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Unique Extension of Atomic Functionals

of JB*-Triples

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Declaration

I confirm that this is my own work and the use of all material from other
sources has been properly and fully acknowledged.

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Abstract

This thesis initiates and proceeds to develop a theory of unique norm preserving extensions of extreme dual ball functionals and their σ-convex sums, in the category of JB*-triples. All such functionals are completely determined by Cartan factors and ℓ_∞-sums of Cartan factors residing as weak* closed ideals in the second dual of a JB*-triple. Implications for structure, particularly involving inner ideals, is a theme running throughout the thesis.

This thesis makes an analysis of inclusions $C \subset D$ of Cartan factors for which there exist an element in $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$. A number of abstract characterisations are given and special cases examined in detail. We prove that, for C to be an inner ideal it is sufficient for a single functional in $S(C_{*,1}) \setminus \partial_e(C_{*,1})$ to have unique norm one extension.

Information gathered on Cartan factors is used to develop a more general theory of unique extension of dual ball extreme points, of Cartan functionals and other atomic functionals, and culminates with an investigation of the extreme, Cartan and atomic extension properties of a separable JB*-subtriple in a JBW*-triple.

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Contents

Introduction

It is the aim of this thesis to study the phenomenon of uniqueness of Hahn-Banach extensions for certain classes of functionals in the category of JB*-triples.

Let A be a JB*-subtriple of a JB*-triple B. In these circumstances A is said to have the *extension property* in B if every element of A^* has a unique norm preserving extension in B^*. Kept in mind throughout the thesis is the following algebraic characterisation proved in [33].

> *A has the extension property in B if and only if*
> *A is a norm closed inner ideal of B.*

This extension property fails in general when A is a c_0-sum of norm closed inner ideals of B. However, in this case it happens that A does have the *extreme extension property* in B, that each element of $\partial_e(A_1^*)$, the set of extreme points of the closed dual ball A_1^*, has a unique extension in $\partial_e(B_1^*)$. But the property of A being a c_0-sum of norm closed inner ideals in B is a strong algebraic condition far from being equivalent to A having the extreme extension property in B.

Broadly speaking, this thesis is an investigation of the implications for the relative structure of A and B when functionals in $\partial_e(A_1^*)$, and associated functionals, have unique Hahn-Banach extensions to corresponding functionals in B^*. The associated functionals we refer to are the atomic functionals and a certain subclass of these which we call Cartan functionals because of their association with Cartan factors (in the ordered category of C*-algebras they correspond to type I factors). We study both *individual* unique Hahn-Banach extensions of these types of functionals as well as the global, *extreme*, *Cartan* and *atomic* extension properties in A and B. The multiplicity of different kinds of Cartan factors is a cause of some quite severe complications. As opposed to the theory of the pure state extension

property for C*-algebras given in [13], the extreme extension property for JB*-triples does not imply the Cartan extension property.

The thesis is organised as follows.

Chapter 1 contains no new results. Its purpose is to collect together the well-known background of JB*-triples which is fundamental for the understanding of the remainder of the thesis. Chapter 1 gives priority to key results and concepts which will be used frequently in the sequel. Known material which is used infrequently is introduced in the place where it is first needed.

Chapter 2, which is largely expository, contains Banach space theory appropriate to the theme of this thesis and sets the scene. In addition to some necessary background material, the second chapter also contains a brief discussion of Banach spaces X for which all closed subspaces have the extension property in X. It is pointed out that Hilbert spaces are the only JB*-triples with this property. The chapter concludes with a brief discussion of c_0-sums of closed subspaces of a Banach space X in relation to the extreme extension property.

The focus of Chapter 3 is on unique norm preserving extension of predual elements of Cartan factors. When C is a Cartan factor and JBW*-subtriple of a Cartan factor D it is shown (Theorem 3.3.4) that the existence of a single element ρ of $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ implies that the same is true of *every* element of $\partial_e(C_{*,1})$ and analysis is made of Cartan factor inclusions that force this property, summarised in Theorem 3.3.11.

When this partial unique extension condition compels C to be an inner ideal is considered and conclusions drawn (for example, Theorem 3.4.8). Turning attention to norm one non-extreme elements in the predual of C,

one of our key results (Theorem 3.5.6) is that unique norm one extension of just one of these forces C to be an inner ideal. The chapter concludes with applications to von Neumann algebras.

Extreme points of the dual ball of a JB*-triple live on Cartan factors contained in the atomic part of the second dual, allowing the work of Chapter 3 to be brought to bear, in Chapter 4, upon unique extension theory of elements of $\partial_e(A_1^*)$ to $\partial_e(B_1^*)$ when A is a JB*-subtriple of a JB*-triple B and consequently upon the situation when A has the extreme extension property in B, and connections with representation theory is made. The *Cartan extension property* is introduced and characterisations given in terms of a stronger unique extension property and of inner ideal structure of atomic parts (Theorem 4.4.5). In an extension of Theorem 3.5.6, a main conclusion of Chapter 4 is that a norm one atomic functional has a unique extension to a norm one atomic functional if and only if it has a unique norm one extension (Theorem 4.5.1) and a discussion of the atomic extension property evolves. A brief review of repercussions for appropriate *state* extensions in the category of JB*-algebras is provided.

Chapter 5 culminates in a solution of the extreme, Cartan and atomic extension properties of a *separable* JB*-subtriple in a JBW*-triple and, in so doing, extends C*-algebra work of [13], [18] to the JB*-triple setting. To acquire this solution we are lead into different areas, obtaining a number of independently interesting results in Banach space properties of JB*-triples along the way. In particular, for every JB*-triple A it is shown that weak sequential convergence in $\partial_e(A_1^*)$ coincides with norm convergence (Theorem 5.3.7). When A is *separable*, exploiting an important result in [19], we deduce that weak* sequential convergence and norm convergence coincide in $\partial_e(A_1^*)$ precisely when A is a weakly compact JB*-triple. This enables the above mentioned solution. The chapter begins with a discussion of weakly compact JB*-triples in which a new 'inner ideal' equivalence is included.

Chapter 1

Jordan Structures

1.1 Introduction

In Hilbert space models of quantum mechanics 'observables' are usually bounded self-adjoint operators acting on a Hilbert space. The product of two such self-adjoint operators is self-adjoint only when the operators commute. However, self-adjointness is preserved by the symmetrised product

$$a \circ b = \frac{1}{2}(ab + ba).$$

In this way Jordan algebras were conceived by Jordan, von Neumann and Wigner [51] as an algebraic model of quantum mechanics.

In this chapter we lay the foundations of Jordan structures which will be utilised later. We first briefly review Jordan algebras and complex Jordan *-triple systems and then, with the addition of a suitably well-behaved norm, the structures known as JB*-algebras and JB*-triples. The principal texts on Jordan operator algebras are [4] and [43]. All unreferenced or unjustified properties of Jordan*-algebras in this chapter are well-known and may be found in these books.

JB*-triples generalise JB*-algebras which in turn generalise C*-algebras, and one reason for interest in JB*-triples lies in its inclusion of such subclasses of significance. An important abstract property of JB*-triples is that they are closed under contractive projections [40], [53], [67], a property not possessed by C*-algebras or JB*-algebras. For a general history of JB*-algebras and JB*-triples, the reader is referred to the surveys of Rodríguez [62] and Russo [64].

The papers [38], [39] and [41] of Friedman and Russo serve as a good introduction to the properties of JB*-triples. 'Concrete' JB*-triples, now known as JC*-triples, were first studied by Harris in [45] and [46] under the name J*-algebras. Although now firmly a branch of functional analysis and operator algebra, JB*-triples have their origin in the theory of bounded symmetric domains. The equivalence of JB*-triples and Banach spaces whose open unit ball is a bounded symmetric domain is due to Kaup [52]. A survey and history of this area (not discussed in this thesis) is given in [20].

The reader is referred to the books of Pedersen [61] and Rudin [63] for the basics of functional analysis. Relevant material on C*-algebras can be found in [58] and [60].

General mathematical notation used throughout this thesis is standard. If X is a Banach space, X_1 and $S(X_1)$ denote the closed unit ball and unit sphere of norm one elements, respectively. In the usual way, X is regarded as being a subspace of X^{**} and X^* is identified with the weak* continuous linear functionals on X^{**}. Our usage of 'σ-convex sum' includes 'convex sum'. Thus a σ-convex sum of a Banach space X is a finite or infinite series $\sum \lambda_n x_n$ where $\lambda > 0$ for all n and $\sum \lambda_n = 1$. If K is a convex set $\partial_e(K)$ denotes the set of extreme points of K. For a locally compact Hausdorff space S, $C_0(S)$ stands for the continuous functionals vanishing at infinity. The notations $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ stand for the real numbers, complex numbers and the quaternions, respectively.

1.2 Algebras

By an *algebra* shall be meant a complex linear space A together with a bilinear map, $\pi : A \times A \to A$, referred to as a *product.* If accompanied by a conjugate linear map of order two, $a \mapsto a^*$, such that

$$(\pi(a, b))^* = \pi(b^*, a^*)$$

then, with respect to π, $a \mapsto a^*$ is an *involution* and A is a **-algebra.* The powers of an element a may be defined inductively by

$$a^{n+1} = \pi(a^n, a), \qquad \text{for all } n \geqslant 1.$$

In this generality no assumption is made concerning commutativity nor associativity of products. In particular, the above definition of powers is one-sided and we need not have $(a^n)^* = (a^*)^n$, in the case of a *-algebra.

1.3 Jordan Algebras

An algebra A with product, $(a, b) \mapsto a \circ b$, is said to be a *Jordan algebra* if

$$a \circ b \;=\; b \circ a \quad \text{and} \quad a \circ (b \circ a^2) \;=\; (a \circ b) \circ a^2.$$

The associated *Jordan triple product* is

$$\{a \circ b \circ c\} \;=\; (a \circ b) \circ c \;+\; (c \circ b) \circ a \;-\; (a \circ c) \circ b.$$

Jordan algebras are commutative but not necessarily associative. An associative Jordan algebra is said to be *abelian.*

Let a belong to A, where A is a Jordan algebra. We have, for $m, n \geqslant 1$,

$$a^{m+n} \;=\; a^m \circ a^n,$$

from which it follows that the Jordan subalgebra generated by a is abelian.

The multiplication operator T_a and quadratic operator U_a are defined on A by

$$T_a(b) = a \circ b \quad \text{and} \quad U_a(b) = \{a \circ b \circ a\}.$$

In particular, $U_a = 2T_a^2 - T_{a^2}$. The elements a and b of A are said to *operator commute* if $T_a T_b = T_b T_a$. All polynomials in a operator commute. The *centre*, $Z(A)$, of A is the set

$$Z(A) = \{x \in A : x \text{ operator commutes with all } y \text{ in } A\},$$

which is an abelian subalgebra of A.

A Jordan algebra A over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) containing an identity element 1 is said to be a *factor* if $Z(A) = \mathbb{F}1$, and $a \in A$ is said to be *invertible* if there exists b in A such that

$$a \circ b = 1 \quad \text{and} \quad a^2 \circ b = a.$$

Let A be a Jordan *-algebra. The set of self-adjoint elements of A,

$$A_{sa} = \{x \in A : x = x^*\},$$

is a real Jordan algebra. The *projections* of A are the elements p in A_{sa} such that $p^2 = p$. Projections p and q in A are said to be *orthogonal* if $p \circ q = 0$.

A significant 'special' class of Jordan algebras arises in associative algebras as follows. Let A be an associative algebra with product $(a, b) \mapsto ab$. The symmetrised product

$$a \circ b = \frac{1}{2}(ab + ba)$$

defined upon A makes A into a Jordan algebra. In this case, we have

$$U_a(b) = aba \quad \text{and} \quad \{a \circ b \circ c\} = \frac{1}{2}(abc + cba).$$

The Jordan algebras that arise in this way, up to Jordan isomorphism, are referred to as *special* Jordan algebras. The Jordan algebras that are not expressible in this way are called *exceptional*.

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An *ideal* of a Jordan algebra A is a linear subspace I for which $A \circ I$ is contained in I. In which case, A/I is canonically a Jordan algebra and I is, in particular, a subalgebra of A. A *quadratic ideal* of a Jordan algebra A is a linear subspace I for which $U_a(I)$ is contained in A, for all elements a in A.

1.4   Complex Jordan *-Triple Systems

1.4.1   Throughout this thesis a *Jordan *-triple system* shall mean a complex linear space A together with a triple product, $\{\cdots\} : A^3 \to A$, linear and symmetric in the outer two variables, conjugate linear in the middle variable and satisfying the *main identity*

$$\{ab\{xyz\}\} \;=\; \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}.$$

Given a and b in A, where A is a Jordan *-triple system, the operators $D(a,b)$ and $Q_{a,b}$ on A are defined by

$$D(a,b)x \;=\; \{abx\} \quad \text{and} \quad Q_{a,b}(x) \;=\; \{axb\}.$$

The first of these operators is linear and the second is conjugate linear. We shall write Q_a for $Q_{a,a}$.

1.4.2   Let A be a Jordan *-triple system. Then A satisfies the *polarisation identities*

(a) $2Q_{a,b} \;=\; Q_{a+b} - Q_a - Q_b$

(b) $4D(a,b) \;=\; \displaystyle\sum_{k=0}^{3} i^k D(a + i^k b, a + i^k b)$

(c) $4\{aba\} \;=\; \displaystyle\sum_{k=0}^{3} (-1)^k \{a + i^k b, a + i^k b, a + i^k b\}.$

Other useful identities (an extensive list can be found in [57]) are

(i) $D(x,y)Q_x = Q_xD(y,x)$

(ii) $D(Q_x(y),y) = D(x,Q_y(x))$

(iii) $Q_{Q_x(y)} = Q_xQ_yQ_x$

(iv) $[D(x,y),D(a,b)] = D(\{xya\},b) - D(a,\{bxy\})$, where $[\cdot,\cdot]$ denotes the commutator.

By means of the above identity (b), and separately that of (iv) – the latter in conjunction with liberal use of the fact that the triple product is symmetric in the outer variables, we have the following.

Proposition 1.4.3

The following are equivalent for a Jordan *-triple system A.

(a) $D(x,x)D(y,y) = D(y,y)D(x,x)$, *for all $x,y \in A$.*

(b) $D(x,y)D(a,b) = D(a,b)D(x,y)$, *for all $a,b,x,y \in A$.*

(c) $\{xy\{abc\}\} = \{x\{yab\}c\}$, *for all $a,b,c,x,y \in A$.*

A Jordan *-triple system satisfying any of the equivalent conditions of Proposition 1.4.3 is said to be *abelian*.

Given an element a in a Jordan *-triple system A, the odd 'powers' of a are defined inductively by

$$a^1 = a, \quad a^{(2n+1)} = \{aa^{(2n-1)}a\}, \quad n \geqslant 1.$$

The set of all odd powers of a is an abelian subtriple of A, the Jordan *-triple in A generated by a.

Let $\pi : A \to B$ be a linear map between Jordan *-triple systems. If

$$\pi(\{abc\}) = \{\pi(a)\pi(b)\pi(c)\}, \quad \text{for all } a,b,c \in A$$

then π is said to be a *triple homomorphism*. In fact, by polarisation, π is a triple homomorphism if

$$\pi(\{aaa\}) = \{\pi(a)\pi(a)\pi(a)\}, \quad \text{for all } a \in A.$$

1.4.4 Significant classes of Jordan *-triple systems arise from Jordan *-algebras and associative *-algebras in a way now described.

(a) Let A be a complex Jordan *-algebra. Then A is a Jordan *-triple system with respect to the triple product

$$\{abc\} \; = \; \{a \circ b^* \circ c\} \; = \; (a \circ b^*) \circ c + a \circ (b^* \circ c) - (a \circ c) \circ b^*.$$

(b) If A is an associative complex *-algebra, so that A is a special Jordan *-algebra via the special Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, then A is a Jordan *-triple system via

$$\{abc\} \; = \; \frac{1}{2}(ab^*c + cb^*a).$$

We note that (b) is a special case of (a).

1.4.5 Let I be a linear subspace of a Jordan *-triple system A. In order of increasing generality, I is said to be

(a) an *ideal* of A if $\{IAA\} + \{AIA\}$ is contained in I;

(b) an *inner ideal* of A if $\{IAI\}$ is contained in I;

(c) a *subtriple* of A if $\{III\}$ is contained in I.

By the polarisation identities of 1.4.2 the conditions (b) and (c) are respectively equivalent to

(b$'$) $\{xAx\} \subset I$, for all $x \in I$;

(c$'$) $\{xxx\} \in I$, for all $x \in I$.

The quotient space A/I is naturally a Jordan *-triple system whenever I is an ideal of A.

We remark that if A is a complex Jordan *-algebra, then the *-ideals of A (that is, those invariant under the given involution) coincide with the ideals of A when A is regarded as a Jordan *-triple system in the way described in 1.4.4(a).

1.4.6 Let A be a Jordan *-triple system. An element e of A is said to be a *tripotent* if $e = \{eee\}$. Let e be a tripotent of A. The linear maps on A given by

$$P_2(e) = Q_e^2, \quad P_1(e) = 2(D(e,e) - Q_e^2) \quad \text{and} \quad P_0(e) = I - 2D(e,e) + Q_e^2$$

satisfy

(a) $P_2(e) + P_1(e) + P_0(e) = I$,

and for $i, j \in \{0, 1, 2\}$,

(b) $P_i(e)P_j(e) = \delta_{ij}P_i(e)$,

(c) $P_j(e)(A) = \{x \in A : 2D(e,e)x = jx\}$.

The $P_i(e)$ are the *Peirce projections* on A. We write $A_j(e) = P_j(e)(A)$ for $j \in \{0, 1, 2\}$.

With the contraction $A_j = A_j(e)$, we have the *Peirce decomposition* and *Peirce rules*

(d) $A = A_2 \oplus A_1 \oplus A_0$,

(e) $\{A_i A_j A_k\} \subset A_{i-j+k}$, whenever $i - j + k \in \{0, 1, 2\}$, and
    $\{A_i A_j A_k\} = 0$ in all other cases,

(f) $\{A_2 A_0 A\} = \{A_0 A_2 A\} = 0$.

It follows from (e) that A_2 and A_0 are inner ideals of A and that A_1 is a subtriple of A.

(g) The tripotent e is said to be a

(i) *minimal* tripotent of A if $A_2(e) = \mathbb{C}e$;

(ii) *complete* tripotent of A if $A_0(e) = \{0\}$;

(iii) *unitary* tripotent of A if $A_2(e) = A$.

(iv) *abelian* tripotent of A if the Jordan *-algebra $A_2(e)$ is associative.

If I is an inner ideal of A and $e \in I$, it follows from the definitions that

(h) if e is a minimal tripotent of I, then e is a minimal tripotent of A.

1.4.7 For each tripotent e in a Jordan *-triple system A, the associated Peirce projections are polynomials in $D(e,e)$.

$$\begin{aligned}
P_2(e) &= 2D(e,e)^2 - D(e,e) \\
P_1(e) &= 4(D(e,e) - D(e,e)^2) \\
P_0(e) &= I - 3D(e,e) + 2D(e,e)^2.
\end{aligned}$$

Conversely,

$$2D(e,e) = 2P_2(e) + P_1(e) = I + P_2(e) - P_0(e).$$

The following is immediate from 1.4.7.

Proposition 1.4.8

*The following conditions are equivalent for tripotents e and f in a Jordan *-triple system A.*

(a) $P_i(e)P_j(f) = P_j(f)P_i(e)$, *for all* $i,j \in \{0,1,2\}$.

(b) $D(e,e)D(f,f) = D(f,f)D(e,e)$.

Proposition 1.4.9

*The following conditions are equivalent for tripotents e and f in a Jordan *-triple system A.*

(a) $\{eef\} = 0$.   (b) $\{ffe\} = 0$.

(c) $D(e,f) = 0$.   (d) $D(f,e) = 0$.

Let e and f be tripotents in a Jordan *-triple system A. When the conditions of Proposition 1.4.8 hold, then e and f are said to be *compatible*. The tripotents e and f are said to be *orthogonal* when the conditions of Proposition 1.4.9 hold and to be *collinear* if

$$f \in A_1(e) \quad \text{and} \quad e \in A_1(f).$$

We write $e \perp f$ when e and f are orthogonal and $e \leqslant f$ when $f - e$ is a tripotent orthogonal to e.

Since, for $j \in \{0, 1, 2\}$, $f \in A_j(e)$ if and only if $2\{eef\} = jf$, the following observation is a consequence of 1.4.2(iv).

Lemma 1.4.10

Tripotents e, f in a Jordan *-triple system are compatible if $f \in A_j(e)$ for some $j \in \{0, 1, 2\}$. In particular, orthogonal tripotents are compatible and collinear tripotents are compatible.

The final statement of this subsection embodies a key localisation process of theoretical and technical importance.

Theorem 1.4.11

Let e be a tripotent in a Jordan *-triple system A.

(a) *Then $A_2(e)$ is a complex Jordan *-algebra with product and involution*

$$a \circ b \ = \ \{aeb\} \quad and \quad a^{\#} \ = \ \{eae\}.$$

Moreover, the triple product induced on $A_2(e)$ by this Jordan *-algebra structure (see (1.4.4(a)) coincides with its original triple product.

(b) *A tripotent f in A is a projection in the Jordan *-algebra $A_2(e)$ if and only if $f \leqslant e$.*

(c) *Projections f and g in the Jordan *-algebra $A_2(e)$ are orthogonal projections if and only if f and g are orthogonal tripotents of A.*

1.5 JB*-Algebras

1.5.1 A *JB*-algebra* is a Jordan *-algebra and Banach space A satisfying the following conditions.

(a) $\|a \circ b\| \leqslant \|a\| \, \|b\|$, for all $a, b \in A$.

(b) $\|a^*\| = \|a\|$, for all $a \in A$.

(c) $\|\{a \circ a^* \circ a\}\| = \|a\|^3$, for all $a \in A$.

A *JB-algebra* is a real Jordan algebra equipped with a complete norm satisfying condition (a) above and

(d) $\|a^2\| = \|a\|^2$, for all $a \in A$;

(e) $\|a^2\| \leqslant \|a^2 + b^2\|$, for all $a, b \in A$.

The self-adjoint part, A_{sa}, of a JB*-algebra A is a JB-algebra. Conversely, the complexification of a JB-algebra has a unique norm organising it as a JB*-algebra [69].

A *JBW*-algebra* M is a JB*-algebra with a predual M_*. JBW*-algebras always have an identity element. If A is a JB*-algebra, then A^{**} is a JBW*-algebra containing A as a JB*-subalgebra. The JB*-subalgebra of A^{**} generated by A and the identity 1 of A^{**} is denoted by $\tilde{A}$.

The Gelfand theory and functional calculus of self-adjoint elements in JB*-algebras is the same as for C*-algebras. If A is a JB*-algebra and $a \in A_{sa}$ then the JB*-subalgebra, $C^*(1, a)$, of A^{**} generated by 1 and a is a commutative C*-algebra and there is a surjective *-isomorphism

$$C^*(1, a) \longrightarrow C(\sigma(a)),$$

where $\sigma(a)$ is the compact subset of $\mathbb{R}$ given by

$$\sigma(a) \ = \ \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } C^*(1, a)\}.$$

The latter set is the spectrum of $a \in A_{sa}$ and coincides with the set

$$\{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } \tilde{A}\}.$$

The class of JB*-algebras includes all C*-algebras. A C*-algebra A is a JB*-algebra via the special Jordan product

$$a \circ b = \frac{1}{2}(ab + ba)$$

and A_{sa} is a JB-algebra. A *JC*-algebra* is a JB*-algebra that is (isometrically) Jordan *-isomorphic to a norm closed Jordan *-subalgebra of a C*-algebra. The self-adjoint part of a JC*-algebra is called a *JC-algebra*. The canonical example of a JB*-algebra that is not a JC*-algebra is the exceptional algebra M_3^8 of the 3×3 hermitian matrices over the complex octonions (see 1.11.1 (6)).

A *JW*-algebra* is a JC*-algebra with a predual. The second dual A^{**} of a JC*-algebra A is a JW*-algebra. The following Gelfand-Naimark theorem shows that every JBW*-algebra decomposes into an orthogonal sum of 'special' and 'exceptional' weak* closed ideals.

