bar{\phi}_2(x) - \phi_1(x)\bar{\phi}_2(t)) .$$

Therefore $\lambda VRWR = CVR - WR$, and a similar calculation shows that $\lambda WRVR = \lambda VRWR$. □

At this stage we need to take stock of the situation. We assumed in (2.1) that χ_1 and χ_2 were such that ϕ_1 and ϕ_2 exist satisfying $(I - \lambda KR)\phi_1 = \chi_1$, $(I - \lambda KR)\bar{\phi}_2 = \bar{\chi}_2$, but we have so far made no further assumptions about χ_1 and χ_2 or about $I - \lambda KR$.

Theorem 2 Suppose that χ_1 and χ_2 belong to $L_2(0,1)$, that $r \in L_\infty(0,1)$, that the operators K and R on $L_2(0,1)$ are defined by

$$(K\phi)(x) = \int_0^1 \chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))\phi(t)dt \qquad (0 \le x \le 1)$$

$$(R\phi)(x) = r(x)\phi(x) \qquad (0 \le x \le 1) ,$$

and that $\lambda \in \mathbb{C}$.

Then if there are functions ϕ_1, $\phi_2 \in L_2(0,1)$ satisfying

$$(I-\lambda KR)\phi_1 = \chi_1, \quad (I-\lambda KR)\bar{\phi}_2 = \bar{\chi}_2$$

it follows that $I-\lambda KR$ is invertible. Its inverse is given by

$$(I-\lambda KR)^{-1} = I+\lambda LR,$$

where $L = \frac{1}{C}L_0$ and $(L_0\phi)(x) = \int_0^1 \phi_1(\min(x,t))\bar{\phi}_2(\max(x,t))\phi(t)dt$, C being the number

$$C = 1+\lambda(r\phi_1, \chi_2),$$

which is necessarily non–zero.

Proof: Assuming for the present that $C \neq 0$, then in the earlier notation we have from Lemma 1,

$$\begin{aligned}(I+\lambda VR)(I-\tfrac{1}{C}\lambda WR) &= (I-\tfrac{1}{C}\lambda WR)(I+\lambda VR)\\ &= I.\end{aligned}$$

Now, using (2.6) and (1.13), for all $\phi \in L_2(0,1)$,

$$\begin{aligned}\lambda KRL_0R\phi &= -\lambda VRL_0R\phi + \lambda(RL_0R\phi, \chi_2)\chi_1\\
&= -\lambda VR(-WR\phi + (R\phi, \phi_2)\phi_1) + (-\lambda RWR\phi + \lambda(R\phi, \phi_2)R\phi_1, \chi_2)\chi_1\\
&= (CVR - WR)\phi + (r\phi, \phi_2)(-\lambda VR\phi_1)\\
&\qquad + (R\phi, -\bar{\lambda}W^*R^*\chi_2)\chi_1 + \lambda(r\phi, \phi_2)(r\phi_1, \chi_2)\chi_1\\
&= -CKR\phi + L_0R\phi + C(r\phi, \chi_2)\chi_1 - (r\phi, \phi_2)\phi_1\\
&\qquad + (r\phi, \phi_2)(\phi_1 - C\chi_1) + (r\phi, \phi_2 - \bar{C}\chi_2)\chi_1\\
&\qquad + \lambda(r\phi, \phi_2)(r\phi_1, \chi_2)\chi_1\\
&= -CKR\phi + L_0R\phi + (r\phi, \phi_2)(1+\lambda(r\phi_1, \chi_2) - C)\chi_1\\
&= -CKR\phi + L_0R\phi,\end{aligned}$$

using (2.5) and (2.8). Similarly $\lambda L_0RKR\phi = -CKR\phi + L_0R\phi$.

Therefore for all $\phi \in L_2(0,1)$,

$$(I - \lambda KR)(I + \frac{\lambda}{C} L_0R)\phi \;=\; (I + \frac{\lambda}{C} L_0R\phi)(I - \lambda KR)\phi \;=\; \phi\,,$$

assuming $C \neq 0$.