Theorem 1.5.2 [66, 3.9]
Every JBW-algebra is an ℓ_∞-sum of the form*

$$M \oplus C(X, M_3^8)$$

where M is a JW-algebra and X is a compact hyperstonean space. Hence, every JB*-algebra is weak* dense in a JBW*-algebra of this form.*

1.5.3 The set of positive elements of a JB*-algebra A,

$$A_+ = \{a^2 : a \in A_{sa}\},$$

is stable under addition and by multiplication by positive scalars. Moreover, $A_{sa} = A_+ - A_+$. The set of positive linear functionals on A is

$$A_+^* = \{\varphi \in A^* : \varphi(A_+) \subset [0, \infty)\}.$$

The *quasi-state space* of A,

$$Q(A) = \{\varphi \in A_+^* : \|\varphi\| \leqslant 1\} = A_+^* \cap A_1^*,$$

is a convex weak*-compact subset of A_1^*. The *state space* of A

$$S(A) = \{\rho \in A^* : \rho(1) = 1 = \|\rho\|\},$$

where 1 is the identity element of A^{**}, is the convex set of positive linear functionals of norm 1. We have

$$\partial_e(Q(A)) \setminus \{0\} = \partial_e(S(A)).$$

This latter set is the set of *pure states* of A and is denoted by $P(A)$.

A positive linear functional φ on a JB*-algebra A is said to be *faithful* if

$$(\ker \varphi) \cap A_+ = \{0\}$$

The *normal states* of a JBW*-algebra M are the weak*-continuous states of M.

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1.6 JB*-Triples

A *JB*-triple* is a complex Banach space and a Jordan *-triple system A such that

(a) $\|\{aaa\}\| = \|a\|^3$, for all $a \in A$;

(b) for each $a \in A$, the operator $D(a, a)$ is hermitian with non-negative spectrum.

Every norm closed subtriple of a JB*-triple is a JB*-triple. A *JBW*-triple* is a JB*-triple M with a predual M_*. By [48, 3.21, 3.24] and [6, 2.1], such a predual is unique and the triple product on a JBW*-triple is separately weak* continuous in each variable. The second dual of a JB*-triple is a JBW*-triple containing A as a JB*-subtriple [26].

If e is a tripotent in a JBW*-triple M the Peirce subspaces $M_j(e)$ are JBW*-subtriples of M for $j = 0, 1$ and 2, and $M_2(e)$ is a JBW*-algebra with the product and involution given in 1.4.4.

A fundamental result concerning triple homomorphisms is the following.

Proposition 1.6.1 [52, 5.5]
Let $\pi : A \to B$ be a map between JB*-triples. Then π is a surjective linear isometry if and only if it is a triple isomorphism.

Every JB*-algebra is a JB*-triple with triple product

$$\{abc\} = \{a \circ b^* \circ c\}.$$

In particular, every JC*-algebra and so every C*-algebra is a JB*-triple with triple product

$$\{abc\} = \frac{1}{2}(ab^*c + cb^*a).$$

Those JB*-triples linearly isometric to a JB*-subtriple of a C*-algebra are termed *JC*-triples*. The JBW*-triples that are JC*-triples are referred to as *JW*-triples*.

For any pair of complex Hilbert spaces H and K the Banach space of operators $B(H, K)$ with triple product

$$\{abc\} = \frac{1}{2}(ab^*c + cb^*a),$$

is realised as a JW*-subtriple of $B(H \oplus K)$ via the triple embedding

$$a \mapsto \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

As a notable special case, the Hilbert space H is a JW*-triple with product

$$\{xyz\} = \frac{1}{2}\left(\langle x, y \rangle z + \langle z, y \rangle x\right).$$

The exceptional JB*-triple factors (see 1.11.1) are M_3^8 and the 1×2 matrices over the complex octonions, denoted by $B_{1,2}$, which can be realised as a JB*-subtriple of M_3^8 [57]. The following Gelfand-Naimark theorem for JB*-triples is proved in [42].

Theorem 1.6.2

(a) *Every JBW*-triple is linearly isometric to an ℓ_∞-sum*

$$A \oplus C(X, B_{1,2}) \oplus C(Y, M_3^8),$$

where A is a JC-triple and X and Y are compact hyperstonean spaces.*

(b) *Every JB*-triple is linearly isometric to a JB*-subtriple of a JB*-algebra.*

1.7 Local JB*-algebra structure

If x is an element of a JB*-triple A, A_x and $A(x)$ respectively denote the JB*-subtriple and norm closed inner ideal of A generated by x. If e is a tripotent in a JB*-triple A, then $A(e) = A_2(e)$. The following is an amalgamation of results found in [14, §2], [31, §3] and [52, §5].

Theorem 1.7.1
Let x be an element of a weak* dense JB*subtriple A of a JBW*-triple M.

(a) *There is a unique tripotent $r(x)$ of M such that*

$$\overline{A(x)}^{w^*} = M_2(r(x))$$

  and that x is a positive element of the JBW*-algebra $M_2(r(x))$.

(b) *Further, when $M_2(r(x))$ is regarded as a JBW*-algebra,*

  (i) *$A(x)$ is a JB*-subalgebra of $M_2(r(x))$ with $x \in A(x)_+$;*

  (ii) *A_x is the commutative C*-subalgebra of $A(x)$ generated by x;*

  (iii) *$\overline{A_x}^{w^*}$ is the commutative von Neumann subalgebra of $M_2(r(x))$ generated by x, and has identity element $r(x)$.*

In Theorem 1.7.1, the tripotent $r(x)$ is the *range tripotent* of x in M.

1.7.2 When x is an element of a JB*-triple A we shall write (as in [14])

$$S(x) = \sigma(x) \setminus \{0\},$$

where $\sigma(x)$ is the spectrum of the positive element x of the JB*-algebra $A(x)$ in the sense of Theorem 1.7.1. This gives rise to the 'triple' Gelfand transform

$$A_x \xrightarrow{\varphi} C^*(x) \xrightarrow{\psi} C_0(\sigma(x)) = C_0(S(x)),$$

which can be interpreted as follows. On the left, A_x is the JB*-subtriple of A generated by x (as defined above), $C^*(x)$ is the commutative C*-algebra generated by the positive element x in the JB*-algebra $A(x)$ and φ is the identity map between them (they are equal as sets) and thus a triple isomorphism with respect to their relative triple products. In particular,

$$\varphi\left(x^{(2n+1)}\right) \;=\; x^{2n+1} \qquad \text{for all } n \geqslant 0.$$

The map ψ is the usual Gelfand transform. Letting

$$y \;=\; \varphi^{-1}\left(x^{\frac{1}{3}}\right),$$

we have $y \in A_x$ with $\{yyy\} = x$. Thus x has a 'cube root' in A_x.

1.8 Ideal Theory

In JB*-triples the algebraic requirements for a norm closed subspace to be an ideal weakens (see 1.4.5(a)).

Proposition 1.8.1 [12, 1.3], [25, 1.4]

Let I be a norm closed subspace of a JB*-triple A. Then the following conditions are equivalent.

(a) *I is an ideal of A.*   (b) *$\{AAI\} \subset I$.*

(c) *$\{AIA\} \subset I$.*   (d) *$\{AII\} \subset I$.*

The important Theorem 1.8.2 is proved in [42], [52].

Theorem 1.8.2

Let J be a norm closed ideal of a JB*-triple A. Then

(a) *A/J is a JB*-triple;*

(b) *A/J is a JC*-triple if A is a JC*-triple.*

The classification of norm closed inner ideals in a JB*-triple by unique extensions is fundamental.

Theorem 1.8.3 [33, 2.5, 2.6]

Let I be a JB*-subtriple of a JB*-triple A. Let J be a JBW*-subtriple of a JBW*-triple M.

(a) *I is an inner ideal of A if and only if each ρ in I^* has unique norm preserving extension in A^*.*

(b) *J is an inner ideal of M if and only if each ρ in J_* has unique norm preserving extension in M_*.*

In the notation of Theorem 1.8.3 it is clear (Hahn-Banach Theorem) that if $\rho \in J_*$ there is a norm preserving extension in M^*, but not clear that such an extension can always be found in M_*. It is true, however.

Theorem 1.8.4 [10]

If M is a JBW*-subtriple of a JBW*-triple N and $\rho \in M_*$, then ρ has a norm preserving extension in N_*.

1.8.5 A projection, $P : A \to A$, on a JB*-triple A is a *structural projection* if

$$P(\{aP(b)c\}) \;=\; \{P(a)bP(c)\}$$

for all $a, b, c \in A$. In which case, by separate weak* continuity of the triple product, the weak* continuous extension

$$P^{**} : A^{**} \longrightarrow A^{**}$$

is a structural projection and thus is contractive (see Theorem 1.8.6 below). In particular, it follows that P is contractive and that $P(A)$ is a norm closed inner ideal of A. If u is a tripotent in A, then $P_2(u)$ and $P_0(u)$ are examples of structural projections on A.

The following is an amalgamation of [30, §5] and [35, §4].

Theorem 1.8.6

Let M be a JBW*-triple.

(a) *Every structural projection on M is weak* continuous and contractive.*

(b) *There is a bijection from the set of all structural projections of M onto the set of all weak* closed inner ideals of M given by*

$$P \longmapsto P(M).$$

1.9 Orthogonality

1.9.1 The notion of orthogonality of tripotents (see remarks following Proposition 1.4.9) in JB*-triples generalises.

Let A be a JB*-triple. Elements a and b in A are said to be *orthogonal* if

$$D(a, b) \; = \; 0.$$

($D(a, b) = 0$ if and only if $D(b, a) = 0$ [34, 3.1].) Let $a, b \in A$ and S and T be subsets of A. We write

(a) $a \perp b$ to denote that a is orthogonal to b,

(b) $a \perp S$ to denote that a is orthogonal to every element of S,

(c) $S \perp T$ to denote that each element of S is orthogonal to each element of T.

We denote the *orthogonal complement of S in A* by

$$S^{\perp} \; = \; \{x \in A : x \perp S\}.$$

The results contained in Theorem 1.9.2 (below) are all well-known and can be found in [14], [34] and [48], for example.

Theorem 1.9.2

Let A be a JB*-triple. Let S be a subset of A, B and C be JB*-subtriples of A, I a norm closed inner ideal of A and let J be a norm closed ideal of A.

(a) *$S^\perp$ is a norm closed inner ideal of A, and is an ideal of A if S is an ideal of A. Moreover, $S \subset S^{\perp\perp}$.*

(b) *$A_2(e)^\perp = A_0(e)$ for each tripotent e in A.*

(c) *If B and C are orthogonal (that is, $B \perp C$), then*

$$B + C \;=\; B \oplus_\infty C$$

is a JB*-subtriple of A containing B and C as norm closed ideals.

(d) *$I \cap J = \{0\}$ if and only if $I \perp J$.*

(e) *If A is a JBW*-triple and J is weak* closed then*

$$A \;=\; J + J^\perp \quad (= J \oplus_\infty J^\perp).$$

Lemma 1.9.3

Let I be a weak* closed inner ideal in a JBW*-triple M and let (J_i) be a family of mutually orthogonal weak* closed ideals of M. Then

$$I \cap \left(\sum J_i \right) \;=\; \sum I \cap J_i.$$

Proof

Let R denote the left hand side, above. Let $x \in R$ and choose $y \in R$ such that $x = \{yyy\}$ (see 1.7.2). We have $y = \sum x_i$, where $x_i \in J_i$ for all i. Therefore, since the x_i are mutually orthogonal

$$x \;=\; \sum \{yx_iy\} \in \sum I \cap J_i,$$

proving that R is contained in $\sum I \cap J_i$. The converse is clear. $\qquad \square$

The orthogonal decomposition of weak* closed ideals in a JBW*-triple described by Theorem 1.9.2(e) is a starting point of an elaborate decomposition theory of 'types' of JBW*-triples [48], [49], [50].

A JBW*-triple M is said to be *type I* if each non-zero weak* closed ideal of M contains a non-zero abelian tripotent, and to be *continuous* if M contains no non-zero abelian tripotents. If M is a JBW*-triple and J is the smallest weak* closed ideal of M containing all abelian tripotents of M then

$$M = J \oplus J^{\perp},$$

where J is type I and $J^{\perp}$ is continuous [48, (4.13)].

Let M be a JBW*-triple. If M has no non-trivial weak* closed ideals it is referred to as a *factor*. (When M is a JBW*-algebra this is equivalent to M having trivial centre.) Since non-zero abelian tripotents in a factor are minimal tripotents [48, (4.9)], the JBW*-triple factors of type I, also called Cartan factors (see Section 1.11 for details), are the factors that posses a minimal tripotent. The cardinality of a maximal orthogonal family of minimal tripotents in a type I factor M is an invariant referred to as the *rank* of M.

Let e and f be orthogonal tripotents of a JB*-triple A. Then $A_2(e)$ is orthogonal to $A_2(f)$. Indeed,

$$f \in A_0(e).$$

Thus, the inner ideal $A_0(e)$ must contain the inner ideal generated by f. That is,

$$A_2(f) \subset A_0(e) = A_2(e)^{\perp}.$$

The following generalises this situation.

Lemma 1.9.4

Let a and b be elements of a JB*-triple A, and let $r(a)$ and $r(b)$ be the corresponding range tripotents in A^{**}. The following are equivalent.

(a) $a \perp b$.

(b) $A(a) \perp A(b)$.

(c) $A_2^{**}(r(a)) \perp A_2^{**}(r(b))$.

(d) $r(a) \perp r(b)$.

Proof

The implications (b)$\Rightarrow$(a), (c)$\Rightarrow$(d) are obvious and (d)$\Rightarrow$(c) was noted above. Moreover (c)$\Rightarrow$(b) since

$$A(a) \subset A_2^{**}(r(a)) \quad \text{and} \quad A(b) \subset A_2^{**}(r(b)).$$

In order to show that (a)$\Rightarrow$(b), let $a \perp b$. Then $a \in \{b\}^{\perp}$, a norm closed inner ideal of A. Hence,

$$A(a) \subset \{b\}^{\perp}.$$

So,

$$b \in \{b\}^{\perp\perp} \subset A(a)^{\perp},$$

giving $b \perp A(a)$. Repeating the argument for each element of $A(a)$, we conclude that

$$A(b) \perp A(a).$$

The implication (b)$\Rightarrow$(c) is a consequence of the equalities following from Theorem 1.7.1(a),

$$A(a)^{**} = A_2^{**}(r(a)), \quad A(b)^{**} = A_2^{**}(r(b))$$

and separate weak* continuity of the triple product. $\square$

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It follows that if a and b are orthogonal elements in a JB*-triple A, so that by Theorem 1.9.2(c) the JB*-subtriple

$$A(a) + A(b) \;=\; A(a) \oplus_\infty A(b),$$

we have that

$$\|a + b\| \;=\; \max(\|a\|, \|b\|).$$

1.10   Support Tripotents

The key notion of support tripotents of functionals in the predual of a JBW*-triple, and much more, was studied in the seminal paper on JBW*-triples [41] which is the main reference for this section.

Let ρ be a normal state of a JBW*-algebra M. The *support projection*, $s(\rho)$, is the least projection upon which ρ takes the value 1. By restriction, ρ is a faithful normal state of $\{s(\rho)Ms(\rho)\}$ $(= P_2(s(\rho))(M))$. The *central support, $c(\rho)$,* of ρ is the least projection $z \in Z(M)$ such that $\rho(z) = 1$.

If v is a tripotent of a JBW*-triple M and ρ is a norm one functional in M_* such that

$$\rho(v) \;=\; 1,$$

then, by restriction, ρ is a normal state of the JBW*-algebra $M_2(v)$.

The important Theorem 1.10.1 is contained in [41, Proposition 2].

Theorem 1.10.1

Let ρ be a norm one functional in the predual of a JBW*-triple M.

(a) *There is a unique tripotent u of M such that $\rho(u) = 1$ and ρ is faithful on $M_2(u)$.*

(b) *If v is a tripotent of M such that $\rho(v) = 1$ then*

$$u \leqslant v \quad and \quad \rho \;=\; \rho \circ P_2(v).$$

In the notation of Theorem 1.10.1, the tripotent u in (a) is said to be the *support tripotent* of ρ and is again denoted by $s(\rho)$.

When $\rho \in S(A_1^*)$, where A is a JB*-triple, unless otherwise stated $s(\rho)$ is understood to be the support tripotent of ρ in A^{**}. The following reflects both usages.

Theorem 1.10.2 [41, Proposition 4]
Let M be a JBW*-triple and A a JB*-triple. Then

$$\rho \longmapsto s(\rho)$$

defines a bijection from

(a) *$\partial_e(M_{*,1})$ onto the set of minimal tripotents of M;*

(b) *$\partial_e(A_1^*)$ onto the set of minimal tripotents of A^{**}.*

Moreover, if $\rho \in \partial_e(M_{*,1})$ then

$$\rho(x)s(\rho) \;=\; P_2(s(\rho))(x)$$

for each x in M.

1.10.3   The corresponding well-known statement for a JBW*-algebra M and JB*-algebra A is that

$$\rho \longmapsto s(\rho)$$

defines a bijection from

(a) *the pure normal states of M onto the set of minimal projections of M;*

(b) *$P(A)$ onto the set of minimal projections of A^{**}.*

In the next statement, (a) is a consequence of Theorem 1.8.3, Theorem 1.10.2 and 1.4.6(h), and (b) is due to Theorem 1.10.2 and the property that if J is a norm closed ideal of A and u is a minimal tripotent of A^{**} then $u \in J^{**}$ or $u \in (J^{**})^{\perp}$.

Corollary 1.10.4

Let A be a JB*-triple. Let I be a norm closed inner ideal of A and let J be a norm closed ideal of A.

(a) $\{\rho \in \partial_e(A_1^*) : s(\rho) \in I^{**}\} \longrightarrow \partial_e(I_1^*) \qquad (\rho \longmapsto \rho|_I)$

  is a bijection, the inverse of which is the norm one unique extension map.

(b) *Let $\rho \in \partial_e(A_1^*)$. Then*

  (i) $s(\rho) \in J^{**}$ *if and only if $\rho(J) \neq \{0\}$.*

  (ii) $s(\rho) \perp J^{**}$ *if and only if $\rho(J) = \{0\}$.*

1.10.5 Two functionals in the predual of a JBW*-triple are said to be *orthogonal* if their support tripotents are orthogonal.

Let $\rho, \tau \in M_*$, where M is a JBW*-triple such that ρ and τ are orthogonal and

$$\|\rho\| = \|\tau\| = 1.$$

Since

$$\rho = \rho \circ P_2(s(\rho)) \quad \text{and} \quad s(\rho) \perp s(\tau),$$

we have

$$\rho(s(\tau)) = 0 = \tau(s(\rho)).$$

Further, since $\|s(\rho) - s(\tau)\| = 1$ and $(\rho - \tau)(s(\rho) - s(\tau)) = 2$, we have that

$$\|\rho - \tau\| = 2.$$

Lemma 1.10.6

Let $\rho \in M_*$, where M is a JBW*-triple. Suppose that ρ is a σ-convex sum

$$\rho \; = \; \sum \lambda_n \rho_n \qquad \left(\sum \lambda_n = 1, \; \lambda_n > 0 \; \text{for all } n \right)$$

of mutually orthogonal norm one functionals $\rho_n \in M_*$. Then

$$\|\rho\| \; = \; 1 \quad \text{and} \quad s(\rho) \; = \; \sum s(\rho_n).$$

Proof

Since $\rho_n(s(\rho_m)) = 0$ whenever $m \neq n$, we have

$$\rho \left(\sum s(\rho_n) \right) \; = \; \sum \lambda_n \; = \; 1.$$

Therefore, $s(\rho) \leqslant \sum s(\rho_n)$, by Theorem 1.10.1 (b). On the other hand

$$\sum \lambda_n \rho_n(s(\rho)) \; = \; 1,$$

giving $\rho_n(s(\rho)) = 1$ and hence $s(\rho_n) \leqslant s(\rho)$, for all n. Thus, since

$$\sum s(\rho_n) \; \leqslant \; s(\rho),$$

the required equality results. □

1.11   Cartan Factors

The Cartan factors (defined below) are exactly the type I JBW*-triple factors [49, 1.8]. There are six *generic types*. Namely, *rectangular, hermitian, symplectic* factors and *spin factors*, all four of which are JC*-triples, and two further *exceptional factors* $B_{1,2}$ and M_3^8.

In the following, complexifications of JB-algebras are assumed to be in possession of the corresponding JB*-algebra norm [69].

Let H and K be complex Hilbert spaces of respective (possibly infinite) orthonormal dimension n and m. Let

$$j : H \longrightarrow H$$

be a conjugation and let $x \mapsto x^t \; = \; jx^*j$ denote the induced transpose on $B(H)$. (It is a *-antiautomorphism of order 2 called a *real flip* [43, §7].)

1.11.1

(1) $B(H, K)$ is the *rectangular* factor, $M_{n,m}$. We write $M_{n,n} = M_n$.

(2) For $n \geqslant 2$, $\{x \in B(H) : x^t = x\}$ is the *hermitian* factor S_n.

(3) For $n \geqslant 4$, $\{x \in B(H) : x^t = -x\}$ is the *symplectic* factor A_n.

Remarks

(a) If $m \leqslant n$, so that K may be regarded as a closed subspace of H, we have

$$M_{m,n} \cong M_{n,m}$$

induced by $x \mapsto x^t$. For finite m and n, $M_{m,n}$ is the space of $m \times n$ complex matrices.

(b) The hermitian factor S_n is a JBW*-algebra Jordan *-isomorphic to

$$B(H_{\mathbb{R}})_{sa} + iB(H_{\mathbb{R}})_{sa},$$

where $H_{\mathbb{R}}$ is the real Hilbert space such that

$$H = H_{\mathbb{R}} + iH_{\mathbb{R}}, \quad \text{given by} \quad H_{\mathbb{R}} = \{h \in H : jh = h\} \quad [43, \S7].$$

For $2 \leqslant n < \infty$, S_n is the space of symmetric $n \times n$ complex matrices.

(c) For any $n \geqslant 2$, A_{2n} is linearly isometric to a JBW*-algebra. Indeed, in this case [43, 7.5.6] there is a conjugate linear isometry

$$v : H \longrightarrow H \quad \text{such that} \quad v^2 = -1$$

inducing a quaternionic Hilbert space structure, $H_{\mathbb{H}}$ on H, and the map

$$A_{2n} \longrightarrow B(H_{\mathbb{H}})_{sa} + iB(H_{\mathbb{H}})_{sa} \quad (x \longmapsto -vjx)$$

is a surjective linear isometry. When n is finite this implies

$$A_{2n} \cong M_n(\mathbb{H})_{sa} + iM_n(\mathbb{H})_{sa}$$

For any finite n, A_n is the space of antisymmetric $n \times n$ complex matrices.

(4) *Spin Factors*

For any cardinal number n, the JBW*-triple *spin factor* V_n is the complexification

$$U_n + iU_n$$

of the JBW-algebra *real* spin factor [43, §6]

$$U_n = L_n \oplus \mathbb{R}1,$$

where L_n is a real Hilbert space of orthonormal dimension $n \geqslant 2$, with product and norm given by

$$(a + \alpha 1) \circ (b + \beta 1) = \beta a + \alpha b + (\langle a, b \rangle + \alpha \beta)1$$

and

$$\|a + \alpha 1\| = \|a\|_2 + |\alpha| \quad (\|\cdot\|_2 \text{ is the Hilbert norm}).$$

The spin factors V_n are reflexive and topologically equivalent to the complex Hilbert space of orthonormal dimension $n + 1$. By a *spin factor V* we shall always mean a *complex* spin factor V_n, for some n, unless otherwise explicitly stated.

The generic types of the two exceptional factors are next given. A clear account of these factors can be found in [36].

(5) $B_{1,2}$. This denotes the sixteen dimensional factor composed of the space of 1×2 matrices over the complex octonions.

(6) M_3^8. This is the 27 dimensional JBW*-algebra of the 3×3 hermitian matrices over the complex octonions.

1.11.2 Type I JBW*-algebra factors

The five kinds of Cartan factor M_n, S_n, A_{2n} (in the sense of (c)), V_n and M_3^8 are the *generic* types of JBW*-algebra factors of type I. There is some overlap with low dimensional spin factors. Namely,

$$V_2 \cong S_2, \quad V_3 \cong M_2, \quad \text{and} \quad V_5 \cong A_4.$$

1.11.3 The rank of a Cartan factor is the cardinality of a maximal family of mutually orthogonal minimal tripotents. The following table gives the ranks of the types of Cartan factors, where $m \leqslant n$.

Cartan factor	$M_{m,n}$	S_n	A_{2n}	A_{2n+1}	V_n	$B_{1,2}$	M_3^8
Rank	m	n	n	n	2	2	3

For a tripotent u in a Cartan factor C the *rank* of u is defined to be rank$(C_2(u))$. Up to linear isometry, Hilbert spaces are the rectangular $M_{1,n}$ factors and are precisely the Cartan factors of rank one.

There are several triple embeddings amongst Cartan factors, discussed in some detail in Chapter 3. Examples are

(a) $B_{1,2} \hookrightarrow M_3^8$: $\quad (x,y) \mapsto \begin{pmatrix} 0 & x & y \\ \bar{x} & 0 & 0 \\ \bar{y} & 0 & 0 \end{pmatrix}$,

and for $m \leqslant s < \infty$ and $n \leqslant t < \infty$, we have

(b) $M_{m,n} \hookrightarrow M_{s,t}$: $\quad X \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$,

(c) $A_n \hookrightarrow A_{n+1}$: $\quad X \mapsto X'$,

where X' is obtained from X by adding a column of zeros on the right and a row of zeros on the bottom.

In particular, (a) realises $B_{1,2}$ as a subtriple of M_3^8. We note that (b) realises $M_{m,n}$ as an inner ideal of $M_{s,t}$ (for $m \leqslant s$ and $n \leqslant t$) and (c) realises A_n as an inner ideal of A_{n+1}.

The smallest non-zero norm closed ideal of a Cartan factor is the *elementary ideal* of C, denoted by $K(C)$ [11].

Note: in this notation, standard in JB*-triple theory, if H is a Hilbert space then $K(H) = H$, and $K(B(H))$ is the C*-algebra of compact operators on H.

The following compilation is drawn from [11], [12], [21] and [41] and summarises salient properties of Cartan factors.

Theorem 1.11.4

(a) *Let C be a Cartan factor.*

    (i) *C is the weak* closed linear span of its minimal tripotents.*

    (ii) *$K(C)$ is the norm closed linear span of the minimal tripotents of C.*

    (iii) *$K(C)^{**} = C$, and $K(C) = C$ if and only if C has finite rank.*

    (iv) *Every weak* closed inner ideal of C is a Cartan factor.*

(b) *Let e be a minimal tripotent in a JBW*-triple M and let f be a minimal tripotent in a JB*-triple A.*

    (i) *The weak* closed ideal of M generated by e is a Cartan factor.*

    (ii) *The norm closed ideal of A generated by f is the elementary ideal of a Cartan factor.*

1.12 Atomic JBW*-Triples and Decomposition

A JBW*-triple is defined to be *atomic* if it is the weak* closed linear span of its minimal tripotents.

In Theorem 1.12.1, part (a) follows from Theorem 1.11.4 (b)(i), and part (b) is [41, Remark 2.8].

Theorem 1.12.1

Let M be an atomic JBW-triple.*

  (a) *M is an ℓ_∞-sum of Cartan factors.*

  (b) *Every norm one functional in the predual of M is a σ-convex sum of mutually orthogonal functionals in $\partial_e(M_{*,1})$.*

If M is a JBW*-triple we denote the smallest weak* closed ideal of M containing the minimal tripotents of M by M_{at} and shall refer to it as the *atomic part* of M. By Theorem 1.9.2(e), the orthogonal decomposition of weak* closed ideals

$$M \; = \; M_{at} \oplus (M_{at})^\perp$$

is automatic, and $(M_{at})^\perp$ contains no minimal tripotents. In light of these remarks the following is crucial.

Theorem 1.12.2 [41, Theorem 2]

If M is a JBW-triple then M_{at} is the weak* closed linear span of the minimal tripotents of M. Hence, M_{at} is an atomic JBW*-triple.*

Let A be a JB*-triple and let $\rho \in A^*$. Then ρ is defined to be an *atomic functional* of A if $s(\rho) \in A_{at}^{**}$. It is immediate from Theorem 1.10.2(b) that ρ is atomic for all $\rho \in \partial_e(A_1^*)$. Consider the orthogonal decomposition

$$A^{**} \; = \; A_{at}^{**} \oplus (A_{at}^{**})^\perp.$$

Let

$$P_{at} : A^{**} \longrightarrow A^{**}_{at}$$

denote the canonical projection. Via Theorem 1.10.1 we have that the set S of atomic functionals of A is given by

$$S \;=\; \{\rho \in A^* : \rho((A^{**}_{at})^{\perp}) = \{0\}\} \;=\; \{\rho \in A^* : \rho = \rho \circ P_{at}\}.$$

In particular, S is a norm closed subspace of A^* and

$$S \longrightarrow (A^{**}_{at})_* \qquad (\rho \longmapsto \rho|_{A^{**}_{at}})$$

is a surjective linear isometry.

Theorem 1.12.3

Let $\rho \in S(A^*_1)$, where A is a JB*-triple. Then the following are equivalent.

(a) *ρ is an atomic functional of A.*

(b) *ρ is a σ-convex sum of mutually orthogonal elements in $\partial_e(A^*_1)$.*

(c) *ρ is a σ-convex sum of elements in $\partial_e(A^*_1)$.*

Proof

The implication (b)$\Rightarrow$(c) is clear. The implication (c)$\Rightarrow$(a) follows from the fact that the atomic functionals of A form a norm closed subspace of A^* containing $\partial_e(A^*_1)$. It remains to see that (a)$\Rightarrow$(b).

Suppose that ρ is atomic. By Theorem 1.12.1(b) and Theorem 1.12.2 we have that, on A^{**}_{at}, ρ is a σ-convex sum

$$\rho \;=\; \sum \lambda_n \rho_n \tag{$*$}$$

where the ρ_n are mutually orthogonal in $\partial_e(A^*_1)$. But ρ vanishes on the orthogonal complement of A^{**}_{at}, since $\rho = \rho \circ P_2(s(\rho))$, and so do all the ρ_n. Hence the above equation ($*$) holds everywhere. $\qquad\square$

Chapter 2

Hahn-Banach Extensions

2.1 Introduction

Let E be a norm closed subspace of a Banach space X and let $\rho \in E^*$. The Hahn-Banach theorem guarantees the existence of a norm preserving extension of ρ, also referred to as a Hahn-Banach extension of ρ, in X^*. Such extensions need not be unique. Moreover, if ρ has two distinct norm preserving extensions φ and ψ in X^*, then it has uncountably many, since each element of the straight line joining φ to ψ in X^*,

$$\{(1 - \alpha)\varphi + \alpha\psi : \alpha \in [0, 1]\},$$

is again a norm preserving extension of ρ.

The question of existence of unique Hahn-Banach extensions seems to be one of considerable complexity, as may be seen by the recent article [5] and references therein. The subject impinges upon JB*-triple theory via a path pioneered in [30], [33], and [35] (see Theorems 1.8.3 and 1.8.6). Involved, broadly speaking, are generalisations of M-ideal theory. Extensive accounts of L- and M-theory can be found in [2], [3], [7] and [44] from which main definitions and some results are recalled. Pertinent Banach space 'smoothness' properties are briefly reviewed and applied in the context of JB*-triples. In addition, useful conclusions are drawn in connection with unique Hahn-Banach extensions of dual ball extreme points.

2.1.1 We shall work with the following definitions. Let X be a Banach space and let E be a norm closed subspace of X.

(a) E is said to have the *extension property* in X if each $\rho \in E^*$ has unique norm preserving extension in X^*.

(b) E is said to have the *extreme extension property* in X if each $\rho \in \partial_e(E_1^*)$ has unique extension in $\partial_e(X_1^*)$.

(c) X is said to have the *extension property* if every norm closed subspace of X has the extension property in X.

(d) X is said to have the *extreme extension property* if every norm closed subspace of X has the extreme extension property in X.

2.2   L-Summands, M-Summands and Ideals

2.2.1   Let E be a closed subspace of a Banach space X. We have the following definitions.

(a) E is an *L-summand* of X if there exists a closed subspace F of X such that $E \oplus F = X$ and

$$\|x + y\| = \|x\| + \|y\|$$

for each $x \in E$ and $y \in F$.

(b) E is an *M-summand* of X if there exists a closed subspace F of X such that $E \oplus F = X$ and

$$\|x + y\| = \max\{\|x\|, \|y\|\}.$$

for each $x \in E$ and $y \in F$.

We also have the following related definitions, where P is a projection on X.

(c) P is said to be an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\|$$

for each $x \in X$.

(d) P is said to be an *M-projection* if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\}$$

for each $x \in X$.

(e) P is said to be *neutral* if P is contractive, and if $x \in X$ such that $\|Px\| = \|x\|$, then $Px = x$.

The complement $I - P$ of an M-projection P is an M-projection. Similarly, complements of L-projections are L-projections. Every L-projection is a neutral projection. The map $P \mapsto P(X)$ defines bijections from the set of

(i) L-projections on X onto the set of L-summands of X;

(ii) M-projections on X onto the set of M-summands of X.

We further recall that

(iii) $P : X \to X$ is an M-projection if and only if $P^* : X^* \to X^*$ is an L-projection;

(iv) $P : X \to X$ is an L-projection if and only if $P^* : X^* \to X^*$ is an M-projection.

2.2.2 Let E be a norm closed subspace of a Banach space X. The topological annihilator of E in X^* is denoted by

$$E^\circ \ = \ \{\rho \in X^* : \rho(E) = \{0\}\}.$$

The *subset* $E^\sharp$ (it need not be a linear subspace) of X^* given by

$$E^\sharp = \{\rho \in X^* : \|\rho|_E\| = \|\rho\|\}$$

satisfies

$$E^\circ \cap E^\sharp \ = \ \{0\} \quad \text{and} \quad X^* = E^\circ + E^\sharp \qquad \text{(set addition)}.$$

(a) E is said to be an *M-ideal* of X if E° is an L-summand of X^*. In which case, $E^\sharp$ is a subspace of X^* and is the complementary L-summand of E° in

$$X^* \ = \ E^\circ \oplus E^\sharp.$$

(b) E is said to be an *N-ideal* of X if $E^\sharp$ is a subspace of X^*. In which case

$$X^* \ = \ E^\circ \oplus E^\sharp$$

and the projection onto $E^\sharp$ is contractive and neutral [35].

(c) E is said to be a *Banach ideal* of X if there is a contractive projection P on X^* with

$$\ker P \ = \ E^\circ.$$

Remarks. The term *Banach ideal* used above is non-standard. Subspaces satisfying (c) are referred to simply as *ideals* of X in [59], for example. But the latter usage conflicts widely with that used elsewhere in this thesis. The term N-ideal was used in [35].

2.2.3 Consider now a norm closed subspace E of a Banach space X. Suppose there is a contractive projection P on X^* such that

$$\ker P \ = \ E^\circ.$$

Let $\rho \in X^*$. By definition, ρ and $P\rho$ agree on E and for each $\tau \in E^\circ$ we have

$$\|P\rho\| \ = \ \|P(\rho - \tau)\| \ \leqslant \ \|\rho - \tau\|$$

so that

$$\|P\rho\| \ \leqslant \ \|\rho + E^\circ\| \ = \ \|\rho|_E\|$$

and thus $\|P\rho\| = \|\rho|_E\|$. Hence, $P\rho$ is a Hahn-Banach extension of $\rho|_E$ and $P(X^*)$ is contained in $E^\sharp$.

Now suppose that, in addition, E has the extension property in X. Then $P\rho$ must be the unique norm preserving extension of $\rho|_E$, and $P(X^*) = E^\sharp$. Moreover, if $\|P\rho\| = \|\rho\|$, then $P\rho = \rho$, by uniqueness. In particular, P is a neutral projection.

On the other hand, if P is a neutral projection on X^* with

$$\ker P \ = \ E^\circ$$

then for $\rho \in E^\sharp$ we have

$$\|\rho\| \ = \ \|\rho|_E\| \ = \ \|P\rho\|,$$

so that $\rho = P\rho$ and hence $E^\sharp = P(X^*)$, implying that E is an N-ideal of X. The following statement summarises the above.

Lemma 2.2.4 [59], [35]

The following are equivalent for a norm closed subspace E of a Banach space X.

(a) *E is a Banach ideal of X with the extension property in X.*

(b) *E is an N-ideal of X.*

(c) *There is a (unique) neutral projection P on X^* with $\ker P = E^\circ$.*

We note that if E is an N-ideal of a Banach space X and

$$\varphi : E^* \longrightarrow X^*$$

denotes the norm preserving unique extension map then

$$\varphi : E^* \longrightarrow E^\sharp \qquad \text{and} \qquad \sigma : E^\sharp \longrightarrow E^* \qquad (\rho \longmapsto \rho|_E)$$

are mutually inverse surjective linear isometries.

The relevance of the above for JB*-triples is that the M-ideals of a JB*-triple are precisely its norm closed ideals [6, Theorem 3.2] and the following variation of Theorem 1.8.3.

Theorem 2.2.5 [30], [33], [35]

The following are equivalent for a JB-subtriple E of a JB*-triple X.*

(a) *E has the extension property in X.*

(b) *E is an N-ideal of X.*

(c) *E is an inner ideal of X.*

2.3 The Extension Property

2.3.1 We recall (for example, see [22], [55]) that a Banach space X is said to be

(a) *smooth* if for each $x \in S(X_1)$ there is a unique $\varphi \in X_1^*$ such that $\varphi(x) = 1$;

(b) *strictly convex* if whenever $x, y \in X$ we have

$$\|x\| \ = \ \|y\| \ = \ \|(x+y)/2\| \ = \ 1 \quad \text{implies} \quad x = y.$$

It is useful to record that the condition (b) is equivalent to

(b′) $S(X_1) = \partial_e(X_1)$.

Each of the conditions (a) and (b) passes to all norm closed subspaces (via the Hahn-Banach theorem in the case of (a)).

There is a well-known (partial) duality between smoothness and strict convexity due to Klee.

Theorem 2.3.2 [54, A1.1]

Let X be a Banach space.

   (a) *If X^* is smooth then X is strictly convex.*

   (b) *If X^* is strictly convex then X is smooth.*

   (c) *If X is reflexive then the converses of* (a) *and* (b) *are true.*

2.3.3 All Hilbert spaces are both smooth and strictly convex. It is clear that neither $\mathbb{C} \oplus_\infty \mathbb{C}$ nor $(\mathbb{C} \oplus_\infty \mathbb{C})^* \cong \mathbb{C} \oplus_1 \mathbb{C}$ is strictly convex and thus, by Theorem 2.3.2(c), neither of these spaces is smooth. Hence, if X is smooth or strictly convex then X cannot contain an isometric copy of $\mathbb{C} \oplus_\infty \mathbb{C}$.

The connection with the extreme extension property comes from the following theorem.

Theorem 2.3.4 [37], [68, Theorem 6]

A Banach space X has the extension property if and only if X^* is strictly convex.

The proceeding elementary result is useful.

Lemma 2.3.5

The following are equivalent for a JB*-triple A.

   (a) *Every norm one element is a tripotent.*

   (b) *No two non-zero elements are orthogonal.*

   (c) *A is a Hilbert space.*

Proof

To see that (a)⇒(b), note that if u and v are non-zero orthogonal tripotents of A then $u + \frac{1}{2}v$ has norm one but is not a tripotent.

In order to show that (b)⇒(a),(c), suppose that (b) holds. Let $x \in A$ such that $\|x\| = 1$. We show that x is a minimal tripotent. The JB*-subtriple A_x generated by x is linearly isometric to a commutative C*-algebra with no non-trivial orthogonality and thus must be one dimensional, proving that x is a tripotent. Given a positive norm one element y in the unital JB*-algebra $A_2(x)$, we have that y is a projection (by above) with $y \leqslant x$. Since $y \perp x - y$, we have $y = x$ and thus $A_2(x) = \mathbb{C}x$. Hence, x is a minimal tripotent.

The norm closed ideal J of A generated by x is such that $J \cong K(C)$, where C is a Cartan factor (Theorem 1.11.4), and the condition (b) gives that $\mathrm{rank}(C) = 1$, so that $K(C) = C$ is a Hilbert space. Since any minimal tripotent in A not in J is orthogonal to J, we deduce that $A = C$.

The implication (c)⇒(b) follows from the remark in 2.3.3, since if x and y are non-zero orthogonal elements, the subspace

$$\mathbb{C}x + \mathbb{C}y \cong \mathbb{C} \oplus \mathbb{C}. \qquad \qquad \square$$

The following deduction is now almost automatic.

Theorem 2.3.6

The following are equivalent for a JB-triple X.*

(a) X^* *is strictly convex.*

(b) X *is smooth.*

(c) X *is a Hilbert space.*

(d) X^* *is smooth.*

(e) X *is strictly convex.*

(f) X *has the extension property.*

Proof

This is a straightforward consequence of Theorem 2.3.2, Theorem 2.3.4 and Lemma 2.3.5. □

We record an immediate corollary.

Corollary 2.3.7

A JB*-triple X is a Hilbert space if and only if all JB*-subtriples of X are inner ideals of X.

Proof

Combine Theorem 2.2.5 and Theorem 2.3.6. □

2.4   Extension of Extreme Functionals

Let E be a norm closed subspace of a Banach space X. Given $\rho \in S(E_1^*)$ and $x \in S(X_1)$ we shall write

$$\mathcal{E}(\rho, E) \; = \; \{\varphi \in X_1^* : \varphi|_E = \rho\}$$

and

$$\mathcal{E}^x \; = \; \{\varphi \in X_1^* : \varphi(x) = 1\}.$$

The Hahn-Banach theorem guarantees that these sets are always non-empty. We shall denote the set of norm attaining elements of $S(X_1^*)$ by $N(X_1^*)$.

Lemma 2.4.1

Let E be a norm closed subspace of a Banach space X and let $\rho \in S(E_1^*)$.

(a) $\mathcal{E}(\rho, E)$ *is a weak* compact convex subset of X_1^*.*

(b) *If $\rho \in \partial_e(E_1^*)$ then*

    (i) $\mathcal{E}(\rho, E)$ *is a face of X_1^*;*

    (ii) ρ *has an extension in $\partial_e(X_1^*)$.*

Proof

(a) Note that $\mathcal{E}(\rho, E)$ is weak* closed, since if φ is the weak* limit of a net (φ_λ) in $\mathcal{E}(\rho, E)$, then

$$\varphi(a) \;=\; \lim \varphi_\lambda(a) \;=\; \rho(a)$$

for all a in E, so that $\varphi \in \mathcal{E}(\rho, E)$. Now, since X_1^* is weak* compact, the same is true of $\mathcal{E}(\rho, E)$.

    For convexity, let $\lambda \in [0, 1]$ and $\varphi, \psi \in \mathcal{E}(\rho, E)$. Then

$$\lambda\varphi + (1 - \lambda)\psi \in X_1^*$$

and

$$(\lambda\varphi + (1 - \lambda)\psi)|_E \;=\; \lambda\varphi|_E + (1 - \lambda)\psi|_E \;=\; \rho$$

so that $\lambda\varphi + (1 - \lambda)\psi \in \mathcal{E}(\rho, E)$, as required.

(b)(i) Suppose $\lambda \in (0, 1)$ and $\varphi, \psi \in X_1^*$ such that

$$\lambda\varphi + (1 - \lambda)\psi \in \mathcal{E}(\rho, E).$$

Since this functional agrees with ρ on E,

$$\lambda\varphi|_E + (1 - \lambda)\psi|_E \;=\; \rho$$

so that $\varphi|_E = \psi|_E = \rho$, because ρ is an extreme point of E_1^*. Therefore φ and ψ lie in $\mathcal{E}(\rho, E)$, as required.

(ii) The Krein-Milman theorem, together with (a), implies that $\partial_e(\mathcal{E}(\rho, E))$ is non-empty. It follows from (i) that $\partial_e(\mathcal{E}(\rho, E))$ is contained in $\partial_e(X_1^*)$. Therefore ρ has an extension in $\partial_e(X_1^*)$. $\qquad\square$

Corollary 2.4.2

Let E be a norm closed subspace of a Banach space X, and let $\rho \in \partial_e(E_1^*)$.

(a) *ρ has a unique extension $\bar{\rho}$ in $\partial_e(X_1^*)$ if and only if $\bar{\rho}$ is the unique extension of ρ in X_1^*.*

(b) *E has the extreme extension property in X if and only if each $\rho \in \partial_e(E_1^*)$ has unique extension in X_1^*.*

Proof

(a) If ρ has unique extension $\bar{\rho}$ in $\partial_e(X_1^*)$, then $\partial_e(\mathcal{E}(\rho, E)) = \{\bar{\rho}\}$ by Lemma 2.4.1 and hence $\mathcal{E}(\rho, E) = \{\bar{\rho}\}$ by the Krein-Milman theorem. Conversely, $\mathcal{E}(\rho, E) = \{\bar{\rho}\}$ implies $\bar{\rho} \in \partial_e(X_1^*)$, again by Lemma 2.4.1.

(b) This is immediate from (a). $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.4.3 Let x be a norm one element of a Banach space X. Even more straightforwardly than in Lemma 2.4.1, it is seen that $\mathcal{E}^x$ is a weak* compact convex subset of X_1^* and, moreover, is always a face of X_1^*. Therefore,

(a) $\partial_e(\mathcal{E}^x) \subset \partial_e(X_1^*)$.

There will be a unique $\varphi \in X_1^*$ attaining its norm at x if and only if $\mathcal{E}^x = \{\varphi\}$. Since $\mathcal{E}^x$ is a face of X_1^*, this is equivalent to saying that $\varphi \in \partial_e(X_1^*)$, by the Krein-Milman theorem. It follows that

(b) X is smooth if and only if $N(X_1^*) \subset \partial_e(X_1^*)$.

We have the following characterisation of smoothness in terms of unique extension conditions.

Proposition 2.4.4

The following are equivalent for a Banach space X.

(a) *X is smooth.*

(b) *Every reflexive subspace of X has the extension property in X.*

(c) *Every reflexive subspace of X has the extreme extension property in X.*

Proof

(a)$\Rightarrow$(b) Assume that X is smooth. Let E be a non-zero reflexive subspace of X and let $\rho \in S(E_1^*)$. Choose $\varphi \in \mathcal{E}(\rho, E)$. Since $E^{**} = E$, there exists $x \in S(X_1)$ such that $\varphi \in \mathcal{E}^x$, so that $\varphi \in \partial_e(X_1^*)$ by 2.4.3(a). Since $\mathcal{E}(\rho, E)$ is convex we have that it coincides with $\{\varphi\}$, proving that E has the extension property in X.

(b)$\Rightarrow$(c) This is immediate from Corollary 2.4.2.

(c)$\Rightarrow$(a) Assume condition (c) holds. Let $x \in S(X_1)$ and consider the (reflexive) subspace $E = \mathbb{C}x(\cong \mathbb{C})$. Obviously, $S(E_1) = \partial_e(E_1)$. Let ρ be the (unique) element of E_1^* with $\rho(x) = 1$. By assumption together with Corollary 2.4.2 ρ has a unique extension $\bar{\rho}$ in X_1^*. Now let $\varphi \in \mathcal{E}^x$. Since $\varphi|_E = \rho$, we must have that $\varphi = \bar{\rho}$ so that $\mathcal{E}^x$ equals $\{\varphi\}$. Therefore, X is smooth. $\square$

We do not know if the extreme extension property implies the extension property in general. We observe the following corollary, however.

Corollary 2.4.5

A Banach space X has the extension property if and only if X has the extreme extension property and $\partial_e(X_1^*)$ is norm closed.

Proof

If X has the extreme extension property then it is certainly smooth, by Proposition 2.4.4, so that

$$N(X_1^*) \subset \partial_e(X_1^*),$$

by 2.4.3(b). If, in addition, $\partial_e(X_1^*)$ is norm closed then we deduce from the Bishop-Phelps theorem ($N(X_1^*)$ is norm dense in $S(X_1^*)$) that

$$\partial_e(X_1^*) = S(X_1^*),$$

so that X^* is strictly convex which, using Theorem 2.3.4, completes the proof. □

2.4.6 We remark that $\partial_e(X_1^*)$ is norm closed whenever X is a JB*-triple [16, Proposition 4]. Without appealing to this fact it is in any case immediate from Theorem 2.3.6 and Proposition 2.4.4 that the extension property and the extreme extension property coincide for JB*-triples.

We shall close this section with three observations used later.