Now suppose $C = 0$. Then by Lemma 1 $\lambda VRWR = -WR$, whence $(I + \lambda VR)WR = 0$. However, VR arises from a Volterra operator so that $I + \lambda VR$ is invertible, showing that $WR = 0$. Since WR is given by

$$(WR\phi)(x) \;=\; \int_0^x (\phi_1(x)\bar{\phi}_2(t) - \phi_1(t)\bar{\phi}_2(x))r(t)\phi(t)dt$$

it follows that $\phi_1(x)\bar{\phi}_2(t)r(t) = \phi_1(t)\bar{\phi}_2(x)r(t)$ for almost all x and t with $0 \leq t \leq x \leq 1$. This in turn shows that $R\phi_1$ and $R\bar{\phi}_2$ are linearly dependent and L_0R and KR are of rank at most 1. Moreover by (2.5) (since $C = 0$ and $WR = 0$) $\phi_1 = \phi_2 = 0$ whence $\chi_1 = \chi_2 = 0$, contradicting the equation $C = 0$. Thus $C \neq 0$.

(To put this in a clearer context, if one pursues the degeneracy that follows from the assumption $C = 0$, KR has rank at most 1 and the condition that $C = 0$ shows that λ is exactly the value making $I - \lambda KR$ singular whence $(I - \lambda KR)\phi_1 = \chi_1$ cannot have a solution. In effect, what prohibits C being zero is the existence of the two particular solutions ϕ_1 and $\bar{\phi}_2$ in (2.1).) □

The modification of this result to deal with operators of the form SKR, where $(S\phi)(x) = s(x)\phi(x)$ $(0 \leq x \leq 1)$ and $s \in L_\infty(0,1)$ is now simple. The key tool is the relation of $I - \lambda SKR$ to $I - \lambda KRS$; since for any two bounded linear maps A and B the non–zero members of the spectra of AB and BA are identical, $I - \lambda SKR$ is invertible if and only if $I - \lambda KRS$ is invertible.

Theorem 3 Suppose that χ_1 and χ_2 belong to $L_2(0,1)$, that r and s belong to $L_\infty(0,1)$, that K, R and S are defined by

$$(K\phi)(x) = \int_0^1 \chi_1(\min(x,t))\bar{\chi}_2(\max(x,t))\phi(t)dt \quad (0 \le x \le 1),$$

$$(R\phi)(x) = r(x)\phi(x), \quad (S\phi)(x) = s(x)\phi(x) \quad (0 \le x \le 1)$$

and that $\lambda \in \mathbb{C}$. Then if there are functions ϕ_1 and ϕ_2 in $L_2(0,1)$ satisfying

$$(I - \lambda KRS)\phi_1 = \chi_1, \quad (I - \lambda KRS)\bar{\phi}_2 = \bar{\chi}_2,$$

$I - \lambda SKR$ is invertible and its inverse is $I + \lambda SLR$ where

$$(L\phi)(x) = \frac{1}{C}\int_0^1 \phi_1(\min(x,t))\bar{\phi}_2(\max(x,t))\phi(t)dt$$

and $C = 1 + \lambda(rs\phi_1, \chi_2)$; C is known to be non–zero.

Alternatively, if there are functions ψ_1 and ψ_2 satisfying

$$(I - \lambda SKR)\psi_1 = S\chi_1, \quad (I - \lambda SKR)\bar{\psi}_2 = S\bar{\chi}_2 \tag{2.10}$$

then $I - \lambda SKR$ is invertible and its inverse is $I + \lambda SLR$ where L is as above, with $\phi_1 = \psi_1/s$, $\bar{\phi}_2 = \bar{\psi}_2/s$ (where these are defined) or $\phi_1 = \chi_1 + \lambda KR\psi_1$, $\bar{\phi}_2 = \bar{\chi}_2 + \lambda KR\bar{\psi}_2$ (in general).

Proof: Suppose that $(I - \lambda KRS)\phi_1 = \chi_1$, $(I - \lambda KRS)\bar{\phi}_2 = \bar{\chi}_2$. Then Theorem 1 applies and we see that $I - \lambda KRS$ is invertible, its inverse being $I + \lambda LRS$. From this it follows that λKRS and λLRS commute and that $\lambda LRS = \lambda KRS + \lambda^2 KRSLRS = \lambda KRS + \lambda^2 LRSKRS$, whence also $\lambda SLRS = \lambda SKRS + \lambda^2 SKRSLRS = \lambda SKRS + \lambda^2 SLRSKRS$.

If S is invertible (or if it has dense image) it follows immediately that $\lambda SLR = \lambda SKR + \lambda^2 SKRSLR = \lambda SKR + \lambda^2 SLRSKR$, which in turn shows that $(I - \lambda SKR)^{-1} = I + \lambda SLR$.