Proposition 2.4.7 [18]

Let X be a Banach space and let E be a closed subspace of X. Suppose that E has the extreme extension property in X. Then the unique extension map

$$\partial_e(E_1^*) \quad \longrightarrow \quad \partial_e(X_1^*)$$
$$\rho \quad \longmapsto \quad \bar{\rho}$$

is weak* continuous.

Proof

Take a net (ρ_α) with weak* limit ρ. It is enough to find a subnet (ρ_β) such that $\bar{\rho}_\beta \to \bar{\rho}$ in the weak* topology. Since E_1^* is weak* compact, there exists a subnet $\bar{\rho}_\beta \to \varphi$, for some $\varphi \in X_1^*$. Therefore

$$\rho = \lim \rho_\beta = \lim \bar{\rho}_\beta|_E = \varphi|_E.$$

Hence $\varphi = \bar{\rho}$, by Corollary 2.4.2. □

Proposition 2.4.8

Let X be a Banach space and let E be a closed subspace of X. Suppose that E has the extreme extension property in X and there exists a contractive projection, Q, from X onto E. Then for each $\rho \in \partial_e(E_1^*)$, $Q^*(\rho)$ is the unique extension of ρ in $\partial_e(X_1^*)$. Moreover, Q is the unique contractive projection from X onto E.

Proof

Let $\rho \in \partial_e(E_1^*)$. Since Q is the identity function on E, $Q^*(\rho)$ extends ρ, and $\|Q^*(\rho)\| = \|\rho\|$ since Q^* is contractive. So, $Q^*(\rho)$ is the unique extension of ρ in X_1^*, by assumption.

To prove uniqueness, suppose that there exists another contractive projection P from X onto E. Then P^* and Q^* agree on $\partial_e(E_1^*)$. But P^* and Q^* are weak* continuous, and so also agree on E_1^*, by the Krein-Milman theorem. Hence, $P^* = Q^*$ and consequently $P = Q$. $\qquad\square$

Corollary 2.4.9

Let I be a norm closed inner ideal of a JB*-triple A.

   (a) *There is at most one contractive projection from A onto I.*

   (b) *If $P : A \to A$ is a contractive projection such that*

$$P(A) \;=\; I,$$

      then P is a structural projection.

Proof

   (a) This follows from Theorem 2.2.5 and Proposition 2.4.8.

   (b) By Theorem 1.8.6, there is a structural projection

$$Q : A^{**} \longrightarrow I^{**},$$

and since I^{**} must have the extension property in A^{**}, being a weak*
closed and thus norm closed inner ideal of A^{**}, we have

$$Q = P^{**},$$

by Proposition 2.4.8. Hence, P is a structural projection. $\qquad\square$

2.5 c_0-sums and M-Orthogonality

2.5.1 Let $(E_i)_{i\in I}$ be a family of Banach spaces. We shall write

(a) $(\sum E_i)_\infty = \{\sum x_i : (\|x_i\|) \in \ell_\infty(I)\}$

and

(b) $(\sum E_i)_0 = \{\sum x_i : (\|x_i\|) \in c_0(I)\}$

to express the respective ℓ_∞ and c_0-sums of the E_i as i ranges over an
indexing set I.

In this way, each E_j is an M-summand of $(\sum E_i)_0$ (and $(\sum E_i)_\infty$) and
we have the natural linear isometry

(c) $E_j^\sharp \cong E_j^*$ $\quad (\rho \mapsto \rho|_{E_j})$.

Similarly, we use

(d) $(\sum E_i)_{\ell_1} = \{\sum x_i : (\|x_i\|) \in \ell_1(I)\}$

for the ℓ_1-sum of the E_i. In this case, each E_i is an L-summand of the
ℓ_1-sum.

We have the further linear isometries

(e) $((\sum E_i)_{\ell_1})^* \cong (\sum E_i^*)_\infty$ $\quad (\rho \mapsto \sum \rho|_{E_i})$;

$$((\sum E_i)_0)^* \cong (\sum E_i^*)_{\ell_1} \quad (\rho \mapsto \sum \rho|_{E_i}),$$

the latter giving the equality

$$\left(\left(\sum E_i\right)_0\right)^* = \left(\sum E_i^\sharp\right)_{\ell_1}.$$

2.5.2 Let $X = (\sum E_i)_{\ell_1}$, where $(E_i)_{i \in I}$ is a family of Banach spaces. Suppose

$$x = \sum x_i, \quad \text{where } x_i \in E_i \text{ for each } i \in I$$

and that $x \in \partial_e(X_1)$. Since $\|x\| = 1$, we can choose i such that $x_i \neq 0$. Let y denote the sum of the remaining x_j's, so that

$$x = x_i + y \ \text{ and } \ 1 = \|x_i\| + \|y\|.$$

If $y \neq 0$, then x is the proper convex sum

$$x = \|x_i\| \left(\frac{x}{\|x_i\|} \right) + \|y\| \left(\frac{y}{\|y\|} \right)$$

in contradiction to the condition that $x \in \partial_e(X_1)$. Hence, $y = 0$. Therefore,

$$\partial_e(X_1) = \bigcup_I \partial_e((E_i)_1) \quad \text{(disjoint union)}.$$

2.5.3 Let X be a Banach space. Norm closed subspaces E and F of X are said to be *M-orthogonal* if

$$E \cap F = \{0\} \text{ and } E + F \text{ is the } \ell_\infty\text{-sum of } E \text{ and } F \text{ (in the norm of } X).$$

In which case $E + F$ is norm closed with M-summands E and F.

If $(E_i)_{i \in I}$ is a family of mutually M-orthogonal norm closed subspaces of X then $(\sum E_i)_0$ is the norm closed subspace of X generated by the E_i, $i \in I$. Thus, if each E_i is an M-ideal of X, then $(\sum E_i)_0$ is an M-ideal of X [44, Proposition 1.11]. The corresponding statement for N-ideals is false.

Perhaps the simplest example is given by the inclusion

$$\mathbb{C}e_1 + \mathbb{C}e_2 \subset M_2(\mathbb{C}),$$

where e_1 and e_2 are the canonical diagonal projections. This is but a very special case of a general result proved next, that we shall make further use of in Chapter 4.

Proposition 2.5.4

A c_0-sum of two or more non-zero M-orthogonal JB*-subtriples of a JBW*-triple factor M cannot have the extension property in M.

Proof

Let (E_i) be a family of non-zero M-orthogonal JB*-subtriples of a JBW*-triple factor M such that

$$\left(\sum E_i\right)_0$$

has the extension property in M. Fix i_0. Then

$$E \oplus F \;=\; \left(\sum E_i\right)_0,$$

where $E = E_{i_0}$ and F is the c_0-sum of the remainder.

Put $A = E \oplus F$. Since it has the extension property in M, A must be a norm closed inner ideal of M, by Theorem 2.2.5. In particular, A is a JB*-triple with orthogonal ideals E and F, since each is an M-ideal of A. Thus,

$$\overline{A}^{w^*} \;=\; \overline{E}^{w^*} \oplus_\infty \overline{F}^{w^*}$$

is a weak* closed inner ideal of M with orthogonal non-zero weak* closed ideals $\overline{E}^{w^*}$ and $\overline{F}^{w^*}$ and so is not a factor. Since every weak* closed inner ideal in a JBW*-triple factor is again a factor, this is a contradiction. $\quad\square$

As consequences we note that given a family $(E_i)_{i\in I}$ of mutually M-orthogonal norm closed subspaces of a Banach space Y,

(a) if each E_i is an N-ideal of Y, $(\sum E_i)_0$ need not be an N-ideal of Y;

(b) if each E_i has the extension property in Y, $(\sum E_i)_0$ need not have the extension property in Y.

The extreme extension property behaves better with respect to c_0-sums.

Proposition 2.5.5

Let $(E_i)_{i \in I}$ be a family of mutually M-orthogonal norm closed subspaces of a Banach space X. Let E_i have the extreme extension property in X, for each $i \in I$. Then $(\sum E_i)_0$ has the extreme extension property in X.

Proof

Put $E = (\sum E_i)_0$, let $\rho \in \partial_e(E_1^*)$ and let $\bar{\rho} \in X_1^*$ be an extension of ρ. By 2.5.1(e) and 2.5.2 we have $\rho \in \partial_e((E_i^\sharp)_1)$, for some unique $i \in I$, and $\rho|_{E_i} \in \partial_e((E_i^*)_1)$, by 2.5.1(c). Now, by assumption, $\bar{\rho}$ is the unique extension of $\rho|_{E_i}$ in X_1^* (where we have used Corollary 2.4.2), whence the result. $\square$

Corollary 2.5.6

If $(E_i)_{i \in I}$ is a family of mutually M-orthogonal norm closed inner ideals in a JB*-triple X, then $(\sum E_i)_0$ has the extreme extension property in X.

Chapter 3

Cartan Factors and Unique Predual Extensions

3.1   Introduction

Given a weak* closed subtriple M of a JBW*-triple N, Theorem 1.8.3 shows that every functional in the predual of M has a unique norm preserving extension in the predual of N if and only if M is an inner ideal of N. In consistency with the general theme of our thesis, in this chapter we shall investigate the consequences of the existence of unique norm preserving extensions in N_* of single functionals in $\partial_e(M_{*,1})$ and of other atomic functionals.

As shall be seen, without any essential loss of information, our investigation quickly reduces to the case where both M and N are Cartan factors. Amongst other things, in this case we show that the existence of one functional in $\partial_e(M_{*,1})$ with unique norm one extension in N_* compels the same condition upon all functionals in $\partial_e(M_{*,1})$. However, this particular unique extension condition does not imply that M is an inner ideal of N because of obstructions involving generic type. Indeed, in these circumstances, it is the case that more often than not the Cartan factors M and N in question have different generic type.

Inspection of finite dimensional examples reveals algebraic and geometric obstructions alluded to above. For instance, given finite $n \geqslant 2$, the triple

embeddings

$$\alpha_1 : S_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \quad \text{and} \quad \alpha_2 : M_n(\mathbb{C}) \longrightarrow A_{2n}(\mathbb{C}),$$

where α_1 is inclusion and α_2 maps x to

$$\begin{pmatrix} 0 & x^T \\ -x & 0 \end{pmatrix}$$

both implement the extreme extension property, as shall be seen in the coming general investigation, as therefore does their composition. However, neither image is an inner ideal in its codomain. The validity of these observations remain unaffected by further (appropriate) composition with the triple embeddings

$$M_{m,n} \longrightarrow M_n, \quad M_n \longrightarrow M_{n,l} \ (m \leqslant n \leqslant l) \quad \text{and} \quad A_{2n} \longrightarrow A_{2n+1},$$

obtained by suitable addition of 0's (see 1.11.3). On the other hand, the triple embeddings

$$\beta_1 : M_n(\mathbb{C}) \longrightarrow S_{2n}(\mathbb{C}) \quad \text{and} \quad \beta_2 : A_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{C}),$$

where β_1 maps x to

$$\begin{pmatrix} 0 & x^T \\ x & 0 \end{pmatrix}$$

and β_2 is inclusion, do not implement the extreme extension property.

It turns out that these observations are characteristic of some general rules. Suppose that M and N are hermitian, rectangular or symplectic Cartan factors and that M is a Cartan subfactor of N for which there exists ρ in $\partial_e(M_{*,1})$ with unique norm one extension in N_*. Then, as shall be proved in the sequel, in terms of generic type we have the following table of the possible generic types of N relative to the generic type of M.

M	N
Hermitian	Hermitian, rectangular or symplectic
Rectangular	Rectangular or symplectic
Symplectic	Symplectic

Given that M and N are Cartan factors with M a JBW*-subtriple of N such that there exists $\rho \in \partial_e(M_{*,1})$ with unique extension in $\partial_e(N_{*,1})$, the above table is to be interpreted as follows.

(a) If M is hermitian, then N can be either hermitian, rectangular or symplectic.

(b) If M is rectangular, then N can be rectangular or symplectic but never hermitian.

(c) If M is symplectic then N must again be symplectic.

There is another point that should be emphasised. We have asserted that each of the canonical embeddings, $M \hookrightarrow N$:

$$S_n(\mathbb{C}) \hookrightarrow M_n(\mathbb{C}), \quad M_n(\mathbb{C}) \hookrightarrow A_{2n}(\mathbb{C}), \quad S_n(\mathbb{C}) \hookrightarrow A_{2n}(\mathbb{C}),$$

$$M_{n,m} \hookrightarrow M_n (m \leqslant n), \quad M_n \hookrightarrow M_{n,l} (n \leqslant l), \quad A_{2n}(\mathbb{C}) \hookrightarrow A_{2n+1}(\mathbb{C})$$

and any combination of these, implement the extreme extension property. In fact, as is proved more elaborately below, *the extreme extension property is independent of the way that M is 'embedded' in N and is a function of the relative generic type and numerical signatures alone.*

To be precise, one of our claims proved in Section 3.3 is that if $M \hookrightarrow N$ is any one of the above canonical embeddings and $X \cong M$, $Y \cong N$ with $X \subset Y$, then X has the extreme extension property in Y.

Let M be a JBW*-subtriple of a JBW*-triple N, where M is a Cartan factor. When there exists $\rho \in \partial_e(M_{*,1})$ with a unique norm preserving

extension in $N_{*,1}$, general rules for deciding when M is an inner ideal of N are derived in Section 3.4. One of the main results of the chapter, proved in Section 3.5, is that if there exists a norm one non-extreme element in M_* with unique norm one extension in N_*, then M is compelled to be an inner ideal of N. The chapter concludes with applications to Cartan subfactors of von Neumann algebras.

We begin with a short preliminary Section 3.2 that lays down some of the general principles involving support tripotents and unique norm preserving extensions from weak* closed inner ideals that run through this thesis.

3.2 Unique Extension from Inner Ideals and Support Tripotents

Lemma 3.2.1

Let I be a weak* closed inner ideal of a JBW*-triple M. Let $\tau \in I_*$ with (unique) norm preserving extension $\bar{\tau}$ in M_*, and let $u = s(\tau)$ be the support tripotent of τ in I. Then $\bar{\tau}$ coincides with the composition

$$M \xrightarrow{P_2(u)} I \xrightarrow{\tau} \mathbb{C}$$

and $s(\tau) = s(\bar{\tau})$.

Proof

We may suppose that $\|\tau\| = 1$. By Theorem 1.10.1(b), $\tau = \tau \circ P_2(u)$ on I. Therefore the composition in the statement extends τ and so must equal $\bar{\tau}$ since it has norm one.

Further, since $\bar{\tau}(u) = 1$, we have that $s(\bar{\tau}) \leqslant u$. In particular, $s(\bar{\tau})$ lies in

$$M_2(u) \subset I.$$

But then

$$1 = \bar{\tau}(s(\bar{\tau})) = \tau(s(\bar{\tau})),$$

giving $u \leqslant s(\bar{\tau})$. □

Corollary 3.2.2

Let I be a weak* closed inner ideal of a JBW*-triple M and let

$$I_* \longrightarrow M_* \qquad (\tau \longmapsto \bar{\tau})$$

denote the norm preserving unique extension map. Then

$$\{\bar{\tau} \in M_* : \tau \in I_*\} \;=\; \{\rho \in M_* : s(\rho) \in I\} \;=\; I^\sharp \cap M_*$$

and

$$I^\sharp \cap M_* \longrightarrow I_* \qquad (\rho \mapsto \rho|_I)$$

is a surjective linear isometry.

Proof

The first equality of the statement follows from Lemma 3.2.1 and the fact that given a norm one element ρ in M_* such that $s(\rho)$ lies in I, then $\rho|_I$ clearly has norm one. The remainder of the statement is a consequence of the remark made after the proof of Lemma 2.2.4. $\qquad\square$

Corollary 3.2.3

Let M be a JBW*-subtriple of a JBW*-triple N and let I be a weak* closed inner ideal of both M and N. Let $\rho \in M_*$ such that $u = s(\rho) \in I$. Then ρ has unique norm preserving extension in N_*, given by the composition

$$N \xrightarrow{P_2(u)} I \xrightarrow{\rho} \mathbb{C}.$$

Proof

If φ is a norm preserving extension of ρ, then φ must also be a norm preserving extension of $\rho|_I$, by Corollary 3.2.2, and therefore has the stated form by Lemma 3.2.1. $\qquad\square$

Proposition 3.2.4

Let M be a JBW*-subtriple of a JBW*-triple N and let I be a weak* closed inner ideal of M. Let $\rho \in M_*$ with $s(\rho) \in I$, and let $\varphi \in N_*$ such that $\|\varphi\| = \|\rho\|$.

(a) *If $\rho|_I = \varphi|_I$, then $\rho = \varphi|_M$.*

(b) *φ is the unique norm preserving extension of ρ in N_* if and only if φ is the unique norm preserving extension of $\rho|_I$ in N_*.*

Proof

(a) Put $u = s(\rho)$, let $\|\rho\| = 1$ and suppose that

$$\rho|_I = \varphi|_I.$$

Since $\varphi(u) = 1$, then Theorem 1.10.1(b) shows that

$$\varphi(x) = \varphi(P_2(u)(x)),$$

for all $x \in N$. Therefore, since $P_2(u)(M) \subset I$, for each x in M we have

$$\varphi(x) = \rho(P_2(u)(x)) = \rho(x),$$

where the second equality again derives from Theorem 1.10.1.

(b) Suppose that φ is the unique norm preserving extension of ρ in N_* and let ψ be any extension of $\rho|_I$ with

$$\|\psi\| = \|\rho|_I\| \qquad (= \|\rho\|).$$

Since $\psi|_I = \rho|_I = \varphi|_I$, (a) implies that

$$\psi|_M = \rho = \varphi|_M,$$

so that $\psi = \varphi$, by assumption. The converse is clear. $\qquad\square$

3.2.5 Let us finally note that if C is a Cartan factor with corresponding elementary ideal $K(C)$, then, since $K(C)^{**} = C$ (Theorem 1.11.4 (a)(iii)), we have $K(C)^* = C_*$, so that upon identifying C_* with the L-summand $K(C)^\sharp$ of C^* we have

$$C^* = C_* \oplus_1 K(C)^\circ \qquad (\ell_1\text{-sum}).$$

Consequently,

$$\partial_e(C_1^*) \;=\; \partial_e(C_{*,1}) \;\cup\; \partial_e((K(C)^\circ)_1) \quad \text{(disjoint union)}$$

and the functionals in $\partial_e(C_{*,1})$ are precisely those in $\partial_e(C_1^*)$ that do not vanish on $K(C)$.

3.3   Unique Extreme Predual Extensions

If $\rho \in \partial_e(M_{*,1})$, where M is a JBW*-triple and C is the weak* closed ideal of M generated by $s(\rho)$, then, by Theorem 1.10.1 together with [21], C is a Cartan factor and ρ vanishes on the orthogonal complement of C in M. Thus, ρ 'lives' on C and we may regard ρ as a member of $\partial_e(C_{*,1})$. In this sense nothing is lost if it is assumed that M coincides with the Cartan factor C when discussing consequences of ρ having a unique norm preserving extension in the predual of a containing JBW*-triple.

We begin by studying unique norm one weak* continuous extensions from the extreme predual of a Cartan subfactor of a JBW*-triple. Since the norm closed predual ball of a JBW*-triple need not be weak* closed, the Krein-Milman theorem is not directly applicable. Essential use is made throughout of Theorem 1.10.2, the bijective correspondence between predual ball extreme points of a JBW*-triple and minimal tripotents. Such is its ubiquity we sometimes employ it tacitly.

Lemma 3.3.1

Let C be a JBW-subtriple of a JBW*-triple M, where C is a Cartan factor. Let $\rho \in \partial_e(C_{*,1})$ such that ρ has a unique norm one extension $\bar\rho$ in M_*. Then $\bar\rho \in \partial_e(M_{*,1})$, $s(\rho) = s(\bar\rho)$ and $s(\rho)$ is a minimal tripotent of M.*

Proof

Suppose that $\bar\rho$ is the unique extension of ρ in $M_{*,1}$, and let $u = s(\rho)$. We shall show that the JBW*-algebra $M_2(u)$ has only one normal state.

Let φ be a normal state of $M_2(u)$ and let $\psi = \varphi \circ P_2(u)$. Then $\psi \in M_{*,1}$ and $\psi(u) = 1$. Therefore $\psi|_C$ belongs to $C_{*,1}$ and is equal to ρ, by Theorem 1.10.2. Hence, by the above assumption, $\psi = \bar{\rho}$. So, on $M_2(u)$, $\varphi = \bar{\rho}$. Therefore φ is the unique normal state of $M_2(u)$ and so the latter must be of dimension one. Hence $M_2(u) = \mathbb{C}u$, giving that u is a minimal tripotent of M and, since $\bar{\rho}(u) = 1$, we have $s(\bar{\rho}) = u$ and $\bar{\rho} \in \partial_e(M_{*,1})$, by further application of Theorem 1.10.2. $\square$

Lemma 3.3.2

Let C be a JBW*-subtriple of a JBW*-triple M, where C is a Cartan factor, such that C and M posses a common minimal tripotent. Then C is contained in a weak* closed ideal D of M, where D is a Cartan factor, and every minimal tripotent of C is a minimal tripotent of D (and hence of M).

Proof

Let u be a minimal tripotent of C such that u is minimal in M, and let D be the weak* closed ideal of M generated by u. Let v be another minimal tripotent of C. Then, by [17, 5.4], there exists a tripotent $w \in C$ such that the automorphism of C

$$\pi(w) = I - 2P_1(w)$$

sends u to a scalar multiple of v. Since $\pi(w)$ is also an automorphism of M and u is minimal in M, we have that v is minimal in M, too. Moreover,

$$v \in (I - 2P_1(w))D \subset D.$$

Thus D contains all minimal tripotents of C and hence contains C. $\square$

Corollary 3.3.3

Let C, M and ρ satisfy the conditions of Lemma 3.3.1. Then every φ belonging to $\partial_e(C_{*,1})$ has a unique extension $\bar{\varphi}$ in $M_{*,1}$, and $\bar{\varphi} \in \partial_e(M_{*,1})$.

Proof

Let $\varphi \in \partial_e(C_{*,1})$. Using Theorem 1.8.4, let τ be any extension of φ in $M_{*,1}$. By Lemma 3.3.1 and Lemma 3.3.2 we have that $s(\varphi)$ is minimal in M. Since $\tau(s(\varphi)) = 1$ we have that, using Theorem 1.10.2, $\tau \in \partial_e(M_{*,1})$ and must be the unique extension of φ in $M_{*,1}$. $\qquad\square$

Theorem 3.3.4

Let C be a Cartan factor contained as a JBW*-subtriple in a Cartan factor D. Then the following are equivalent.

(a) *There exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $D_{*,1}$.*

(b) *There exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$.*

(c) *Every $\rho \in \partial_e(C_{*,1})$ has unique extension in $D_{*,1}$.*

(d) *Every $\rho \in \partial_e(C_{*,1})$ has unique extension in $\partial_e(D_{*,1})$.*

(e) *There is a minimal tripotent of C that is minimal in D.*

(f) *Every minimal tripotent of C is minimal in D.*

(g) *$K(C)$ has the extreme extension property in D.*

Proof

The implications (c)$\Rightarrow$(a), (d)$\Rightarrow$(b) and (f)$\Rightarrow$(e) are clear. The implications (a)$\Rightarrow$(b),(e) and (c)$\Rightarrow$(d),(f) follow from Lemma 3.3.1 and (e)$\Rightarrow$(f) is proved in Lemma 3.3.2.

(f)$\Rightarrow$(d) This follows as in Corollary 3.3.3.

(b)$\Rightarrow$(a) Let $\rho \in \partial_e(C_{*,1})$ have unique extension $\bar{\rho}$ in $\partial_e(D_{*,1})$. Let φ be any extension of ρ in $D_{*,1}$. By Theorem 1.12.1, φ can be written as a σ-convex sum

$$\varphi = \sum \lambda_n \tau_n$$

where the $\tau_n \in \partial_e(D_{*,1})$. But then ρ is the σ-convex sum

$$\rho = \sum \lambda_n \left(\tau_n|_C \right).$$

Therefore, since $\rho \in \partial_e(C_{*,1})$, the sum is degenerate and all the $\tau_n|_C$ equal ρ. Hence, by the assumption, all the τ_n equal $\bar{\rho}$. Hence, $\varphi = \bar{\rho}$.

(d)$\Rightarrow$(c) This follows from the above proof of (b)$\Rightarrow$(a).