In the general case, we can choose a sequence s_n of functions in $L_\infty(0,1)$ such that the essential infimum of $|s_n|$ is positive and such that $\text{ess sup}|s_n - s| \longrightarrow 0$ as $n \longrightarrow \infty$. If the corresponding multiplication operator is S_n, then S_n is invertible and $\|S_n - S\| \longrightarrow 0$ as $n \longrightarrow \infty$. Because $I - \lambda KRS$ is invertible, so is $I - \lambda KRS_n$ for large n, and so there are functions $\phi_1^{(n)}$ and $\phi_2^{(n)}$ in $L_2(0,1)$ which satisfy $(I - \lambda KRS_n)\phi_1^{(n)} = \chi_1$, $(I - \lambda KRS_n)\bar{\phi}_2^{(n)} = \bar{\chi}_2$. Then if we construct L_n from $\phi_1^{(n)}$ and $\bar{\phi}_2^{(n)}$, as before, $I + \lambda S_n L_n R$ is the inverse of $I - \lambda S_n KR$. Now, because $(I - \lambda KRS_n)^{-1} \longrightarrow (I - \lambda KRS)^{-1}$ as $n \longrightarrow \infty$, it follows that $\|\phi_1^{(n)} - \phi_1\| \longrightarrow 0$ and $\|\bar{\phi}_2^{(n)} - \bar{\phi}_2\| \longrightarrow 0$ as $n \longrightarrow \infty$; hence $\|L_n - L\| \longrightarrow 0$ as $n \longrightarrow \infty$, showing that $(I - \lambda SKR)^{-1} = I + \lambda SLR$ in this case also.

For the alternative version, suppose that ψ_1 and ψ_2 satisfy $(I - \lambda SKR)\psi_1 = S\chi_1$, $(I - \lambda SKR)\bar{\psi}_2 = S\bar{\chi}_2$. From these equations it follows that ψ_1 and $\bar{\psi}_2$ are in the image of S. Moreover, if we set $\phi_1 = \chi_1 + \lambda KR\psi_1$ and $\bar{\phi}_2 = \bar{\chi}_2 + \lambda KR\bar{\psi}_2$ it is easily checked that $(I - \lambda KRS)\phi_1 = \chi_1$ and $(I - \lambda KRS)\bar{\phi}_2 = \bar{\chi}_2$ reducing the situation to that of the first case. Finally, we notice that

$$S\chi_1 = S(I - \lambda KRS)\phi_1 = (I - \lambda SKR)S\phi_1 ,$$

which by the invertibility of $I - \lambda SKR$ shows that $\psi_1 = S\phi_1$, and similarly $\bar{\psi}_2 = S\bar{\phi}_2$, whence the expression for ϕ_i in terms of ψ_i $(i = 1,2)$. □

The second version of Theorem 3 gives us the result closest to Theorem 2, namely that the equation $(I - \lambda SKR)\phi = f$ has a solution for all f and $(I - \lambda SKR)$ is invertible if (and, trivially, only if) the equation is soluble in two particular cases, $f = S\chi_1$ and $f = S\bar{\chi}_2$. The existence of solutions of $(I - \lambda SKR)\phi = f$ in the cases $f = \chi_1$ and $f = \bar{\chi}_2$ is not sufficient to show the invertibility where s is not a constant. (This may be seen by considering the case where $\chi_1 = \chi_{[0,\frac{1}{4}]}$, $\chi_2 = \chi_{[\frac{3}{4},1]}$, and $\lambda = 4$

$$r(x) = s(x) = \begin{cases} 1 & (0 \le x \le \frac{1}{8} \text{ or } \frac{7}{8} \le x \le 1) \\ -1 & (\frac{1}{8} < x < \frac{7}{8}) , \end{cases}$$

where χ_A denotes the function which takes the value 1 on the set A and zero elsewhere.)

The inverse of the operator $(I - \lambda SKR)$ is therefore known once we can solve two particular cases. This is rather similar to the situation with other classes of integral operators (see Porter (2) and Porter and Stirling (4)), where the resolvent operator was also generated by solving finitely many particular cases. Moreover, as in the work cited, we have more than one route for finding the resolvent. For simplicity we shall take $S = I$ here; the general case can be deduced from this by the methods of Theorem 3.