This shows that (a), (b), (c), (d), (e) and (f) are equivalent. The condition (f) implies that $K(C) \subset K(D)$ and the condition (d) now implies that each element of $\partial_e(K(C)_1^*)$ has unique extension in $\partial_e(K(D)_1^*)$, so that $K(C)$ has the extreme extension property in $K(D)$ and hence in D, giving (g).

(g)$\Rightarrow$(c) Assume (g). Then each $\rho \in \partial_e(C_{*,1})$ has unique extension in D_1^* and hence in $D_{*,1}$ (since each such ρ has an extension in $D_{*,1}$). This completes the proof. $\square$

We shall proceed to develop an analysis of Cartan subfactors of Cartan factors.

Proposition 3.3.5

Let C be a Cartan factor JBW*-subtriple of a Cartan factor D. Let u be a tripotent of C such that $\mathrm{rank}(u) = n$, where $n < \infty$. Then the following are equivalent.

(a) *There exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$.*

(b) *$C_2(u)$ has the extreme extension property in $D_2(u)$ (respectively, D).*

(c) *$D_2(u)$ has rank n (a type I_n JBW*-algebra factor).*

Proof

(a) $\Rightarrow$ (b) Assume (a). Then, by Theorem 3.3.4 (b)$\Rightarrow$(f), all minimal tripotents of C are minimal in D, a condition that must be inherited by the inclusion $C_2(u) \subset D_2(u)$, so that $K(C_2(u))$ has the extreme extension property in $D_2(u)$, by Theorem 3.3.4 (f)$\Rightarrow$(g). But $K(C_2(u)) = C_2(u)$, since the latter has finite rank.

(b) $\Rightarrow$ (c) Similarly, given condition (b), Theorem 3.3.4 (g)$\Rightarrow$(f) implies that u is a tripotent of rank n with respect to D, as required.

(c) $\Rightarrow$ (a) We have $u = u_1 + \ldots u_n$, where the u_i are mutually orthogonal minimal tripotents of C. If u_1 is not minimal in D, so that $u_1 = v_1 + w_1$, for certain non-zero orthogonal tripotents of D, we must have

$$\mathrm{rank}(D_2(u)) \;\geqslant\; n + 1.$$

Therefore (c) implies (a) by these remarks together with Theorem 3.3.4(e)$\Rightarrow$(b). $\qquad\square$

We shall need the following lemma.

Lemma 3.3.6

Let M be a JBW*-subalgebra of a JBW*-algebra N with the same identity. Let M be a type I_m factor and N a type I_n factor, where $n < \infty$. Then n is a multiple of m.

Proof

Choose orthogonal minimal projections $e_1, \ldots, e_m$ in M with sum 1. We have that $e_1, \ldots, e_m$ are Jordan equivalent in M [4, p86] and hence in N. Thus, for some k we have that

$$\mathrm{rank}\left(\{e_i \circ N \circ e_i\}\right) \;=\; k$$

for all i. It follows that there exists km orthogonal minimal projections in N with sum 1. Hence, $n = km$. □

Proposition 3.3.7

Let C be a Cartan factor JBW*-subtriple of a Cartan factor D such that

$$\text{rank}(D) \ < \ 2\,\text{rank}(C) \ < \ \infty.$$

Then C has the extreme extension property in D. In particular, this holds when

$$\text{rank}(C) \ = \ \text{rank}(D) \ < \ \infty.$$

Proof

Suppose that $\text{rank}(C) = n$ and choose a tripotent u in C such that $C_2(u)$ has rank n, and put $m = \text{rank}(D_2(u))$. By assumption we have

$$n \leqslant m < 2n.$$

Consider the inclusion of type I finite JBW*-algebra factors

$$C_2(u) \subset D_2(u).$$

Since $C_2(u)$ is a type I_n factor, that $D_2(u)$ is a type I_m factor and that u is the common identity of $C_2(u)$ and $D_2(u)$, we have that $m = kn$, for some integer k, by Lemma 3.3.6. It follows that $m = n$ so that C has the extreme extension property in D by Proposition 3.3.5 and the implication (b)$\Rightarrow$(d) of Theorem 3.3.4. □

Prior to a more general statement given in Theorem 3.3.11 below, we note the following useful corollary.

Corollary 3.3.8

Let C be a Cartan factor JBW*-subtriple of a Cartan factor D. Then C has the extreme extension property in D in each of the following cases:

(a) *C and D are spin factors,*

(b) *$C \cong B_{1,2}$ and $D = M_3^8$,*

(c) *rank$(C) = 2$ and rank$(D) = 2$ or 3.*

Proof

Since all spin factors have rank 2 and rank$(B_{1,2}) = 2$, and rank$(M_3^8) = 3$, the statements (a) and (b) are immediate from the statement (c) which, in turn, is immediate from Proposition 3.3.7. □

The *classification scheme* of Dang and Friedman [21, p305] shows that the generic type of a Cartan factor C of rank greater than one is completely determined up to linear isometry by the (spin factor) structure of the Peirce 2-space $C_2(u)$ associated with a rank 2 tripotent u of C according to the following table

3.3.9

C	V_n	S_n	$M_{n,k}$	$A_n, n \geqslant 4$	$B_{1,2}$	M_3^8
$C_2(u)$	V_n	V_2	V_3	V_5	V_7	V_9

where n and k are (possibly infinite) cardinal numbers with $2 \leqslant n \leqslant k$. We note that, since V_7 has the extreme extension property in V_9, the final two columns in 3.3.9 in conjunction with Proposition 3.3.5 provides an alternative proof of the final statement of Corollary 3.3.8.

Corollary 3.3.10

Let C and D be Cartan factors where C is a JBW*-subtriple of D. Let u be a rank 2 tripotent of C such that $C_2(u) = D_2(u)$.

(a) *If D is a spin factor, then $C = D$.*

(b)   (i) *If C is exceptional, then $C = D$.*

    (ii) *If $D = M_3^8$, then $C \cong V_9$ or $C = D$.*

(iii) *If $D = B_{1,2}$, then $C \cong V_7$ or $C = D$.*

(c) *If D is hermitian, rectangular or symplectic, then C is hermitian, rectangular or symplectic and C and D have the same generic type.*

Proof

The condition imposed by the second sentence of the statement implies that u has rank 2 in D.

(a) If D is a spin factor, then u must be a unitary tripotent of D (that is, $P_2(u)$ is the identity on D) and hence a unitary tripotent of C. Therefore,

$$C = C_2(u) = D_2(u) = D.$$

Alternatively, we could appeal directly to the second column of table 3.3.9.

(b) (i) Let C be exceptional. Then D must be exceptional. If $C \cong B_{1,2}$, then the penultimate column of table 3.3.9 gives

$$V_7 \cong C_2(u) = D_2(u)$$

so that $D \cong B_{1,2}$ and thus $C = D$ (both having dimension 16). Similarly, or purely from dimensional considerations $C = D$ if $C \cong M_3^8$.

(ii), (iii) These are derived from similar appeals to the final two columns of table 3.3.9.

(c) The result can be read directly from columns two, three and four of table 3.3.9. (It is possible for C and D to be a spin factor V_2, V_3 or V_5.) $\qquad\square$

The following is a conflation of Proposition 3.3.5, Proposition 3.3.7 and the table 3.3.9.

Theorem 3.3.11

Let C and D be non-exceptional Cartan factors, where C is a JW*-subtriple of D.

(a) *Suppose that C a spin factor.*

    (i) *If $C \cong V_n$, where $n \geqslant 6$, then C has the extreme extension property in D if and only if $D \cong V_m$, $m \geqslant n$.*

    Suppose now that C has the extreme extension property in D.

    (ii) *If $C \cong V_n$, where $n = 4$ or 5, then D is symplectic or $D \cong V_m$ where $m \geqslant n$.*

    (iii) *If $C \cong V_3$, then D is rectangular or symplectic or $D \cong V_m$, $m \geqslant 3$.*

    (iv) *If $C \cong V_2$, then D is hermitian, rectangular or symplectic or $D \cong V_m$, $m \geqslant 2$.*

(b) *Let $2 \leqslant n < \infty$. If the structure of D relative to that of C (all up to linear isometry) corresponds to the following table, then C has the extreme extension property in D.*

C	D
S_n	S_m or $M_{m,l}$ or A_{2m} or A_{2m+1} : $\quad n \leqslant m < 2n$, $m \leqslant l \leqslant \infty$
$M_{n,l}$	$M_{m,k}$: $\quad n \leqslant m < 2n$, $\quad n \leqslant l$, $\quad m \leqslant k$, $\quad l \leqslant k \leqslant \infty$
$M_{n,n}$	A_{2m} or A_{2m+1} : $\quad n \leqslant m < 2n$
A_{2n}	A_{2m} or A_{2m+1} : $\quad n \leqslant m < 2n$
A_{2n+1}	A_k : $\quad 2n+1 \leqslant k < 4n$

(c) *Suppose that there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$, where C and D are hermitian, rectangular or symplectic (of possibly infinite) rank $\geqslant 2$.*

(i) *For prescribed C, the (only) possible generic type of D is as follows.*

C	D
Hermitian	*Hermitian, rectangular or symplectic*
Rectangular	*Rectangular or symplectic*
Symplectic	*Symplectic*

(ii) *For prescribed D, the (only) possible generic type of C is as follows.*

D	C
Hermitian	*Hermitian*
Rectangular	*Hermitian or rectangular*
Symplectic	*Hermitian, rectangular or symplectic*

Proof

(a) Suppose that $C \cong V_n$, where $n \geqslant 6$. If D is also a spin factor, then C has the extreme extension property in D, by Corollary 3.3.8(a).

In order to prove the converse, suppose that $C \cong V_n$ ($n \geqslant 6$) has the extreme extension property in D and let u be a rank 2 tripotent of C. It follows from Proposition 3.3.5 that $C_2(u)$ has the extreme extension property in $D_2(u)$. By table 3.3.9, $V_n \cong C = C_2(u)$, so that

$$D_2(u) \cong V_m, \text{ where } m \geqslant n \geqslant 6.$$

If D is not a spin factor then, again by table 3.3.9,

$$D_2(u) \cong V_2, V_3 \text{ or } V_5,$$

which is impossible. Hence $D \cong V_m$, where $m \geqslant 6$.

This proves (i). The remaining statements of part (a) are consequences of the table 3.3.9.

(b) The table follows from Proposition 3.3.7 and that $M_{n,m}$, S_n, A_{2n} and A_{2n+1} all have rank n.

For example, let $n \leqslant m < 2n < \infty$ and let $C = S_n$ and $D = S_m$. Then we have

$$\operatorname{rank}(D) \;=\; m \;<\; 2n \;=\; 2\operatorname{rank}(C) \;<\; \infty$$

so that C has the extreme extension property in D by Proposition 3.3.7. The rest of the table follows similarly.

(c) We prove the second row of the first table (the proof of the other two rows is almost identical). The second table follows immediately from the first table.

Let u be a rank 2 tripotent of C and suppose that C is rectangular. Then, by Proposition 3.3.5 together with table 3.3.9,

$$C_2(u) \;\cong\; V_3$$

and $C_2(u)$ has the extreme extension property in $D_2(u)$. Hence, we must have that

$$D_2(u) \;\cong\; V_3 \quad \text{or} \quad D_2(u) \;\cong\; V_5$$

and it follows, again by table 3.3.9, that D is either rectangular or symplectic. □

3.4   Inner Ideals

3.4.1   In this section we shall consider the question of when the 'single' unique extension condition from $\partial_e(C_{*,1})$ to $\partial_e(D_{*,1})$, introduced and discussed in the previous section, forces C to be an inner ideal of D, where C and D are Cartan factors and C is a JBW*-subtriple of D. When C has rank 1 (that is, is a Hilbert space) this question is easily answered by use of

the result of Dang and Friedman [21, p306] that the closed subspace generated by a family of mutually collinear minimal tripotents in a JBW*-triple M is a Hilbert space and an inner ideal of M. In order to be clear we shall formally record this result.

Theorem 3.4.2 [21, p306]

Let M be a JBW-triple and let (e_α) be a family of mutually collinear minimal tripotents of M. Then the closed linear span of the (e_α) is linearly isometric to a Hilbert space and is a weak* closed inner ideal of M.*

Conversely, every weak* closed inner ideal of M linearly isometric to a Hilbert space arises in this way.

Corollary 3.4.3

If C is a Hilbert space JBW-subtriple of a Cartan factor D and there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$, then C is an inner ideal of D.*

Proof

This follows from Theorem 3.3.4 (b)⇔(f) and Theorem 3.4.2. □

Let v be a minimal tripotent in a Cartan factor C. By the *classification scheme* of [21, p305] (see A: Case 1, Case 2), the Peirce 1 space $C_1(v)$ is a Hilbert space if and only if C is a Hilbert space or is hermitian and in that case C contains a tripotent u such that $u \in C_1(v)$ and $v \in C_1(u)$ (that is, u, v are collinear) only when C is a Hilbert space. In particular, an hermitian Cartan factor cannot contain two collinear minimal tripotents. The following lemma is a restatement of this property.

Lemma 3.4.4

If H is a Hilbert space and an inner ideal of an hermitian Cartan factor, then $\dim(H) = 1$.

In 3.4.5 and 3.4.6 below C and D are Cartan factors where C is a JBW*-subtriple of D.

3.4.5 *Spin Factors and Exceptional Factors*

Let D be a spin factor or an exceptional factor. By Proposition 3.3.7 and Corollary 3.3.8, all rank 2 subfactors of D satisfy the extreme extension property in D, as do all rank 3 subfactors when $D = M_3^8$.

(a) If D is a spin factor, then all proper inner ideals are of rank 1, by [34, Lemma 5.5].

   If D is exceptional, every proper rank 2 inner ideal of D is of the form $D_2(u)$, where u is a (rank 2) tripotent of D, and hence linearly isometric to the spin factor V_7 or V_9 according to whether $D = B_{1,2}$ or $D = M_3^8$, respectively. On the other hand, if $V_7 \cong C$ and u is a unitary tripotent in C, then

$$V_7 \cong C_2(u) \subset D_2(u) \cong V_7,$$

   so that $C = D_2(u)$. Similarly, if $D = M_3^8$ and $C \cong V_9$, then $C = D_2(u)$ for some rank 2 tripotent of D. Summarising, we have:

(b) if $\mathrm{rank}(C) = 2$, then C is an inner ideal of

   (i) $B_{1,2}$ if and only if $C \cong V_7$;

   (ii) M_3^8 if and only if $C \cong V_9$.

We now turn to the situation where the vast majority of cases occur.

3.4.6 Suppose now that C and D are hermitian, rectangular or symplectic (but not necessarily of the same generic type) and that

$$\mathrm{rank}(C) \geqslant 2.$$

(We are allowing C or $D \cong V_2, V_3$ or V_5.) If C is an inner ideal of D then for any rank 2 tripotent u of C we have

$$C_2(u) \;=\; D_2(u) \;\cong\; V_2, V_3 \text{ or } V_5$$

so that C and D have the same generic type (see table 3.3.9). Our present intention is to prove, conversely, that if C and D have the same generic type and there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$, then C must be an inner ideal of D.

Lemma 3.4.7

Let M be a JBW*-subtriple of a JBW*-triple N such that $M_2(u)$ is an inner ideal of N for each complete tripotent of M. Then M is an inner ideal of N.

Proof

Let $x \in M$. We must show that $\{xNx\} \subset M$. By the triple functional calculus, x lies in $M_2(r(x))$ (Theorem 1.7.1). In turn, via [49, 3.12], choose a complete tripotent u of M such that $r(x)$ lies in $M_2(u)$. This gives

$$x \in M_2(r(x)) \;\subset\; M_2(u).$$

Since $M_2(u)$ is an inner ideal of N we have that $M_2(u)$ contains $N_2(u)$ and so $M_2(u) = N_2(u)$, giving $x \in N_2(u)$. Hence,

$$\{xNx\} \;\subset\; N_2(u) \;\subset\; M$$

as required. $\qquad\square$

We are now in a position to state and prove an inner ideal characterisation in terms of a 'single' extreme extension property.

Theorem 3.4.8

Let C and D be hermitian, rectangular or symplectic Cartan factors of rank $\geqslant 2$ and of the same generic type, where C is a JBW*-subtriple of D. Then there exists ρ in $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ if and only if C is an inner ideal of D. In which case, every element of C^* has a unique norm preserving extension in D^*.

Proof

Suppose there exists ρ in $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$. Let u be a complete tripotent of C and consider the type I JBW*-algebra factors $C_2(u)$ and $D_2(u)$ denoted by E and F, respectively. Let e be a finite rank n projection of E. We have

$$E_2(e) \; = \; C_2(e) \quad \text{and} \quad F_2(e) \; = \; D_2(e).$$

By Proposition 3.3.5, $E_2(e)$ has the extreme extension property in $F_2(e)$, both these factors have rank n, and they have the same generic type because C and D do. Hence,

$$E_2(e)_{sa} \; \cong \; M_n(\mathbb{F})_{sa} \; \cong \; F_2(e)_{sa}, \quad \text{where } \mathbb{F} = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H},$$

so that $E_2(e) = F_2(e)$, by finite dimensionality. In particular, if u has finite rank, then $C_2(u) = D_2(u)$. Otherwise,

$$u \; = \; \sum_{i \in I} e_i,$$

where $\{e_i : i \in I\}$ is an infinite orthogonal family of mutually orthogonal minimal projections in E, and we have that (e_F) is a net of (finite rank) projections with weak* limit u, where

$$e_F \; = \; \sum_{i \in F} e_i$$

and F ranges over all finite subsets of I.

Let $x \in D_2(u)_{sa}$. Then x is the weak* limit of $\{e_F x e_F\}$. But, for each F, e_F has finite rank, so that

$$e_F x e_F \in D_2(e_F) \;=\; C_2(e_F) \;\subset\; C_2(u).$$

Therefore, $x \in C_2(u)$. Hence, $C_2(u) \;=\; D_2(u)$ which, by Lemma 3.4.7, proves that C is an inner ideal of D. The converse and final statement is an application of Theorem 1.8.3. $\qquad\square$

We shall proceed to derive a number of corollaries.

Corollary 3.4.9

Let C and D be Cartan factors, where C is a JBW*-subtriple of D. If there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ and C is symplectic or D is hermitian, then C is an inner ideal of D.

Proof

If C is a symplectic Cartan factor then, by Theorem 3.3.11(c), D must also be symplectic. Similarly, C must be hermitian if D is hermitian. In which case C and D are of the same generic type, so that C is an inner ideal of D by Theorem 3.4.8. $\qquad\square$

Corollary 3.4.10

Let M be a JBW*-triple of a JBW*-triple N. Suppose there exists $\rho \in \partial_e(M_{*,1})$ with unique extension in $N_{*,1}$. Suppose further that the weak* closed ideal of M generated by $s(\rho)$ is symplectic. Then the weak* closed ideal of M generated by $s(\rho)$ is an inner ideal of N.

Proof

Let C denote the weak* closed ideal of M generated by $s(\rho)$. By assumption, C is a symplectic Cartan factor. Further, by Lemma 3.3.1 and Lemma 3.3.2 there exists a weak* closed ideal D of N, where D is a Cartan factor, such

that $C \subset D$ and we may suppose that $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$. Hence, C is an inner ideal of D, by Corollary 3.4.9, and therefore C is an inner ideal of N. □

Corollary 3.4.11

Let C and D be Cartan factors, where C is a JBW*-subtriple of D. If C and D posses a common unitary tripotent such that

(a) *C and D are hermitian, rectangular or symplectic and of the same generic type and,*

(b) *there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$,*

then $C = D$.

Proof

In this case C is an inner ideal of D, by Theorem 3.4.8, and C contains a unitary tripotent u of D, so that $C = C_2(u) = D_2(u) = D$. □

Our final corollary of this section does not make explicit mention of any unique extension condition and may be seen as a contribution purely to inner ideal theory.

Corollary 3.4.12

Let C and D be Cartan factors, where C is a JBW*-subtriple of D. C is an inner ideal of D whenever, for finite $n \geqslant 2$,

(a) *$C \cong S_n$ and $D \cong S_m$, where $n \leqslant m < 2n$;*

(b) *$C \cong M_{n,l}$ and $D \cong M_{m,k}$, where $n \leqslant m < 2n$, $m \leqslant k$ and $l \leqslant k \leqslant \infty$;*

(c) *$C \cong A_{2n}$ and $D \cong A_{2m}$ or A_{2m+1}, where $n \leqslant m < 2n$;*

(d) *$C \cong A_{2n+1}$ and $D \cong A_k$, where $2n + 1 \leqslant k < 4n$.*

Proof

In each case, by Theorem 3.3.11, C has the extreme extension property in D, and C and D are of the same generic type. Hence, Theorem 3.4.8 applies to give the desired conclusions. $\qquad\square$

3.5 Unique Non-Extreme Extensions

3.5.1 In this section we investigate unique norm preserving extensions of single mixed functionals in the predual of a Cartan factor to the predual of a second Cartan factor that contains the first as a JBW*-triple. Building upon the work of previous sections, we shall see that this leads to a new geometric characterisation of weak* closed inner ideals in Cartan factors.

3.5.2 Given a real (JW-algebra) spin factor, $U = L \oplus \mathbb{R}1$, where L is a real Hilbert space, for each $h \in L$ let

$$\rho_h^U : U \longrightarrow \mathbb{R}$$

be given by

$$\rho_h^U(\alpha 1 + h') \;=\; \alpha + \langle h, h' \rangle, \quad \text{for each } \alpha \in \mathbb{R} \text{ and } h' \in L.$$

By [65, 2.2.2, 2.2.3], we have

(a) $S(U) = \{\rho_h^U \mid h \in L, \|h\| \leqslant 1\}$;

(b) $P(U) = \{\rho_h^U \mid h \in L, \|h\| = 1\}$,

where $S(U)$ denotes the state space of U and $P(U)$ denotes its set of pure states.

Lemma 3.5.3

Let U and U' be real spin factors where U is a JW-subalgebra of U'. Let $\rho \in S(U) \setminus P(U)$ and let ρ have unique extension in $S(U')$. Then $U = U'$.

Proof

In order to obtain a contradiction, assume that U is properly contained in U'. We have

$$U = L \oplus \mathbb{R}1 \quad \text{and} \quad U' = L' \oplus \mathbb{R}1$$

where L and L' are real Hilbert spaces with L being a proper closed subspace of L'. In particular, the Hilbert space orthogonal complement of L in L' is non-zero.

By 3.5.2, we have

$$\rho = \rho_h^U, \text{ for some } h \in L \text{ with } \|h\| < 1.$$

Choose any non-zero element h' in L' orthogonal to L satisfying

$$\|h'\| < (1 - \|h\|^2)^{1/2}$$

so that

$$\|h + h'\|^2 = \|h\|^2 + \|h'\|^2 < 1.$$

Now, for any such h', of which there are infinitely many, we have

$$\rho_{h+h'}^{U'}(\alpha 1 + k) = \alpha + \langle h, k \rangle = \rho_h^U(\alpha 1 + k),$$

for each $\alpha \in \mathbb{R}$ and $k \in L$. Thus $\rho_{h+h'}^{U'}$ extends ρ_h^U and, by 3.5.2, lies in $S(U')$. This contradicts the assumption that ρ has unique extension in $S(U')$. $\qquad\square$

Recall that for a Banach space X, $S(X_1)$ denotes the norm one elements of X.

Lemma 3.5.4

Let V be a JBW*-subtriple of W, where V and W are (complex) spin factors. Suppose there exists $\rho \in S(V_1^*) \setminus \partial_e(V_1^*)$ with unique norm preserving extension in W_1^*. Then $V = W$.

Proof

Let $\rho \in S(V_1^*) \setminus \partial_e(V_1^*)$ such that ρ has a unique extension in W_1^*. Let u be the support tripotent of ρ in V. By Theorem 1.10.2, u cannot be minimal in V and so must be unitary in both V and W. Let E and F, respectively, denote the real spin factors $V_2(u)_{sa}$ and $W_2(u)_{sa}$. Then, by restriction, $\rho \in S(E) \setminus P(E)$ with unique extension in $S(F)$. Hence, by Lemma 3.5.3, $E = F$ and thus $V = W$. $\qquad\qquad\square$

Proposition 3.5.5

Let M be a JBW*-subtriple of a JBW*-triple N. Let $\rho = \sum \lambda_n \rho_n$ be a σ-convex sum of mutually orthogonal norm one functionals $\rho_n \in M_*$. Let ρ have unique extension in $N_{*,1}$. Then every partial sum $\sum_1^k \lambda_n \rho_n$ has a unique norm preserving extension in N_*. Moreover, each ρ_n has unique extension in $N_{*,1}$.