We notice first that equation (2.8) shows us that $\phi_1 + \lambda VR\phi_1 = C\chi_1$, so that ϕ_1 is a multiple of $(I + \lambda VR)^{-1}\chi_1$. Since VR is a Volterra operator it is automatic that $I + \lambda VR$ has an inverse and this inverse $(I + \lambda VR)^{-1}$ is given by the sum of its

Neumann series $\Sigma_{n=0}^{\infty}(\lambda VR)^n$. In some cases it may prove practicable to obtain a close approximation to ϕ_1 by truncating the series after a few terms. Alternatively, in certain special cases, it may be convenient to convert the Volterra integral equation into a differential equation and thus find ϕ_1. $\bar{\phi}_2$ can also be obtained from (2.8) by solving the corresponding equation or by observing that $(I - \bar{\lambda}V^*R^*)\phi_2 = \bar{C}\chi_2$.

Suppose now that we choose an arbitrary $g_1 \in L_2(0,1)$ and θ_1 satisfying $(I - \lambda KR)\theta_1 = g_1$. Then by (1.13) we have

$$\theta_1 - g_1 \;=\; -\lambda VRg_1 + \lambda(rg_1, \chi_2)\chi_1 + \lambda KR(\theta_1 - g_1)$$

whence

$$(I - \lambda KR)(\theta_1 - g_1) \;=\; -\lambda VRg_1 + \lambda(rg_1, \chi_2)(I - \lambda KR)\phi_1\,,$$

from the definition of ϕ_1. Therefore, if $(I - \lambda KR)$ is invertible,

$$\theta_1 - g_1 \;=\; -\lambda\,\Psi_1 + \lambda(rg_1, \chi_2)\phi_1$$

where $(I - \lambda KR)\Psi_1 = VRg_1$. Therefore, provided $\lambda(rg_1, \chi_2) \neq 0$ we can determine ϕ_1 from

$$\lambda(rg_1, \chi_2)\phi_1 \;=\; \theta_1 - g_1 + \lambda\Psi_1\,.$$

To find ϕ_1 in this case involves solving the two equations $(I - \lambda KR)\theta_1 = g_1$ and $(I - \lambda KR)\Psi_1 = VRg_1$ if g_1 is prescribed. If, on the other hand, we prescribe θ_1 and calculate g_1 then only the one equation needs to be solved. This may prove advantageous if a suitable choice of g_1 (or θ_1) yields a free term which renders the integral equation more convenient to solve than $(I - \lambda KR)\phi_1 = \chi_1$. A similar technique yields an equation for ϕ_2 in the form $\bar{\lambda}(g_2, r\chi_1)\phi_2 = \theta_2 + \bar{\lambda}\,\Psi_2 - g_2$ where

$(I - \lambda KR)\bar{\theta}_2 = \bar{g}_2$ and $(I - \lambda KR)\bar{\Psi}_2 = \overline{V^*R^*g_2}$; by choosing $\bar{g}_2 = g_1$ (hence $\bar{\theta}_2 = \theta_1$) this process requires only one additional integral equation to be solved to find ϕ_2. Provided that g_1 and g_2 are chosen so that $\lambda(rg_1, \chi_2)$ and $\lambda(g_2, r\chi_1)$ are non–zero, then the existence of θ_1, θ_2, Ψ_1 and Ψ_2 guarantees that ϕ_1 and ϕ_2 exist and, in turn, that $I - \lambda KR$ is invertible. It is possible, even if $I - \lambda KR$ is invertible, to choose "unsuitable" g_1 and g_2 violating these requirements.

A further point emerges from (2.8), where we saw that (provided ϕ_1 and ϕ_2 exist) $(I + \lambda VR)\phi_1 = C\chi_1$ and $(I + \lambda VR)\bar{\phi}_2 = \bar{\chi}_2 + \lambda(r\bar{\phi}_2, \chi_2)\chi_1$. Since we know that $I + \lambda VR$ is invertible, let us define

$$\Phi_1 = (I + \lambda VR)^{-1}\chi_1, \quad \bar{\Phi}_2 = (I + \lambda VR)^{-1}\bar{\chi}_2 .$$

Then $(I - \lambda KR)\Phi_1 = (I + \lambda VR)\Phi_1 - \lambda(r\Phi_1, \chi_2)\chi_1 = (1 - \lambda(r\Phi_1, \chi_2))\chi_1$ whence $\phi_1 = (1 - \lambda(r\Phi_1, \chi_2))^{-1}\Phi_1$ provided that $1 - \lambda(r\Phi_1, \chi_2) \neq 0$, in which case $C = (1 - \lambda(r\Phi_1, \chi_2))^{-1}$. That is, ϕ_1 does not exist if $1 - \lambda(r\Phi_1, \chi_2) = 0$ or, more graphically but less precisely, if $C = 0$. Similarly, if ϕ_2 exists it can be determined analogously.