Proof

We shall suppose the sum is an infinite σ-convex sum. (The finite convex case is marginally simpler.) We have $\lambda_n > 0$ for all n, $\sum \lambda_n = 1$ and $\|\rho\| = 1$. By Theorem 1.8.4 we can choose a norm one extension τ_n in N_* for each ρ_n. Then

$$\bar{\rho} = \sum \lambda_n \tau_n$$

extends ρ in N_* and has norm one, since

$$1 = \|\rho\| \leqslant \|\bar{\rho}\| \leqslant \sum \lambda_n = 1.$$

Let $k \geqslant 1$. Put

$$\varphi = \sum_1^k \lambda_n \rho_n, \quad \psi = \sum_{k+1}^{\infty} \lambda_n \rho_n \quad \text{and} \quad v = \sum_{k+1}^{\infty} s(\rho_n).$$

We have

$$\sum_{k+1}^{\infty} \lambda_n \geqslant \|\psi\| \geqslant \psi(v) = \sum_{k+1}^{\infty} \lambda_n.$$

Thus,

$$\|\psi\| \;=\; \sum_{k+1}^{\infty} \lambda_n \quad \text{and similarly} \quad \|\varphi\| = \sum_{1}^{k} \lambda_n.$$

It follows that

$$\bar{\varphi} \;=\; \sum_{1}^{k} \lambda_n \tau_n \quad \text{and} \quad \bar{\psi} \;=\; \sum_{k+1}^{\infty} \lambda_n \tau_n$$

are norm preserving extensions of φ and ψ, respectively, in N_*.

Now let φ' be any extension of φ in N_* such that $\|\varphi'\| = \|\varphi\|$. Then $\varphi' + \bar{\psi}$ extends ρ and has norm one, since

$$1 \;=\; \|\rho\| \;\leqslant\; \|\varphi'\| + \|\bar{\psi}\| \;=\; \|\varphi\| + \|\psi\| \;=\; 1.$$

Hence, by uniqueness, we have

$$\varphi' + \bar{\psi} \;=\; \bar{\rho} \;=\; \bar{\varphi} + \bar{\psi}.$$

Therefore, $\varphi' = \bar{\varphi}$, which proves that $\bar{\varphi}$ is the unique norm preserving extension of φ in N_*.

Finally, it follows from the above that, for each n, $\lambda_n \tau_n$ is the unique norm preserving extension of $\lambda_n \rho_n$. Hence, τ_n is the unique norm one extension of each ρ_n. This completes the proof. $\qquad\square$

We can now prove the key result of this section. In the statement below we note that $\mathrm{rank}(C) > 1$.

Theorem 3.5.6

Let C be a Cartan factor and JBW*-subtriple of a JBW*-triple N. Suppose there exists $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ with unique extension in $N_{*,1}$. Then C is an inner ideal of N.

Proof

Let $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ such that ρ has unique norm one extension $\bar{\rho}$ in $N_{*,1}$. Since every Cartan factor is atomic (Theorem 1.11.4(a)(i)), by Theorem 1.12.1(b) ρ is a σ-convex sum

$$\rho = \sum \lambda_n \rho_n, \quad \text{where } \lambda_n > 0 \text{ for each } n \text{ and } \sum \lambda_n = 1 \qquad (*)$$

of at least two mutually orthogonal elements $\rho_n \in \partial_e(C_{*,1})$.

Consider now

$$\tau = \lambda_1 \rho_1 + \lambda_2 \rho_2 \quad \text{and} \quad \sigma = \tau/\|\tau\|.$$

By Proposition 3.5.5, ρ_1 has a unique extension in $N_{*,1}$. Therefore, by Lemma 3.3.1 and Lemma 3.3.2, there is a Cartan factor and weak* closed ideal D of N such that D contains C. It follows that ρ_1 has unique extension in $D_{*,1}$. Hence, the inclusion

$$C \subset D$$

satisfies the equivalent conditions of Theorem 3.3.4. In particular, with

$$u = s(\rho_1) + s(\rho_2), \quad \text{we have,} \quad C_2(u) \subset D_2(u)$$

is an inclusion of spin factors.

Further, by Proposition 3.5.5, σ has a unique norm one extension $\bar{\sigma}$ in D_*. Since $\sigma(u) = \bar{\sigma}(u) = 1$, we have

$$\sigma = \sigma \circ P_2(u) = \bar{\sigma} = \bar{\sigma} \circ P_2(u)$$

and we see that $\bar{\sigma}|_{D_2(u)}$ is the unique norm one extension of $\sigma|_{C_2(u)}$ in $D_2(u)_*$. Hence, by Lemma 3.5.4, $C_2(u) = D_2(u)$.

Now, applying Corollary 3.3.10, we see that either $C = D$ or C and D are hermitian, rectangular or symplectic and of the same generic type. In the latter event, C is an inner ideal of D by Theorem 3.4.8. Hence, C is an inner ideal of N, as required. $\qquad\qquad \square$

In the context of Cartan factors, the results of sections 3.3 and 3.4 combined with Theorem 3.5.6 settle a number of unique extension questions and provide new characterisations of weak* closed inner ideals.

Theorem 3.5.7

Let C be a Cartan factor and JBW*-subtriple of a JBW*-triple N. Suppose that rank$(C) > 1$. Then the following are equivalent.

(a) *There exists $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ with unique extension in $S(N_{*,1})$.*

(b) *Every $\rho \in S(C_{*,1})$ has unique extension in $S(N_{*,1})$.*

(c) *Every $\rho \in S(C_1^*)$ has unique extension in $S(N_1^*)$.*

(d) *C is an inner ideal of N.*

Proof

The implication (a)$\Rightarrow$(b) follows from Theorem 3.5.6. The conditions (b), (c) and (d) are equivalent by Theorem 1.8.3. □

3.6 Von Neumann Algebras

We shall proceed to interpret our above unique extension results in the context of Cartan subfactors of von Neumann algebras. We recall the following proved in [32, Lemma 3.2].

Lemma 3.6.1

Let u be a partial isometry in a von Neumann algebra W. Then

$$W_2(u) \ = \ uu^*Wu^*u \quad (= uWu)$$

is a von Neumann algebra with respect to multiplication and involution given, respectively, by

$$a.b \ = \ au^*b \ \ and \ \ a^\sharp \ = \ ua^*u.$$

Theorem 3.6.2

Let W be a von Neumann algebra and C a JBW*-subtriple of W, where C is a Cartan factor. Suppose there exists $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ with unique extension in $W_{*,1}$. Then we have the following.

(a) $C = eWf$, where e and f are projections in W.

(b) *If C is linearly isometric to a JBW*-algebra then $C = eWf$ where e and f are von Neumann equivalent projections of W, and C is linearly isometric to a von Neumann algebra.*

(c) *If C is a JBW*-subalgebra of W, then $C = eWe$ where e is a projection in W. Hence, C is an hereditary von Neumann subalgebra of W.*

(d) *If C contains the identity element of W, then $C = W$.*

Proof

By Theorem 3.5.6, C is a weak* closed inner ideal of W.

(a) All weak* closed inner ideals of W are of the described form [32, Theorem 3.16].

(b) Let C be linearly isometric to a JBW*-algebra. Then C has a unitary tripotent u. Thus

$$C \; = \; C_2(u) \; = \; W_2(u),$$

where the second equality follows from the fact (above) that C is an inner ideal of W. Hence, by Lemma 3.6.1, C is linearly isometric to a von Neumann algebra and $C = eWf$, where $e = uu^*$ and $f = u^*u$.

(c) In this case, letting the projection e be the identity element of C, we have

$$C \; = \; C_2(e) \; = \; W_2(e) \; = \; eWe.$$

(d) This is immediate from (c). $\qquad\square$

Theorem 3.6.3

Let $D = B(H)$, where H is a complex Hilbert space, and let C be linearly isometric to a von Neumann algebra. Then there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ if and only if $C = eB(H)f$ where e and f are von Neumann equivalent projections in D.

Proof

Since C and D have the same generic type (rectangular), C is an inner ideal of D by Theorem 3.4.8. In addition, by assumption, C contains a unitary tripotent u. Therefore,

$$C \;=\; C_2(u) \;=\; D_2(u).$$

But u is a partial isometry of $B(H)$ and we have

$$D_2(u) \;=\; uu^*B(H)u^*u. \qquad \square$$

Theorem 3.6.4

Let W be a von Neumann algebra and C be a JBW*-subtriple of W, where C is a Cartan factor of rectangular type. Suppose there exists $\rho \in \partial_e(C_{*,1})$ with unique extension in $W_{*,1}$. Then the statements (a), (b), (c) and (d) of Theorem 3.6.2 hold true.

Proof

Using Theorem 3.3.4, we have that there exists ρ in $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ where $C \subset D$ and D is a weak* closed ideal of W and a Cartan factor. In particular, D is *-isomorphic to some $B(H)$ and so has rectangular type. Hence, by Theorem 3.6.3, C is an inner ideal of D and hence is an inner ideal of W. The remainder of the proof now proceeds as in the proof of Theorem 3.6.2. $\qquad \square$

Chapter 4

Unique Dual Ball Extensions

4.1 Introduction

Let C be a Cartan factor contained as a JBW*-subtriple in a JBW*-algebra M. In Chapter 3, it was shown that the existence of a single functional ρ in $\partial_e(C_{*,1})$ with unique extension in $M_{*,1}$ is sufficient to show that *every* element of $\partial_e(C_{*,1})$ has unique extension in $M_{*,1}$ (Corollary 3.3.3). This result, amongst others, is used in this chapter to investigate the 'local' extreme extension property.

Let A be a JB*-subtriple of a JB*-triple B and let $\rho \in \partial_e(A_1^*)$ with unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$. Then the weak* closed ideal A_ρ^{**} of A^{**} generated by $s(\rho)$ is a Cartan factor (see below). Moreover, we can consider ρ as an element of $\partial_e((A_\rho^{**})_{*,1})$. Thus, many of the results of Chapter 3, including Theorem 3.3.4, are available to us. It turns out that A_ρ^{**} has the extreme extension property in $B_{\bar{\rho}}^{**}$, so that A has the extreme extension property locally in B. One of our main results (Theorem 4.2.7) examines all possible cases of the relative structure of A_ρ^{**} and $B_{\bar{\rho}}^{**}$. This theorem and the results of Chapter 3 involving inner ideals are used to classify when A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$.

Much of the remainder of the chapter is taken up with the global extreme, Cartan and atomic extension properties, the latter two being introduced for the first time. Structure space connections and 'inner ideal' conclusions are made. A main result concerns individual unique norm one extension of atomic functionals. In the final section implications for states in the ordered theory of JB*-algebras are discussed.

4.2 Local Theory

4.2.1 Let A be a JB*-triple and let $\rho \in \partial_e(A_1^*)$. Since $s(\rho)$ is a minimal tripotent of A^{**}, the weak* closed ideal, A_ρ^{**}, of A^{**} generated by $s(\rho)$ is a Cartan factor [21]. Let

$$P_\rho : A^{**} \longrightarrow A_\rho^{**}$$

be the canonical weak* continuous M-projection from A^{**} onto A_ρ^{**}. We have

$$A^{**} = A_\rho^{**} \oplus_\infty (\ker P_\rho).$$

Thus, when $(A_\rho^{**})_*$ is identified with its canonical isometric image in A^* we have

$$A^* = (A_\rho^{**})_* \oplus_1 J_*$$

where $J = \ker P_\rho$, the orthogonal complementary weak* closed ideal of A_ρ^{**} in A^{**}. We shall often make such identification.

4.2.2 Let $\rho, \tau \in \partial_e(A_1^*)$. Since A_ρ^{**}, A_τ^{**} are weak* closed ideals of A^{**} and are Cartan factors we have

(a) $A_\rho^{**} = A_\tau^{**}$ or $A_\rho^{**} \cap A_\tau^{**} = \{0\}$.

Thus we either have equality or A_ρ^{**} is orthogonal to A_τ^{**}. Further, for the same reasons we have

(b) $A_\rho^{**} = A_\tau^{**}$ if and only if $s(\rho) \in A_\tau^{**}$.

(c) If $A_\rho^{**} = A_\tau^{**}$ we shall say that ρ and τ are equivalent and write $\rho \sim \tau$.

4.2.3 4.2.2(c) defines an equivalence relation $\sim$ on $\partial_e(A_1^*)$. The equivalence class of $\rho \in \partial_e(A_1^*)$ shall be denoted by $[\rho]$. When, for $\rho \in \partial_e(A_1^*)$, the predual of A_ρ^{**} is canonically identified as an ℓ_1-summand of A^* as indicated in 4.2.1, this gives

$$[\rho] = \partial_e((A_\rho^{**})_{*,1}).$$

4.2.4   Since $\{s(\rho) : \rho \in \partial_e(A_1^*)\}$ is the set of minimal tripotents of A^{**}, 4.2.2(a) implies that

(a) A_{at}^{**} is the ℓ_∞-sum of the distinct A_ρ^{**} as ρ ranges over $\partial_e(A_1^*)$.

Let I be a norm closed inner ideal of A. Since all minimal tripotents of I^{**} are minimal in A^{**} we have

(b) $I_{at}^{**} = A_{at}^{**} \cap I^{**}$.

Let $\rho \in \partial_e(I_1^*)$ and identify ρ with its unique extension in $\partial_e(A_1^*)$. Since, by Theorem 1.11.4 (a)(iv), $A_\rho^{**} \cap I^{**}$ is a Cartan factor and contains I_ρ^{**} as a weak* closed ideal we have

(c) $I_\rho^{**} = A_\rho^{**} \cap I^{**}$.

Lemma 4.2.5

Let A be a JB*-subtriple of a JB*-triple B. Let $\rho \in \partial_e(A_1^*)$ and suppose that ρ has unique extension $\bar\rho \in \partial_e(B_1^*)$. Then $s(\rho) = s(\bar\rho)$, $A_\rho^{**} \subset B_{\bar\rho}^{**}$ and every minimal tripotent of A_ρ^{**} is minimal in $B_{\bar\rho}^{**}$.

Proof

We have that A_ρ^{**} is a Cartan subfactor of the JBW*-triple B^{**} and, by assumption, (identifying ρ as an element of $\partial_e((A_\rho^{**})_{*,1})$) ρ has a unique norm one extension $\bar\rho$ in B^*. Therefore, by Lemma 3.3.1 and proof $s(\rho)$ is a minimal tripotent of B^{**} with $s(\rho) = s(\bar\rho)$. Thus, A_ρ^{**} and B^{**} posses a common minimal tripotent in $s(\rho)$ $(= s(\bar\rho))$. Hence, the Cartan factor D appearing in Lemma 3.3.2 is $B_{\bar\rho}^{**}$,

$$A_\rho^{**} \ \subset \ B_{\bar\rho}^{**}$$

and every minimal tripotent of A_ρ^{**} is a minimal tripotent of $B_{\bar\rho}^{**}$.   □

Let A be a JB*-subtriple of a JB*-triple B. Lemma 4.2.5 shows that whenever there exists $\rho \in \partial_e(A_1^*)$ with unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$, then every element of $\partial_e(A_1^*)$ equivalent to ρ has unique extension in $\partial_e(B_1^*)$. We record a list of equivalent conditions in the statement below, which is companion to Theorem 3.3.4.

Theorem 4.2.6

Let $\rho \in \partial_e(A_1^*)$, where A is a JB-subtriple of a JB*-triple B. Let $\bar{\rho}$ be an extension of ρ in B_1^*. Then the following are equivalent.*

(a) *$\bar{\rho}$ is the unique extension of ρ in B_1^*.*

(b) *$\bar{\rho}$ is the unique extension of ρ in $\partial_e(B_1^*)$.*

(c) *$s(\rho) = s(\bar{\rho})$.*

(d) *$A_\rho^{**} \subset B_{\bar{\rho}}^{**}$ and every minimal tripotent of A_ρ^{**} is minimal in $B_{\bar{\rho}}^{**}$.*

(e) *$K(A_\rho^{**})$ has the extreme extension property in $K(B_{\bar{\rho}}^{**})$.*

(f) *Every element of $[\rho]$ has unique extension in $\partial_e(B_1^*)$.*

Proof

(a)$\Leftrightarrow$(b) The equivalence of these conditions is is given by Corollary 2.4.2.

(b)$\Rightarrow$(c), (c)$\Rightarrow$(d). The required arguments are given by Lemma 4.2.5.

(d)$\Rightarrow$(e) This follows as in the proof of Theorem 3.3.4 (f)$\Rightarrow$(g).

(e)$\Rightarrow$(f) Assume that (e) holds. In particular, $A_\rho^{**} \subset B_{\bar{\rho}}^{**}$ by taking weak* closures and using Theorem 1.11.4 (a)(iii). Thus, (e) implies that every minimal tripotent of A_ρ^{**} is minimal in $B_{\bar{\rho}}^{**}$. Now let $\tau \in [\rho]$ and let $\bar{\tau} \in \partial_e(B_1^*)$ extend τ. Then $s(\tau)$ is minimal in $B_{\bar{\rho}}^{**}$ and $\bar{\tau}(s(\tau)) = 1$ so that, by Proposition 1.10.1, $\bar{\tau}$ is the unique element of $\partial_e(B_1^*)$ supporting $s(\tau)$.

(f)$\Rightarrow$(a) This is clear. $\square$

If A, B, ρ and $\bar{\rho}$ are as in Theorem 4.2.6 and satisfy any of its equivalent conditions, then the results of Chapter 3 can be appealed to in order to study the relative structures of the arising Cartan factors A_ρ^{**} and $B_{\bar{\rho}}^{**}$. In particular, let us note that in these circumstances, the inclusion

$$A_\rho^{**} \subset B_{\bar{\rho}}^{**}$$

satisfies the equivalent conditions of Theorem 3.3.4, a fact employed several times in the proof of Theorem 4.2.7, below.

Since there are many cases to consider giving rise to a formal statement of inordinate length, we shall prove each case in the list prior to the statement and proof of the one that follows.

Theorem 4.2.7

Let A be a JB*-subtriple of a JB*-triple B. Let $\rho \in \partial_e(A_1^*)$ such that ρ has unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$.

Case I A_ρ^{**} or $B_{\bar{\rho}}^{**}$ exceptional.

(a) *If $A_\rho^{**} \cong B_{1,2}$, then $B_{\bar{\rho}}^{**} \cong B_{1,2}$ or M_3^8.*

(b) *If $A_\rho^{**} \cong M_3^8$, then $B_{\bar{\rho}}^{**} \cong M_3^8$.*

(c) *If $B_{\bar{\rho}}^{**} \cong B_{1,2}$, then A_ρ^{**} is a rank 2 subfactor of $B_{\bar{\rho}}^{**}$ or a Hilbert space inner ideal of $B_{\bar{\rho}}^{**}$.*

(d) *If $B_{\bar{\rho}}^{**} \cong M_3^8$, then A_ρ^{**} is a rank 2 or 3 subfactor of $B_{\bar{\rho}}^{**}$ or a Hilbert space inner ideal of $B_{\bar{\rho}}^{**}$.*

Proof

(a),(b) If A_ρ^{**} is exceptional, then $B_{\bar{\rho}}^{**}$ must also be exceptional. Hence, if $A_\rho^{**} \cong B_{1,2}$ then $B_{\bar{\rho}}^{**} \cong B_{1,2}$ or M_3^8. If instead $A_\rho^{**} \cong M_3^8$ then the only possibility is $B_{\bar{\rho}}^{**} \cong M_3^8$.

(c) Suppose that $B_{\bar{\rho}}^{**} \cong B_{1,2}$. By Proposition 3.3.7, any rank 2 sub-factor of $B_{\bar{\rho}}^{**}$ will have the extreme extension property in $B_{\bar{\rho}}^{**}$. Hence A_{ρ}^{**} must either be of this form, or A_{ρ}^{**} has rank 1 and hence is a Hilbert space. In the latter case, A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, by Corollary 3.4.3.

(d) This is almost identical to the previous case. If $B_{\bar{\rho}}^{**} \cong M_3^8$, then there is the added possibility that A_{ρ}^{**} can be a rank 3 subfactor of $B_{\bar{\rho}}^{**}$. The rest of the proof is identical to that of (c). $\quad\square$

Case II A_{ρ}^{**} and $B_{\bar{\rho}}^{**}$ non-exceptional, A_{ρ}^{**} is a Hilbert space.

In this case, A_{ρ}^{**} must be an inner ideal of $B_{\bar{\rho}}^{**}$. If $\dim(A_{\rho}^{**}) \geqslant 2$, then $B_{\bar{\rho}}^{**}$ cannot be hermitian (but can be a spin factor or a rectangular or symplectic factor).

Proof

A_{ρ}^{**} has the extreme extension property in $B_{\bar{\rho}}^{**}$ and hence A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, by Corollary 3.4.3. If $\dim(A_{\rho}^{**}) \geqslant 2$ then A_{ρ}^{**} cannot be hermitian, by Lemma 3.4.4. In turn, $B_{\bar{\rho}}^{**}$ cannot be hermitian by Theorem 3.3.11(c). $\quad\square$

Case III A_{ρ}^{**} and $B_{\bar{\rho}}^{**}$ non-exceptional, A_{ρ}^{**} a spin factor.

(a) *If $A_{\rho}^{**} \cong V_n$, where $n \geqslant 6$, then $B_{\bar{\rho}}^{**} \cong V_m$, $m \geqslant n$.*

(b) *If $A_{\rho}^{**} \cong V_n$, where $n = 4$ or 5, then $B_{\bar{\rho}}^{**}$ is symplectic or $B_{\bar{\rho}}^{**} \cong V_m$ where $m \geqslant n$.*

(c) *If $A_{\rho}^{**} \cong V_3$, then $B_{\bar{\rho}}^{**}$ is rectangular or symplectic or $B_{\bar{\rho}}^{**} \cong V_m$, $m \geqslant 3$.*

(d) *If $A_{\rho}^{**} \cong V_2$, then $B_{\bar{\rho}}^{**}$ is hermitian, rectangular or symplectic or $B_{\bar{\rho}}^{**} \cong V_m$, $m \geqslant 2$.*

Proof

Since A_ρ^{**} has the extreme extension property in $B_{\bar\rho}^{**}$, we have that (a), (b), (c) and (d) all follow immediately from Theorem 3.3.11(a). $\qquad\square$

Case IV A_ρ^{**} and $B_{\bar\rho}^{**}$ hermitian, rectangular or symplectic and rank$(A_\rho^{**}) \geqslant 2$.

(a) *For prescribed A_ρ^{**} the possible structure of $B_{\bar\rho}^{**}$ is as follows.*

A_ρ^{**}	$B_{\bar\rho}^{**}$
Hermitian	*Hermitian, rectangular or symplectic*
Rectangular	*Rectangular or symplectic*
Symplectic	*Symplectic*

(b) *For prescribed $B_{\bar\rho}^{**}$, the possible structure of A_ρ^{**} is as follows.*

$B_{\bar\rho}^{**}$	A_ρ^{**}
Hermitian	*Hermitian*
Rectangular	*Hermitian or rectangular*
Symplectic	*Hermitian, rectangular or symplectic*

Proof

Since the inclusion $A_\rho^{**} \subset B_{\bar\rho}^{**}$ satisfies the equivalent conditions of Theorem 3.3.4, these follow from Theorem 3.3.11(c). $\qquad\square$

Case V $B_{\bar\rho}^{**}$ a Hilbert space or a spin factor.

(a) *If $B_{\bar\rho}^{**}$ is a Hilbert space, then A_ρ^{**} is a Hilbert space and inner ideal of $B_{\bar\rho}^{**}$.*

(b) *If $B_{\bar\rho}^{**}$ is a spin factor, then either A_ρ^{**} is a Hilbert space and inner ideal of $B_{\bar\rho}^{**}$ or A_ρ^{**} is a spin factor.*

Proof

(a) If $B_{\bar{\rho}}^{**}$ is a Hilbert space, then it has rank 1 and hence A_{ρ}^{**} must also be a Hilbert space.