3. Approximation of the Inverse of $I - \lambda SKR$

In the notation of the preceding sections, we can construct the inverse, $I + \lambda SLR$, of $I - \lambda SKR$ once we have found the solutions ϕ_1 and ϕ_2 arising from the two equations

$$(I - \lambda KRS)\phi_1 = \chi_1, \quad (I - \lambda KRS)\bar{\phi}_2 = \bar{\chi}_2 . \tag{3.1}$$

In many situations of interest these equations will not readily admit an exact solution, so we are forced to find some approximate solution. Suppose that, by some approximation technique, we have found $\tilde{\phi}_1$ and $\tilde{\phi}_2$ which approximate ϕ_1 and ϕ_2 respectively. We can then calculate the corresponding functions $\tilde{\chi}_1$ and $\tilde{\chi}_2$, approximations to χ_1 and χ_2 respectively, by

$$(I-\lambda KRS)\tilde{\phi}_1 = \tilde{\chi}_1 , \quad (I-\lambda KRS)\bar{\tilde{\phi}}_2 = \bar{\tilde{\chi}}_2 . \tag{3.2}$$

The fact that $\|\chi_1-\tilde{\chi}_1\|$ is small does not of itself tell us that $\|\phi_1-\tilde{\phi}_1\|$ is small; indeed, if we have not solved (3.1) exactly we do not yet know that $I-\lambda KRS$ is invertible.

Let $\tilde{L}$ be the operator formed by using $\tilde{\phi}_1$ and $\tilde{\phi}_2$ in place of ϕ_1 and ϕ_2, that is,

$$(\tilde{L}\phi)(x) = \frac{1}{\tilde{C}}\int_0^1 \tilde{\phi}_1(\min(x,t))\bar{\tilde{\phi}}_2(\max(x,t))\phi(t)dt \qquad (0 \le x \le 1) ,$$

where $\tilde{C}_1 = 1+\lambda(rs\tilde{\phi}_1, \chi_2)$, $\tilde{C}_2 = 1+\lambda(rs\chi_1, \tilde{\phi}_2)$ and $\tilde{C} = (\tilde{C}_1+\tilde{C}_2)/2$. (In this case $\tilde{C}_1$ and $\tilde{C}_2$ need not be equal.) Then $\lambda S\tilde{L}R - \lambda SKR - \lambda^2 S\tilde{L}RSKR$ will not be the zero operator since ϕ_1 and ϕ_2 have not been found exactly. However, its kernel can be computed directly so that, if we denote the corresponding operator by Δ,

$$(I-\lambda SKR)(I+\lambda S\tilde{L}R) = I+\Delta . \tag{3.3}$$

Provided the kernel generating Δ is small enough, we will have $\|\Delta\| < 1$, that $I+\Delta$ is invertible and $\|(I+\Delta)^{-1}\| \le (1-\|\Delta\|)^{-1}$. Moreover, equation (3.3) shows that $I-\lambda SKR$ is surjective if $I+\Delta$ is invertible, so that equations (2.9) do possess solutions ϕ_1 and $\bar{\phi}_2$ as required, showing that $I-\lambda SKR$ is indeed invertible. Then

$$\|(I - \lambda SKR)^{-1}\| \leq \|I + \lambda S\tilde{L}R\| \|(I + \Delta)^{-1}\|$$

$$\leq \frac{\|I + \lambda S\tilde{L}R\|}{1 - \|\Delta\|}, \tag{3.4}$$

giving an explicit bound for the norm of the inverse involving quantities which can be directly calculated. In practical terms, this means that equations (3.1) have to be solved approximately so that the resulting approximate solutions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ yield an operator Δ of small norm, necessarily less than 1 for this estimate to be valid. Using (3.3) we deduce that if $\tilde{\phi} = f + \lambda S\tilde{L}Rf$, the approximate solution obtained by using $\tilde{\phi}_1$ and $\tilde{\phi}_2$ in lieu of ϕ_1 and ϕ_2, then

$$\|\tilde{\phi} - \phi\| \leq \frac{\|I + \lambda S\tilde{L}R\| \, \|\Delta f\|}{1 - \|\Delta\|}. \tag{3.5}$$