(b) Suppose that $B_{\bar{\rho}}^{**}$ is a spin factor. Then $B_{\bar{\rho}}^{**}$ has rank 2, hence A_{ρ}^{**} has rank 1 or 2. If A_{ρ}^{**} has rank 1, then (as in previous cases) it is a Hilbert space and inner ideal of $B_{\bar{\rho}}^{**}$. Otherwise A_{ρ}^{**} has rank 2 and hence is a spin factor. $\qquad\square$

Let $\rho \in \partial_e(A_1^*)$ have unique extension $\bar{\rho} \in \partial_e(B_1^*)$, where A is a JB*-subtriple of the JB*-triple B. We now use the results of Chapter 3 in conjunction with the previous theorem to deduce precisely when A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$.

Corollary 4.2.8

Let A be a JB-subtriple of a JB*-triple B. Let $\rho \in \partial_e(A_1^*)$ such that ρ has unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$.*

(a) *If A_{ρ}^{**} is exceptional, it is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if $A_{\rho}^{**} = B_{\bar{\rho}}^{**}$.*

(b) *If $B_{\bar{\rho}}^{**}$ is exceptional, A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if one of the following occurs.*

   (i) *A_{ρ}^{**} is a Hilbert space.*

   (ii) *$A_{\rho}^{**} \cong V_7$ and $B_{\bar{\rho}}^{**} \cong B_{1,2}$.*

   (iii) *$A_{\rho}^{**} \cong V_9$ and $B_{\bar{\rho}}^{**} \cong M_3^8$.*

(c) *When neither A_{ρ}^{**} nor $B_{\bar{\rho}}^{**}$ are exceptional then A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if one of the following occurs.*

   (i) *A_{ρ}^{**} or $B_{\bar{\rho}}^{**}$ are Hilbert spaces.*

(ii) $A_\rho^{**} = B_{\bar{\rho}}^{**} = V$, where V is a spin factor.

(iii) A_ρ^{**} and $B_{\bar{\rho}}^{**}$ are of the same generic hermitian, rectangular or symplectic type and $rank(A_\rho^{**}) \geqslant 2$.

(iv) A_ρ^{**} is symplectic or $B_{\bar{\rho}}^{**}$ is hermitian.

Proof

(a) If $A_\rho^{**} \cong M_3^8$, then A_ρ^{**} is equal to $B_{\bar{\rho}}^{**}$ by Theorem 4.2.7 Case I(b). Since $B_{1,2}$ is not an inner ideal of M_3^8, if $A_\rho^{**} \cong B_{1,2}$ then $A_\rho^{**} = B_{\bar{\rho}}^{**}$ by Theorem 4.2.7 Case I(a).

(b) If A_ρ^{**} is a Hilbert space then by the proof of Theorem 4.2.7 Case I(c) it is an inner ideal of $B_{\bar{\rho}}^{**}$.

Let $B_{\bar{\rho}}^{**} \cong B_{1,2}$ and suppose that A_ρ^{**} has rank 2 (the rank 1 case is given by the above). By 3.4.5(b)(i), A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if $A_\rho^{**} \cong V_7$.

Let $B_{\bar{\rho}}^{**} \cong M_3^8$ and suppose that A_ρ^{**} has rank 2. In this case, 3.4.5(b)(ii) shows that A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if $A_\rho^{**} \cong V_9$, giving the desired result.

(c) In case (i), A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, since it must be a Hilbert space.

If we are in neither of the cases (i) and (ii) then, since a spin factor is an inner ideal of another spin factor only when there is equality, it follows from Theorem 3.4.8 and Corollary 3.4.9 (or by Theorem 4.2.7 case IV) that A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ if and only if we are in case (iii) or (iv). $\qquad\square$

4.3 The Extreme Extension Property and Structure Spaces

In this section we examine the extreme extension property of a JB*-subtriple A in a JB*-triple B in connection with certain 'structure spaces' of primitive ideals and equivalence classes of dual ball extreme points associated with A and B.

We begin by formally stating in Theorem 4.3.1 a list of conditions equivalent to the JB*-triple extreme extension property. The proof can be seen by inspection of the *individual* extreme extension property given in Theorem 4.2.6 and only brief indications are required. However, we shall continue to raid Theorem 4.2.6 for information when studying the full extreme extension property.

Theorem 4.3.1

Let A be a JB-subtriple of a JB*-triple B. Then the following are equivalent.*

(a) *A has the extreme extension property in B.*

(b) *Each $\rho \in \partial_e(A_1^*)$ has a unique extension in B_1^*.*

(c) *If $\rho \in \partial_e(A_1^*)$, then $s(\rho)$ is a minimal tripotent of B^{**}.*

(d) *Each $\rho \in \partial_e(A_1^*)$ has an extension $\bar{\rho} \in \partial_e(B_1^*)$ such that $s(\rho) = s(\bar{\rho})$.*

(e) *The minimal tripotents of A^{**} are minimal in B^{**}.*

Proof

(a)$\Leftrightarrow$(b)$\Leftrightarrow$(d)   See Theorem 4.2.6, (b)$\Leftrightarrow$(a)$\Leftrightarrow$(c).

(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e)   These follow from applications of Theorem 1.10.2.   □

4.3.2 Let A be a JB*-triple. A norm closed ideal I of A is said to be *primitive* if it is the largest norm closed ideal contained in $\ker \rho$, for some $\rho \in \partial_e(A_1^*)$. We write $\mathrm{Prim}(A)$ for the set of primitive ideals of A with the *structure topology* (or hull-kernel topology), derived as follows. Given $E \subset A$ and $S \subset \mathrm{Prim}(A)$, the *hull* of E is

$$h(E) \ = \ \{I \in \mathrm{Prim}(A) : E \subset I\}$$

and the *kernel* of S is

$$k(S) = \bigcap_{I \in S} I.$$

The hulls form the closed sets of the *structure topology* on $\mathrm{Prim}(A)$ and the closure of a subset $S \subset \mathrm{Prim}(A)$ is $hk(S)$ [3, Proposition 3.2]. The *structure space* of a JB*-triple A is $\mathrm{Prim}(A)$ endowed with the structure topology.

The map $I \mapsto h(I)$ defines a bijection between the norm closed ideals of a JB*-triple A and the closed sets of $\mathrm{Prim}(A)$. For each norm closed ideal I of A we have the homeomorphisms

$$h(I) \ \longrightarrow \ \mathrm{Prim}(A)$$
$$P \ \longmapsto \ P/I$$

and

$$\mathrm{Prim}(A) \setminus h(I) \ \longrightarrow \ \mathrm{Prim}(I)$$
$$P \ \longmapsto \ P \cap I$$

In fact, the second map is a homeomorphism even when I is a norm closed inner ideal of A [14, 3.3].

4.3.3 A *Cartan factor representation* of a JB*-triple A is a triple homomorphism $\pi : A \to C$, where C is a Cartan factor and $\overline{\pi(A)}^{w^*} = C$. We write $\mathcal{C}(A)$ for the set of Cartan factor representations of A.

Let A be a JB*-triple. Recall that the weak*-closed ideal A_ρ^{**} in A^{**} generated by $s(\rho)$ is a Cartan factor and we have the canonical weak* con-

tinuous M-projection

$$P_\rho : A^{**} \longrightarrow A_\rho^{**}$$

of A^{**} onto A_ρ^{**} (4.2.1). The restriction

$$\pi_\rho : A \longrightarrow A_\rho^{**}$$

of P_ρ is a Cartan factor representation of A. Essentially, all Cartan factor representations arise in this way (see Theorem 4.3.4(b) below). We note that P_ρ is the unique weak* continuous extension, to A^{**}, of π_ρ.

The next theorem is a combination of [14, 3.2] and [6, 3.6].

Theorem 4.3.4

Let A be a JB*-triple.

(a) *If $\rho \in \partial_e(A_1^*)$ then $\ker \pi_\rho$ is the largest norm closed ideal of A in $\ker \rho$.*

(b) *Let $\pi : A \to C$ be a Cartan factor representation of A. Then there exists $\rho \in \partial_e(A_1^*)$ and a surjective isometry $\varphi : A_\rho^{**} \to C$ such that $\varphi \circ \pi_\rho = \pi$.*

(c) $\mathrm{Prim}(A) = \{\ker \pi_\rho \,:\, \rho \in \partial_e(A_1^*)\} = \{\ker \pi : \pi \in \mathcal{C}(A)\}.$

4.3.5 Let A be a JB*-triple and let $\rho, \tau \in \partial_e(A_1^*)$. Recall that either A_ρ^{**} is equal to A_τ^{**}, in which case $\rho \sim \tau$, or A_ρ^{**} is orthogonal to A_τ^{**} (4.2.2). Hence, we have the following list of equivalent conditions.

(a) $P_\rho = P_\tau$.   (b) $\pi_\rho = \pi_\tau$.

(c) $A_\rho^{**} = A_\tau^{**}$.   (d) $s(\rho) \in A_\tau^{**}$.

(e) $s(\tau) \in A_\rho^{**}$.

We use $\hat{A}$ to denote the set of equivalence classes of A which arise from the equivalence relation $\sim$ on A.

4.3.6   The function

$$\theta_A : \hat{A} \longrightarrow \operatorname{Prim}(A)$$

$$[\rho] \longmapsto \ker \pi_\rho$$

is well-defined. Indeed, if $\rho, \tau \in \partial_e(A_1^*)$ with $[\rho] = [\tau]$, then $\pi_\rho = \pi_\tau$ (4.3.5) and it follows that $\ker \pi_\rho = \ker \pi_\tau$. Clearly θ_A is surjective. Define a topology on $\hat{A}$ to be $\{\theta_A^{-1}(U) : U \text{ is open in } \operatorname{Prim}(A)\}$. Then, by definition, θ_A is open and continuous.

Now define

$$\varphi_A : \partial_e(A_1^*) \longrightarrow \hat{A}$$

$$\rho \longmapsto [\rho]$$

and put $\psi_A = \theta_A \circ \varphi_A$. We shall need Proposition 4.3.7, which can be found in [17].

Proposition 4.3.7

If A is a JB*-triple then $\psi_A : \partial_e(A_1^*) \to \operatorname{Prim}(A)$ is open and continuous.

Lemma 4.3.8

Let A be a JB*-triple. Then $\varphi_A : \partial_e(A_1^*) \to \hat{A}$ is open and continuous.

Proof

Let U be an open subset of $\operatorname{Prim}(A)$. Then

$$\varphi_A^{-1}(\theta_A^{-1}(U)) = (\theta_A \circ \varphi_A)^{-1}(U) = \psi_A^{-1}(U)$$

is open in $\partial_e(A_1^*)$, proving continuity.

Now let U be an open subset of $\partial_e(A_1^*)$. Then

$$\theta_A(\varphi_A(U)) = \psi_A(U)$$

is open in $\operatorname{Prim}(A)$. Hence,

$$\varphi_A(U) = \theta_A^{-1}(\psi_A(U))$$

is open in $\hat{A}$ and it follows that φ_A is open. $\square$

Lemma 4.3.9

Let A be a JB*-subtriple of a JB*-triple B and suppose that A has the extreme extension property in B. Then the functions

$$\beta : \hat{A} \; \longrightarrow \; \hat{B}$$
$$[\rho] \; \longmapsto \; [\bar{\rho}]$$

and

$$\gamma : Prim(A) \; \longrightarrow \; Prim(B)$$
$$\ker \pi_\rho \; \longmapsto \; \ker \pi_{\bar{\rho}}$$

are well-defined.

Proof

Let $\rho, \tau \in \partial_e(A_1^*)$ such that $\rho \sim \tau$. Since A has the extreme extension property in B,

$$s(\bar{\rho}) \; = \; s(\rho) \in A_\rho^{**} \; = \; A_\tau^{**} \; \subset \; B_{\bar{\tau}}^{**}$$

by 4.3.5 and Lemma 4.2.5 and proof. Hence, $\bar{\rho} \sim \bar{\tau}$ and β is well-defined.

For γ, let $\rho, \tau \in \partial_e(A_1^*)$ such that $\ker \pi_\rho = \ker \pi_\tau$. It follows that $\rho \sim \tau$ and, as in the previous paragraph, we have $\bar{\rho} \sim \bar{\tau}$. Thus $\ker \pi_{\bar{\rho}} = \ker \pi_{\bar{\tau}}$, which completes the proof. $\qquad \square$

Lemma 4.3.10

Let U, V, X, Y be topological spaces. If the diagram below commutes, where f and g are open and continuous, and h is continuous, then k is continuous.

Proof

Let $A \subset Y$ be open. By continuity of h and g, $h^{-1}(g^{-1}(A))$ is open. Thus, since the diagram commutes,

$$f^{-1}(k^{-1}(A)) = (k \circ f)^{-1}(A) = (g \circ h)^{-1}(A) = h^{-1}(g^{-1}(A))$$

is open. Finally, since f is open,

$$k^{-1}(A) = f(f^{-1}(k^{-1}(A)))$$

is open. □

Theorem 4.3.11

Let A be a JB*-subtriple of a JB*-triple B and suppose that A has the extreme extension property in B. Then the following diagram commutes.

$$
\begin{array}{ccccc}
\partial_e(A_1^*) & \xrightarrow{\varphi_A} & \hat{A} & \xrightarrow{\theta_A} & Prim(A) \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
\partial_e(B_1^*) & \xrightarrow{\varphi_B} & \hat{B} & \xrightarrow{\theta_B} & Prim(B)
\end{array}
$$

All of the maps are continuous and the horizontal maps are open.

Proof

Subsection 4.3.6 and Lemma 4.3.8 show that the horizontal maps are open and continuous. By Proposition 2.4.7, α is continuous. Finally, two applications of Lemma 4.3.10 gives that β and γ are continuous. □

We remark that Theorem 4.3.11 extends [13, Proposition 4.1].

4.4 Unique Extension of Cartan Factor and Atomic Functionals

4.4.1 Cartan Factor Functionals

Definition Let $\tau \in S(A_1^*)$, where A is a JB*-triple. Then τ is defined to be a *Cartan factor functional* of A if the weak* closed ideal of A^{**} generated by $s(\tau)$ is a Cartan factor.

The set of all Cartan factor functionals of A is denoted by $\mathcal{C}_e(A_1^*)$. We note that

$$\partial_e(A_1^*) \subset \mathcal{C}_e(A_1^*).$$

Definition Let A be a JB*-subtriple of a JB*-triple B. Then A is said to have the *Cartan extension property in B* if each element of $\mathcal{C}_e(A_1^*)$ has unique extension in $\mathcal{C}_e(B_1^*)$.

Since the weak* closed Cartan factor ideals of A^{**} are the A_ρ^{**} as ρ ranges over $\partial_e(A_1^*)$, and because of Theorem 1.12.3, the next statement is straightforward.

Lemma 4.4.2

Let $\tau \in S(A_1^*)$, where A is a JB-triple. Then the following are equivalent.*

(a) *$\tau \in \mathcal{C}_e(A_1^*)$.*

(b) *There exists $\rho \in \partial_e(A_1^*)$ such that $s(\tau) \in A_\rho^{**}$.*

(c) *There exists $\rho \in \partial_e(A_1^*)$ such that τ is a σ-convex sum of mutually orthogonal elements of $[\rho]$.*

It is seen from Lemma 4.4.2 that if A_{at}^{**} is the ℓ_∞-sum of Cartan factors

$$A_{at}^{**} = \left(\sum C_i \right)_\infty$$

then via the usual identification

$$C_{i,*} = K(C_i)^* = \{\rho \in A^* : s(\rho) \in C_i\}$$

we have

$$\mathcal{C}_e(A_1^*) = \bigcup S(C_{i,*,1}).$$

Since a Cartan factor C is a Hilbert space if and only if all elements of $S(C_{*,1})$ are extreme points, the next proposition is clear from the above remarks and the Cartan factor representation theory stated in Theorem 4.3.4 (b).

Proposition 4.4.3

Let A be a JB*-triple. Then $\mathcal{C}_e(A_1^*) = \partial_e(A_1^*)$ if and only if all Cartan factor representations of A are onto Hilbert spaces.

Proposition 4.4.4

Let A be a JB*-subtriple of a JB*-triple B. Let $\tau \in \mathcal{C}_e(A_1^*)$ such that $s(\tau) \in A_\rho^{**}$, where $\rho \in \partial_e(A_1^*)$. Suppose that τ has unique extension $\bar{\tau}$ in B_1^*. We have the following.

(a) ρ has unique extension $\bar{\rho}$ in B_1^*.

(b) If $\tau \notin \partial_e(A_1^*)$, then A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$.

(c) $s(\tau) = s(\bar{\tau})$.

(d) $\bar{\tau} \in \mathcal{C}_e(B_1^*)$.

Proof

(i) If $\tau \in \partial_e(A_1^*)$, then $\tau \in [\rho]$ and (a), (c) and (d) (since $\bar{\tau} \in \partial_e(B_1^*)$) are immediate from the equivalent conditions of Theorem 4.2.6.

(ii) Suppose now that $\tau \notin \partial_e(A_1^*)$. Since $s(\tau) \in C$, where $C = A_\rho^{**}$, we may suppose that

$$\tau \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$$

with unique extension $\bar{\tau}$ in the predual ball of B^{**}. Therefore, by Theorem 3.5.6, A_ρ^{**} is an inner ideal of B^{**}. In particular, by Lemma 3.2.1 we have

$$s(\bar{\tau}) = s(\tau) \in A_\rho^{**},$$

and ρ has unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$. Theorem 4.2.6 now gives that

$$A_\rho^{**} \subset B_{\bar{\rho}}^{**},$$

so that $s(\bar{\tau}) \in B_{\bar{\rho}}^{**}$. Hence, $\bar{\tau} \in \mathcal{C}_e(B_1^*)$, by definition. $\qquad\square$

Theorem 4.4.5

Let A be a JB*-subtriple of a JB*-triple B. Then the following are equivalent.

(a) *Every $\tau \in \mathcal{C}_e(A_1^*)$ has unique extension in B_1^*.*

(b) *A has the Cartan extension property in B.*

(c) *A_{at}^{**} is an ℓ_∞-sum of weak* closed inner ideals of B_{at}^{**}.*

(d) *A_ρ^{**} is an inner ideal of B_{at}^{**} for each $\rho \in \partial_e(A_1^*)$.*

Proof

(a)$\Rightarrow$(b) This follows from (the conclusion (d) of) Proposition 4.4.4.

(b)$\Rightarrow$(c) Let (b) be true. Since $\partial_e(A_1^*) \subset \mathcal{C}_e(A_1^*)$, $\partial_e(B_1^*) \subset \mathcal{C}_e(B_1^*)$ and each $\rho \in \partial_e(A_1^*)$ has an extension in $\partial_e(B_1^*)$, we conclude that A must have the extreme extension property in B. To prove (c), it is enough to show that A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$ for each $\rho \in \partial_e(A_1^*)$.

Let $\rho \in \partial_e(A_1^*)$ and let $\bar{\rho}$ be its unique extension in $\partial_e(B_1^*)$ so that, by application of Theorem 4.2.6 we have

$$C \subset D,$$

where $C = A_\rho^{**}$ and $D = B_{\bar{\rho}}^{**}$.

Let $\varphi \in S(C_{*,1})$ and let ψ be an extension of φ in $S(D_{*,1})$. Then regarding $S(C_{*,1})$ and $S(D_{*,1})$ as being contained in $\mathcal{C}_e(A_1^*)$ and $\mathcal{C}_e(B_1^*)$, respectively, as noted above via the usual identifications, we see that ψ must be the unique extension of φ in $\mathcal{C}_e(B_1^*)$. Consequently, ψ is the unique norm one extension of φ in D_*. Hence, by Theorem 1.8.3, C is an inner ideal of D and therefore is an inner ideal of B_{at}^{**}, as required.

(c)$\Rightarrow$(a) Suppose that (c) holds. We have

$$A_{at}^{**} = \left(\sum J_i\right)_\infty,$$

where each J_i is a weak* closed inner ideal of B_{at}^{**}. Each J_i is a weak* closed ideal of A_{at}^{**}. Let $\tau \in \mathcal{C}_e(A_1^*)$. Then $s(\tau) \in C$, and so we may suppose $\tau \in S(C_{*,1})$, for some Cartan factor $C = A_\rho^{**}$ with $\rho \in \partial_e(A_1^*)$. Now there exists i such that

$$C \cap J_i \neq \{0\} \quad \text{and so} \quad C \subset J_i.$$

Therefore, C is a weak* closed ideal of B^{**}. Hence, τ has unique extension in B_1^*.

(c)$\Leftrightarrow$(d) This is clear from the above. $\qquad\square$

Numerous examples have been seen of a JB*-subtriple A having the extension property in a JB*-triple B when condition (c) of Theorem 4.4.5 fails and so that A fails to have the Cartan extension property in B.

In each of the following sample of corollaries, the proofs of which can be read from Corollary 4.2.8 and Theorem 4.4.5(c),

A is a JB*-subtriple of a JB*-triple B.

The first can also be deduced using Proposition 4.4.3

Corollary 4.4.6

Suppose that all Cartan factor representations of A are onto Hilbert spaces. Then A has the extreme extension property in B if and only if A has the Cartan extension property in B.

Corollary 4.4.7

Let A have the extreme extension property in B, let all Cartan factor representations of A be onto factors of rank ≥ 2 and let all Cartan factor representations of B be symplectic. Then A has the Cartan extension property in B if and only if all Cartan factor representations of A are symplectic.

Corollary 4.4.8

If A has the extreme extension property in B, then A has the Cartan extension property in B in each of the following cases.

(a) *All Cartan factor representations of A are symplectic.*

(b) *All Cartan factor representations of B are hermitian.*

4.5 The Atomic Extension Property

Recall that, as defined in Section 1.12, the set of atomic functionals of a JB*-triple A is the norm closed subspace of A^*

$$\{\rho \in A^* : s(\rho) \in A_{at}^{**}\}.$$

In particular, all Cartan factor functionals of A are atomic functionals.

We begin with a unique extension criterion for *individual* atomic functionals.

Theorem 4.5.1

Let A be a JB*-subtriple of a JB*-triple B. Let ρ be an atomic functional in A^*. Then ρ has a unique norm preserving extension to an atomic functional in B^* if and only if ρ has a unique norm preserving extension in B^*.

Proof

Some of the ingredients are already contained in the proof of Proposition 3.5.5. To be sure, let $\|\rho\| = 1$ and using Theorem 1.12.3 write ρ as a σ-convex sum

$$\rho \;=\; \sum \lambda_n \rho_n$$

of mutually orthogonal elements ρ_n in $\partial_e(A_1^*)$. For each n, let $\tau_n \in \partial_e(B_1^*)$ be an extension of ρ_n (using Lemma 2.4.1(b)) and put

$$\bar{\rho} \;=\; \sum \lambda_n \tau_n.$$

Then $\bar{\rho}$ is a norm one atomic functional in B^* extending ρ. In particular, we note this shows that

$(*)$ every atomic functional in A^* has a norm preserving atomic extension in B^*.

Suppose now that ρ has unique norm one atomic extension in B^*. With

$$\varphi \;=\; \sum_{1}^{k} \lambda_n \rho_n, \qquad \psi \;=\; \sum_{n>k} \lambda_n \rho_n$$

and

$$\bar{\varphi} \;=\; \sum_{1}^{k} \lambda_n \tau_n, \qquad \bar{\psi} \;=\; \sum_{n>k} \lambda_n \tau_n$$

for each k, as in the proof of Proposition 3.5.5 we have

$$\|\varphi\| \;=\; \|\bar{\varphi}\| \;=\; \sum_{1}^{k} \lambda_n, \qquad \|\psi\| \;=\; \|\bar{\psi}\| \;=\; \sum_{n>k} \lambda_n$$

so that $\bar{\varphi}$ and $\bar{\psi}$ are *atomic* extensions of φ and ψ, respectively, in B^*.

Now, using $(*)$, let φ' be any norm preserving atomic extension of φ in B^*. Thus (see the proof of Proposition 3.5.5), $\varphi' + \bar{\psi}$ and $\bar{\varphi} + \bar{\psi} = \tau$ are two norm one *atomic* extensions of ρ, and are therefore equal by our supposition. Hence,

$$\varphi' = \bar{\varphi}.$$

Therefore, for each k, $\sum_1^k \lambda_n \tau_n$ is the unique norm preserving atomic extension of $\sum_1^k \lambda_n \rho_n$ in B^*. It follows now that, for each n, $\tau_n \in \partial_e(B_1^*)$ is the unique atomic extension of $\rho_n \in \partial_e(A_1^*)$, and so

$$A_{\rho_n}^{**} \subset B_{\tau_n}^{**} \subset B_{at}^{**},$$

by Theorem 4.2.6 (b)$\Leftrightarrow$(d). Thus

$$M \subset B_{at}^{**},$$

where M is the ℓ_∞-sum of the disjoint members of the family $\{A_{\rho_n}^{**}\}$. Using Lemma 1.10.6 we have

$$s(\rho) = \sum s(\rho_n) \in M.$$

In particular, $s(\rho) \in B_{at}^{**}$.