It is possible to obtain an estimate of $\|\Delta\|$ in terms of the quantities calculated. Let $k(x,t)$ be the kernel defining $\tilde{L}RSK$ (so that $(\tilde{L}RSK\phi)(x) = \int_0^1 k(x,t)\phi(t)dt$). Then

$$\tilde{C}k(x,t) = \int_0^1 \chi_1(\min(x,u))\bar{\chi}_2(\max(x,u))r(u)s(u)\tilde{\phi}_1(\min(u,t))\bar{\tilde{\phi}}_2(\max(u,t))du$$

giving

$$\tilde{C}k(x,t) = \begin{cases} \bar{\chi}_2(x)\tilde{C}(\tilde{L}RS\chi_1)(t) + (KRS\bar{\tilde{\phi}}_2)(x)\phi_1(t) - (rs\chi_1,\tilde{\phi}_2)\bar{\chi}_2(x)\tilde{\phi}_1(t) & (t < x) \\ \chi_1(x)\tilde{C}(\tilde{L}RS\bar{\chi}_2)(t) + (KRS\tilde{\phi}_1)(x)\bar{\tilde{\phi}}_2(t) - (rs\tilde{\phi}_1,\chi_2)\chi_1(x)\bar{\tilde{\phi}}_2(t) & (t > x) \end{cases}$$

If we then set $\hat{\phi}_1 = (I + \lambda\tilde{L}RS)\chi_1$ and $\bar{\hat{\phi}}_2 = (I + \lambda\tilde{L}RS)\bar{\chi}_2$ and use the notations $\delta\chi_i = \tilde{\chi}_i - \chi_i$, $\delta\tilde{\phi}_i = \phi_i - \tilde{\phi}_i$ $(i = 1,2)$, then we see that the kernel of the operator $\lambda S\tilde{L}R - \lambda SKR - \lambda^2 SKRS\tilde{L}R$ (that is, Δ) is, as a function of x and t,

$$\frac{\lambda}{\tilde{C}} s(x) \left\{ \tilde{\phi}_1(t)(\overline{\delta\chi_2})(x) - \tilde{C}(\delta\tilde{\phi}_1)(t)\bar{\chi}_2(x) + (\tilde{C}_2 - \tilde{C})\bar{\chi}_2(x)\tilde{\phi}_1(t) \right\} \qquad (t < x)$$

and

$$\frac{\lambda}{\tilde{C}} s(x) \left\{ \bar{\tilde{\phi}}_2(t)(\delta\chi_1)(x) - \tilde{C}(\overline{\delta\tilde{\phi}_2})(t)\chi_1(x) + (\tilde{C}_1 - \tilde{C})\chi_1(x)\bar{\tilde{\phi}}_2(t) \right\} \qquad (t > x)$$

whence

$$\|\Delta\| \le \frac{|\lambda|\,\|R\|\,\|S\|}{|\tilde{C}|} \Big\{ \|\tilde{\phi}_1\|\|\delta\chi_2\| + \|\tilde{\phi}_2\|\|\delta\chi_1\| + |\tilde{C}|\,\|\delta\tilde{\phi}_1\|\|\chi_2\| + |\tilde{C}|\,\|\delta\tilde{\phi}_2\|\|\chi_1\| + |C - \tilde{C}_1|\,\|\tilde{\phi}_2\|\|\chi_1\| + |C - \tilde{C}_2|\,\|\tilde{\phi}_1\|\|\chi_2\| \Big\}.$$

The upper bound for $\|\Delta\|$ just given is likely to be pessimistic (i.e. larger than the true value). However, as long as this upper bound is less than 1 the inequality (3.5) is available and this can be used to find the norm of the difference between the exact solution operator and the approximate solution operator. Since this involves $\|\Delta\|$ it, too, is likely to be pessimistic. Another approach is to notice that $\tilde{\phi}_1 - \phi_1 = (I + \lambda LRS)(\tilde{\chi}_1 - \chi_1) = (I - \lambda KRS)^{-1}\delta\chi_1$. From this and (3.4) we see that

$$\|\tilde{\phi}_1 - \phi_1\| \le \frac{\|I + \lambda\tilde{L}RS\|}{1 - \|\Delta_1\|} \|\delta\chi_1\|$$

where $\Delta_1 = \lambda\tilde{L}RS - \lambda KRS - \lambda KRS\tilde{L}RS$, and the analogous result for $\|\tilde{\phi}_2 - \phi_2\|$ is proved in the same way. This provides a direct estimate of $\|L - \tilde{L}\|$.

4. The Inverse of $I - \lambda S\hat{K}R$

The method we have used in the preceding sections extends to a slightly wider class of operators, so that we can solve equations of the form $(I - \lambda S\hat{K}R)\phi = f$ where S and R are as before and $\hat{K}$ is of the form

$$\hat{K}\phi = -V\phi + f(\phi)\chi_1 + g(\phi)\bar{\chi}_2 \tag{4.1}$$

for given linear functionals f, g on $L_2(0,1)$; V here is, as before, given by $V\phi(x) = \int_0^x (\chi_1(x)\bar{\chi}_2(t) - \chi_1(t)\bar{\chi}_2(x))\phi(t)dt$. This includes the operator defined in (1.10). The significance of this apparently minor extension is that it includes some cases of practical importance. For example, Chamberlain (1) reduces a wave scattering problem to one of solving the integral equation

$$\phi(x) - \lambda \int_0^1 \sin(k_0|x-t|)r(t)\phi(t)dt = f(x) \quad (0 \le x \le 1),$$

where $k_0 \in \mathbb{R}$. This equation is an example of $\phi = f + \lambda\hat{K}R\phi$ in which the operator $\hat{K}$ arises with $\chi_1(x) = 2\cos k_0x$, $\chi_2(x) = \sin k_0x$, $f(\phi) = \frac{1}{2}(\phi, \chi_2)$, $g(\phi) = -\frac{1}{2}(\phi, \chi_1)$.

As in the earlier work, we shall consider the simpler situation in which $S = I$ first. Suppose, then, that $\hat{K}$ is defined by (4.1) and that the two equations

$$(I - \lambda\hat{K}R)\phi_1 = \chi_1, \quad (I - \lambda\hat{K}R)\bar{\phi}_2 = \bar{\chi}_2 \tag{4.2}$$

have solutions $\phi_1, \bar{\phi}_2 \in L_2(0,1)$. Then

$$\begin{pmatrix} \phi_1(x) \\ \bar{\phi}_2(x) \end{pmatrix} = \begin{pmatrix} 1 - \lambda\int_0^x \bar{\chi}_2 r\phi_1 + \lambda f(r\phi_1) & \lambda\int_0^x \chi_1 r\phi_1 + \lambda g(r\phi_1) \\ -\lambda\int_0^x \bar{\chi}_2 r\bar{\phi}_2 + \lambda f(r\bar{\phi}_2) & 1 + \lambda\int_0^x \bar{\chi}_1 r\bar{\phi}_2 + \lambda g(r\bar{\phi}_2) \end{pmatrix} \begin{pmatrix} \chi_1(x) \\ \bar{\chi}_2(x) \end{pmatrix}$$

where, as before, the determinant of the matrix is constant, its value being $\hat{C} = (1 + \lambda f(r\phi_1))(1 + \lambda g(r\bar{\phi}_2)) - \lambda^2 f(r\bar{\phi}_2)g(r\phi_1)$.

Then it follows that

$$\left.\begin{aligned} \hat{C}\chi_1 - \lambda WR\chi_1 &= \phi_1 + \lambda g(r\bar{\phi}_2)\phi_1 - \lambda g(r\phi_1)\bar{\phi}_2 \\ \hat{C}\bar{\chi}_2 - \lambda WR\bar{\chi}_2 &= \bar{\phi}_2 - \lambda f(r\bar{\phi}_2)\phi_1 + \lambda f(r\phi_1)\bar{\phi}_2 \end{aligned}\right\} \qquad (4.3)$$

where

$$(W\phi)(x) = \int_0^x (\phi_1(x)\bar{\phi}_2(t) - \phi_1(t)\bar{\phi}_2(x))\phi(t)dt \qquad (0 \le x \le 1)\,.$$

From this we deduce (in the same spirit as before) that

$$\lambda VRWR = \lambda WRVR = \hat{C}VR - WR\,.$$

This equation shows us that if $\hat{C} = 0$, then $WR = 0$, which (by (4.3)) shows that ϕ_1 and $\bar{\phi}_2$ are linearly dependent, whence so are χ_1 and $\bar{\chi}_2$. In this trivial case we soon see that a contradiction arises if $\hat{C} = 0$ as long as (4.2) holds.

In the general case, then, where (4.2) has solutions and $\tilde{C} \neq 0$, we set

$$\hat{L}\phi = -\frac{1}{\hat{C}} W\phi + f((I - \frac{\lambda}{\hat{C}} RW)\phi)\phi_1 + g((I - \frac{\lambda}{\hat{C}} RW)\phi)\bar{\phi}_2$$

so that $\hat{L}$ is related to W in a similar way to that in which $\hat{K}$ is related to V. Then some straightforward but tedious calculations show that

$$I + \lambda\hat{L}R = (I - \lambda\hat{K}R)^{-1}\,.$$

The principal structural point here is that, provided the two special cases in (4.2) can be solved, $I - \lambda\hat{K}R$ is invertible and the solution of $(I - \lambda\hat{K}R)\phi = f$ is given in terms of the solution of (4.2).

The case of $(I - \lambda S\hat{K}R)^{-1}$ can be deduced from that of $I - \lambda\hat{K}R$ in the same way as in §2, the equations to be solved in this case being

$$(I - \lambda S\hat{K}R)\psi_1 = S\chi_1 \ , \quad (I - \lambda S\hat{K}R)\bar{\psi}_2 = S\bar{\chi}_2 \, .$$

Approximations to $\hat{L}$ and associated error bounds follow as in the case of L.

5. Conclusions

This paper extends the class of integral equations whose solutions are known to be finitely generated to include equations of Sturm–Liouville type. Until recently, investigations of the sort of structural results we have established here were confined to operators with difference kernels. Porter and Stirling (4) have shown that such operators are particular examples of a wider class. The present contribution provides a further generalisation in a different direction.

It has been shown that the solution of $\phi = f + \lambda SKR\phi$ (where K, R and S are defined in (1.1), (1.6) and (1.15) respectively) is determined for all admissible f by the solutions corresponding to two particular cases of f. A similar result applies to the related situation where $\hat{K}$ (as in (1.10)) replaces K.

The most immediate practical application of the theory developed arises because the class of equations considered includes all those equivalent to the general Sturm–Liouville boundary–value problem given by (1.2) and (1.3). Where approximation methods have to be used to solve the Sturm–Liouville problem, the present approach not only provides a method of approximating the solution, but one with an explicit error bound available, without imposing undue restrictions on the quantities involved. Existing approximation techniques rely, for example, on the assumption that K is self–adjoint, that r be of constant sign or that $I - \lambda KR$ be a positive operator. We have not required such assumptions.

This work was partly motivated by Chamberlain's (1) investigation into surface water wave scattering by uneven beds. In that paper Chamberlain used existing Hilbert space techniques to approximate the scattered wave amplitude and encountered the difficulty that good error estimates were available only under certain restricting assumptions. Even in the case where the function r was of one sign, allowing the problem to be re–cast into a form using a self–adjoint operator, an explicit error bound was not available. Roughly speaking, the awkwardness arose from the lack of an effective bound on $\|(I - \lambda K)^{-1}\|$ if $I - \lambda K$ was invertible but possessed both positive and negative eigenvalues. We have found such a bound, and moreover in our case the operators involved need not be self–adjoint. The techniques above can be used to extend the cases in which Chamberlain's results can be applied and will be of practical value in other problem areas.

REFERENCES

(1) P.G. CHAMBERLAIN Wave Scattering over Uneven Depth Using the Mild-Slope Equation *Wave Motion* **17** (1993), 267–285.

(2) D. PORTER The Solution of Integral Equations with Difference Kernels *J. Integral Equations and Appls.* **3** (1991), 429–454.

(3) D. PORTER and D.S.G. STIRLING *Integral Equations* (Cambridge University Press, 1990).

(4) D. PORTER and D.S.G. STIRLING Finitely Generated Solutions of Certain Integral Equations, to appear in *Proc. Edinburgh Math. Soc.* (1994).

(5) K.A. STROMBERG *An Introduction to Classical Real Analysis* (Wadsworth International, Belmont, California, 1981).