Finally, let τ be any extension of ρ in $S(B_1^*)$. By Theorem 1.10.1(b) and Theorem 1.4.11(b) we have

$$s(\tau) = \{s(\rho)s(\tau)s(\rho)\} \in B_{at}^{**}.$$

Hence, τ is a norm one atomic functional of B^* and so, by our uniqueness assumption above, $\tau = \bar{\rho}$.

Conversely, if ρ has unique extension in $S(B_1^*)$ then it has unique norm one atomic extension in B^* (since it always has at least one). $\qquad \square$

We now consider a further strengthening of the extreme extension property.

4.5.2 Definition Let A be a JB*-subtriple of a JB*-triple B. Then A is said to have the *atomic extension property in B* if each atomic functional in A^* has a unique norm preserving extension to an atomic functional in B^*.

Equivalently, by Theorem 4.5.1, A has the atomic extension property in B if and only if every atomic functional in $S(A_1^*)$ has unique extension in B_1^*.

It is clear from the equivalence of (a), (b) and (c) in Theorem 4.3.1 that if A has the atomic extension property in B, then A has the extreme extension property in B and

$$A_{at}^{**} \subset B_{at}^{**},$$

in which case, it follows that every element in the predual of A_{at}^{**} must have unique norm preserving extension in the predual of B_{at}^{**}. The following is an immediate consequence of these remarks and Theorem 1.8.3.

Proposition 4.5.3

Let A be a JB-subtriple of a JB*-triple B. Then A has the atomic extension property in B if and only if A_{at}^{**} is an inner ideal of B_{at}^{**}.*

Corollary 4.5.4

Let A be a JB-subtriple of a JB*-triple B such that all Cartan factor representations of A and B are onto the same fixed finite dimensional Cartan factor C. If A has the extreme extension property in B, then A has the atomic extension property in B.*

Proof

Suppose that A has the extreme extension property in B and let $\bar{\rho}$ be the unique extension in $\partial_e(B_1^*)$ of $\rho \in \partial_e(A_1^*)$. Then since $A_\rho^{**} \subset B_{\bar{\rho}}^{**}$ (Theorem 4.2.6) and both have the same finite dimension, we have equality. This

implies that A_{at}^{**} is an ideal of B_{at}^{**} and hence that A has the atomic extension property in B, by Proposition 4.5.3. □

Corollary 4.5.5

Let A be a JB*-subtriple of a JB*-triple B. If A has the atomic extension property in B, then A has the Cartan extension property in B.

Proof

This follows immediately from Theorem 4.4.5 and Proposition 4.5.3. □

We shall proceed to characterise the atomic extension property in terms of the Cartan factor extension property and the structure map

$$\beta : \hat{A} \longrightarrow \hat{B}$$

of Lemma 4.3.9.

Given a JB*-subtriple A of a JB*-triple B and a norm one atomic functional ρ in A^*, by Theorem 1.12.3 we have that ρ is a σ-convex sum

$$\rho = \sum \lambda_i \rho_i$$

of $\rho_i \in \partial_e(A_1^*)$. Choosing an extension $\bar{\rho}_i \in \partial_e(B_1^*)$ for each ρ_i, we have that

$$\sum \lambda_i \bar{\rho}_i$$

is a norm one atomic extension in B^*.

Theorem 4.5.6

Let A be a JB*-subtriple of a JB*-triple B. Then the following are equivalent.

(a) *Every atomic functional in A^* has a unique norm preserving extension in B^*.*

(b) *A has the atomic extension property in B.*

(c) *A has the Cartan extension property in B and $\beta : \hat{A} \to \hat{B}$ is injective.*

Proof

(a)⇔(b) This follows from Theorem 4.5.1.

(b)⇒(c) Assume (b). We know that A has the Cartan extension property in B, by Corollary 4.5.5. Consider the extreme point unique extension map

$$\partial_e(A_1^*) \longrightarrow \partial_e(B_1^*) \quad (\rho \longmapsto \bar{\rho})$$

and the corresponding map

$$\beta : \hat{A} \longrightarrow \hat{B} \quad ([\rho] \longmapsto [\bar{\rho}])$$

of Lemma 4.3.9. Let $\rho, \tau \in \partial_e(A_1^*)$ such that $[\bar{\rho}] = [\bar{\tau}]$.

$$A_\rho^{**}, A_\tau^{**} \subset B_{\bar{\rho}}^{**} = B_{\bar{\tau}}^{**},$$

where we have used Theorem 4.2.6.

Suppose that $[\rho] \neq [\tau]$. Then

$$A_\rho^{**} \neq A_\tau^{**}.$$

Therefore, using 4.2.2, we have that

$$A_\rho^{**} + A_\tau^{**} = A_\rho^{**} \oplus_\infty A_\tau^{**}$$

is a JBW*-subtriple of $B_{\bar{\tau}}^{**}$. But, by assumption and Proposition 4.5.3, we have that $A_\rho^{**} \oplus_\infty A_\tau^{**}$ is a weak* closed inner ideal of the Cartan factor $B_{\bar{\tau}}^{**}$. This contradicts Proposition 2.5.4, proving that $[\rho] = [\tau]$ and hence that β is injective.

(c)⇒(b) Assume (c). Since A has the Cartan extension property in B and by Theorem 4.2.6 together with Theorem 4.4.5, we have that A_ρ^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, for each $\rho \in \partial_e(A_1^*)$, where

$$\partial_e(A_1^*) \longrightarrow \partial_e(B_1^*) \quad (\rho \longmapsto \bar{\rho})$$

denotes the unique extension map. Moreover, since

$$\beta : \hat{A} \longrightarrow \hat{B} \quad ([\rho] \longmapsto [\bar{\rho}])$$

is injective, we have that, for $\rho, \tau \in \partial_e(A_1^*)$,

$$A_\rho^{**} \neq A_\tau^{**} \quad \text{implies} \quad B_{\bar{\rho}}^{**} \neq B_{\bar{\tau}}^{**}.$$

Thus, writing $\partial_e(A_1^*)$ as a union of mutually disjoint classes,

$$\partial_e(A_1^*) = \bigcup [\rho_i],$$

we have

$$A_{at}^{**} = \left(\sum A_{\rho_i}^{**} \right)_\infty$$

and putting

$$J = \left(\sum B_{\bar{\rho}_i}^{**} \right)_\infty,$$

we have that J is an ℓ_∞-sum of mutually orthogonal weak* closed ideals of B_{at}^{**}.

Consider an element of A_{at}^{**}

$$x = \sum x_i,$$

where each $x_i \in A_{\rho_i}^{**}$. Then, by the above,

$$\{xB^{**}x\} \subset \sum \{x_i B_{\bar{\rho}_i}^{**} x_i\} \subset J.$$

Therefore, A_{at}^{**} is an inner ideal of J and hence of B_{at}^{**}. $\qquad \square$

4.5.7 We remark that in Theorem 4.5.6 the condition (c) cannot be replaced with the condition

(c′) A has the extreme extension property in B and $\beta : \hat{A} \to \hat{B}$ is injective.

For example, with $2 \leqslant n \leqslant \infty$, if $A = S_n(\mathbb{C})$ and $B = M_n(\mathbb{C})$ the condition (c′) is clearly satisfied. But A is not an inner ideal of B.

4.6 JB*-Algebras and Unique Extension of Classes of States

4.6.1 The previous discussion applies to JB*-algebras (and C*-algebras) since these are examples of JB*-triples. The global order structure, as well as algebraic structure, of JB*-algebras leads to considerations that do not arise in general JB*-triples.

Let A be a JB*-algebra. Recall (1.10.3) the bijective correspondence

$$\rho \longmapsto s(\rho)$$

between the set $P(A)$ of pure states of A and the minimal projections of A^{**}. We have

$$P(A) \;=\; S(A) \cap \partial_e(A_1^*) \;\subset\; \partial_e(A_1^*)$$

and, as ρ ranges over $P(A)$ the weak* closed Cartan factor ideals of A^{**} are the type I factors $A^{**} \circ c(\rho)$, where $c(\rho)$ is the central support projection of ρ in A^{**}. In particular,

$$A_\rho^{**} \;=\; A^{**} \circ c(\rho), \quad \text{for each } \rho \in P(A).$$

If $\tau \in \partial_e(A_1^*)$, then there exists $\rho \in P(A)$ such that

$$A_\tau^{**} \;=\; A^{**} \circ c(\rho) \;=\; A_\rho^{**}$$

so that (in the notation of 4.3.5) $\tau \sim \rho$. As ρ ranges over $P(A)$ the sum of all of the mutually orthogonal projections $c(\rho)$ arising is the central atomic projection, z_A, of A^{**} and we have

$$A_{at}^{**} \;=\; A^{**} \circ z_A.$$

The *type I factor states* of A are those states ρ for which

$$A^{**} \circ c(\rho)$$

is a type I factor. The set of all type I factor states of A coincides with the set

$$S(A) \cap \mathcal{C}_e(A_1^*).$$

The norm one atomic functionals of A that are states of A, the *atomic states* of A, comprises the subset of $S(A)$,

$$\{\rho \in S(A) : \rho(z_A) = 1\}.$$

4.6.2 Let A be a JB*-algebra. A *type I factor representation* of A is a *-Jordan homomorphism

$$\pi : A \to M,$$

where M is a type I JBW*-algebra factor and $\pi(A)$ is weak* dense in M. In particular, every type I factor representation of A is a Cartan factor representation of A. Given $\rho \in P(A)$, in the notation introduced in 4.3.3, the M-projection

$$P_\rho : A^{**} \longrightarrow A^{**} \circ c(\rho) \ (= A_\rho^{**})$$

and its restriction

$$\pi_\rho : A \longrightarrow A^{**} \circ c(\rho)$$

are given by the assignment

$$a \longmapsto a \circ c(\rho).$$

Moreover, in direct correspondence with Theorem 4.3.4(b), if

$$\pi : A \longrightarrow M$$

is a type I factor representation of A, then there exists $\rho \in P(A)$ and a surjective Jordan *-isomorphism

$$\varphi : A_\rho^{**} \longrightarrow M$$

such that $\varphi \circ \pi_\rho = \pi$.

4.6.3   Let A be a JB*-subalgebra of a JB*-algebra B. Then A is said to have

(a) the *pure extension property in B* if every pure state of A has unique extension to a pure state of B;

(b) the *type I factor extension property in B* if every type I factor state of A has unique extension to a type I factor state of B;

(c) the *atomic state extension property in B* if every atomic state of A has unique extension to an atomic state of B.

Lemma 4.6.4

Let A be a JB-subalgebra of a JB*-algebra B and let $\varphi \in S(A)$. Then every extension of φ in B_1^* is a state of B.*

Proof

Let $\psi \in B_1^*$ such that ψ extends φ. For arguments sake let p denote the identity element of A^{**} and 1 the identity element of B^{**}. Since $\varphi(p) = 1$ and ψ extends φ, we have $\psi(p) = 1$ and hence $\psi = \psi \circ P_2(p)$. So,

$$\psi(1) \;=\; \psi(p) \;=\; 1,$$

proving that $\psi \in S(B)$. □

Lemma 4.6.5

Let A be a JB-subalgebra of a JB*-algebra B. Let $\rho \in P(A)$ with unique extension $\bar{\rho}$ in $P(B)$. Then*

(a) $A^{**} \circ c(\rho) \subset B^{**} \circ c(\bar{\rho})$;

(b) $c(\rho) \leqslant c(\bar{\rho})$;

(c) *Every minimal projection of $A^{**} \circ c(\rho)$ is a minimal projection of $B^{**} \circ c(\bar{\rho})$.*

Proof

Let τ be an extension of ρ in $\partial_e(B_1^*)$. Using Lemma 4.6.4, we have

$$\tau \in S(B) \cap \partial_e(B_1^*) \;=\; P(B).$$

Hence, $\tau = \bar{\rho}$, by assumption. Thus, $\bar{\rho}$ is the unique extension of ρ in $\partial_e(B_1^*)$. Lemma 4.2.5 now gives that $A_\rho^{**} \subset B_{\bar\rho}^{**}$ and that the minimal tripotents of A_ρ^{**} are minimal in $B_{\bar\rho}^{**}$. However, as noted above,

$$A_\rho^{**} \;=\; A^{**} \circ c(\rho) \quad \text{and} \quad B_{\bar\rho}^{**} \;=\; B^{**} \circ c(\bar\rho),$$

from which (a), (b) and (c) now follow. □

In view of the above, short steps only are required to obtain, for JB*-algebras, ordered analogues of such as Theorem 4.2.6 and Theorem 4.3.1 and even to add to equivalent conditions found there. We shall record some of these for completeness.

Proposition 4.6.6

Let A be a JB*-subalgebra of a JB*-algebra B. The following are equivalent for $\rho \in P(A)$.

(a) *ρ has unique extension in $P(B)$.*

(b) *ρ has unique extension in B_1^*.*

(c) *ρ has unique extension in $S(B)$.*

(d) *$s(\rho)$ is a minimal projection of B^{**}.*

Proof

That (b) and (c) are equivalent is immediate from the fact that, by Lemma 4.6.4, norm one extensions of states must be states. The remainder is a consequence of this latter observation together with Theorem 4.2.6. The

details are that, in this way, (b)$\Leftrightarrow$(d) by Theorem 4.2.6(a)$\Leftrightarrow$(c), and that (a)$\Rightarrow$(b) since (as stated in the proof of Lemma 4.6.5) ρ has unique extension in $P(B)$ implies ρ has unique extension in $\partial_e(B_1^*)$ and hence in B_1^*, by Theorem 4.2.6(b)$\Rightarrow$(a). $\qquad\square$

Lemma 4.6.7

Let A be a JB*-subalgebra of a JB*-algebra B and let A have the pure extension property in B. Then A has the extreme extension property in B

Proof

Let u be a minimal tripotent of A^{**}. The weak* closed ideal of A^{**} generated by u is a type I JBW*-algebra factor of the form $A^{**} \circ c(\rho)$, for some $\rho \in P(A)$. Let $\bar{\rho}$ be the unique extension of ρ in $P(B)$. By Lemma 4.6.5, u is a minimal tripotent of $B^{**} \circ c(\bar{\rho})$ and hence is a minimal tripotent of B^{**}. Therefore, A has the extreme extension property in B by Theorem 4.3.1(e)$\Rightarrow$(a). $\qquad\square$

The following theorem is now obtained by combining Theorem 4.3.1 with Proposition 4.6.6 and Lemma 4.6.7.

Theorem 4.6.8

The following conditions are equivalent, for a JB*-subalgebra A of a JB*-algebra B.

(a) *A has the extreme extension property in B.*

(b) *A has the pure extension property in B.*

(c) *Each $\rho \in P(A)$ has unique extension in $S(B)$.*

(d) *If $\rho \in P(A)$, then $s(\rho)$ is a minimal projection of B^{**}.*

(e) *Each $\rho \in P(A)$ has an extension $\bar{\rho} \in P(B)$ such that $s(\rho) = s(\bar{\rho})$.*

(f) *The minimal projections of A^{**} are minimal in B^{**}.*

Proof

The equivalence of (a), (c) and (d) is a direct consequence of Proposition 4.6.6 (a)⇔(c)⇔(d). The equivalence of (d) and (f) follows from applications of Subsection 1.10.3.

Suppose A has the extreme extension property in B and let $\rho \in P(A)$. By Lemma 4.6.4, ρ has unique extension in B_1^* and hence in $P(B)$, using Proposition 4.6.6. Thus, A has the pure extension property in B, proving (a)⇒(b). The converse is Lemma 4.6.7.

Now assume that condition (a) holds and let $\rho \in P(A)$. In particular, $\rho \in \partial_e(A_1^*)$ so that there exists an extension $\bar{\rho}$ in $\partial_e(B_1^*)$ such that $s(\rho) = s(\bar{\rho})$. But $\bar{\rho}$ is a state of B, by Lemma 4.6.4. Hence $\bar{\rho} \in P(B)$, so that condition (e) holds.

Finally, we show that (e)⇒(d). Let $\rho \in P(A)$. Condition (e) implies that there exists an extension $\bar{\rho}$ in $P(B)$ such that $s(\rho) = s(\bar{\rho})$. But $s(\bar{\rho})$ is a minimal projection of B^{**} and hence condition (d) holds. □

Let A be a JB*-algebra. A JB*-subalgebra I of A is an *hereditary* JB*-subalgebra of A if and only if whenever $x \in A$ and $y \in I$ such that

$$0 \leqslant x \leqslant y,$$

then $x \in I$. The *hereditary JB-subalgebras* of A_{sa} are the I_{sa} where I is an hereditary JB*-subalgebra of A.

It is well-known that the hereditary JB*-subalgebras of A are the subalgebras of the form

$$\{p \circ A^{**} \circ p\} \cap A \qquad (*)$$

where p is a projection of A^{**} [28], [29]. In particular, they are inner ideals of A. On the other hand, if I is a norm closed inner ideal of A and I is a JB*-subalgebra of A, then I has the form $(*)$ where p is the identity element

of I^{**}. The following 'ordered' version of Theorem 1.8.3 is contained in [33]. It was proved for C*-algebras in [56].

Lemma 4.6.9

The following are equivalent for a JB*-subalgebra I of a JB*-algebra A.

(a) *I is an hereditary JB*-subalgebra of A.*

(b) *I is an inner ideal of A.*

(c) *Every $\rho \in S(I)$ has unique extension in $S(A)$.*

We shall now state the 'ordered' versions of Theorems 4.4.5, 4.5.1 and 4.5.6.

Theorem 4.6.10

The following are equivalent for a JB*-subalgebra A of a JB*-algebra B.

(a) *A has the Cartan extension property in B.*

(b) *Every type I factor state of A has unique extension in $S(B)$.*

(c) *A has the type I factor extension property in B.*

(d) *$A^{**} \circ c(\rho)$ is an hereditary subalgebra of $B^{**} \circ z_B$, for all $\rho \in P(A)$.*

(e) *$A^{**} \circ z_A$ is an ℓ_∞-sum of hereditary subalgebras of B.*

Proof

(a)$\Rightarrow$(b) Assume (a). It follows from Lemma 4.6.4 and Theorem 4.4.5 that every $\tau \in S(A) \cap \mathcal{C}_e(A_1^*)$ has unique extension in $S(B)$.

(b)$\Rightarrow$(c) Assume (b). Let $\tau \in S(A) \cap \mathcal{C}_e(A_1^*)$. Then condition (b) plus Lemma 4.6.4 implies that τ has unique extension $\bar{\tau}$ in B_1^*. In which case, using Proposition 4.4.4(d),

$$\bar{\tau} \in S(B) \cap \mathcal{C}_e(B_1^*).$$

(c)$\Rightarrow$(d) Assume (c). Then A has the pure extension property in B (pure states are type I factor states and extend to pure states). Let $\rho \in P(A)$ with unique extension $\bar{\rho} \in P(B)$. Lemma 4.6.5 gives

$$A^{**} \circ c(\rho) \ \subset \ B^{**} \circ c(\bar{\rho}) \ \subset \ B^{**} \circ z_B.$$

Since all normal states of $A^{**} \circ c(\rho)$ and $B^{**} \circ c(\bar{\rho})$ correspond to type I factor states of A and B respectively (4.6.2), every normal state of $A^{**} \circ c(\rho)$ has unique extension to a normal state of $B^{**} \circ c(\bar{\rho})$. Hence, by Lemma 4.6.9, $A^{**} \circ c(\rho)$ is an hereditary subalgebra of $B^{**} \circ c(\bar{\rho})$ and hence of $B^{**} \circ z_B$.

(d)$\Rightarrow$(e) This is clear.

(e)$\Rightarrow$(a) This follows from Lemma 4.6.9 and Theorem 4.4.5. □

Theorem 4.6.11

Let A be a JB*-subalgebra of a JB*-algebra B. Let ρ be an atomic state of A. Then ρ has unique extension to an atomic state of B if and only if ρ has unique extension in $S(B)$.

Proof

This follows from Lemma 4.6.4 and Theorem 4.5.1. □

Theorem 4.6.12

The following are equivalent for a JB*-subalgebra A of a JB*-algebra B.

(a) *A has the atomic extension property in B.*

(b) *A has the atomic state extension property in B.*

(c) *Every atomic state of A has unique extension in $S(B)$.*

(d) *$A^{**} \circ z_A$ is an hereditary subalgebra of $B^{**} \circ z_B$.*

(e) *A has the type I factor extension property in B and $\beta : \hat{A} \to \hat{B}$ is injective.*

Proof

(a)$\Rightarrow$(b) This follows from Theorem 4.5.6 and Lemma 4.6.4.

(b)$\Rightarrow$(c) This is also a consequence of Theorem 4.5.6 and Lemma 4.6.4.

(c)$\Rightarrow$(d) If (c) holds then A has the pure extension property in B so that

$$A^{**} \circ z_A \ \subset \ B^{**} \circ z_B,$$

using Lemma 4.6.5. But (c) now implies that every normal state of $A^{**} \circ z_A$ has unique extension to a normal state of $B^{**} \circ z_B$. In which case, (d) results from Lemma 4.6.9.

(d)$\Rightarrow$(a) This is immediate from Lemma 4.6.9 and Theorem 4.5.6.

(a)$\Leftrightarrow$(e) This follows from (a)$\Leftrightarrow$(b) above, Theorem 4.5.6 (b)$\Leftrightarrow$(c) and Theorem 4.6.10 (a)$\Leftrightarrow$(c). $\qquad\square$

Chapter 5

Weak Sequential Convergence,
Weakly Compact JB*-Triples
and Unique Extensions

5.1 Introduction

The ostensible purpose of this chapter is to investigate and to resolve the extreme extension property of a separable JB*-triple A in a JBW*-triple M. As shall be shown, eventually, the JB*-triple in this case A turns out to be a c_0-sum of elementary JB*-triples. JB*-triples of this kind were studied under the name of *weakly compact* in [12], where several equivalent conditions were found. Along the way to establishing the weak compactness (in the above sense) of a separable JB*-triple having the extreme extension property in a JBW*-triple, and in order to bring into appropriate relief various structural implications, we prove a number of results of independent interest.

Weakly compact JB*-triples are aired in section 5.2. A new 'inner ideal' characterisation of weakly compact JB*-triples is given, appropriately in line with the unique extension theme of this thesis. The extreme extension property in the context of weakly compact JB*-triples is investigated and the links to the work of earlier chapters are exposed. The final section of the chapter is almost entirely devoted to a study of weak *sequential* convergence and weak* *sequential* convergence in the topological space $\partial_e(A_1^*)$ of a JB*-

triple A. It is shown that weak sequential convergence in $\partial_e(A_1^*)$ always implies norm convergence, first by proving the truth of this for all spin factors and then by exploiting work of [15] on the so-called Kadec-Klee property on the unit sphere of JB*-triple duals. Subsequent application of a deep result of [19] allows us to conclude that, for every JBW*-triple M, any sequence converging in $\partial_e(M_1^*)$ with respect to the weak* topology must converge in norm.

Concentrating upon separable JB*-triples thereafter, it is shown that in these cases the weak* topology is homeomorphic to the norm topology on $\partial_e(A_1^*)$ precisely when A is a weakly compact JB*-triple. This enables the taking of the final step in order to characterise the extreme extension property, and the atomic extension property, of separable JB*-triples in JBW*-triples.

5.2   JB*-Triples Generated by Minimal Tripotents

5.2.1   Given a JB*-triple A we shall denote by $K(A)$ the norm closed linear span of the minimal tripotents of A. If A has no minimal tripotents we write $K(A) = \{0\}$. This is consistent with the notation $K(C)$ for the elementary ideal of a Cartan factor C (but differs slightly from the usage in [12]). We remark that $K(A)$ is sometimes referred to as the *socle* of A. The ideal $K(A)$ and particularly the case when $K(A) = A$ (that is, when A is generated as a Banach space by its minimal tripotents) was investigated in [12]. See also [8] for a more general and algebraic study of related Jordan Banach *-algebras. We recall that for some family (possibly empty) (C_i) of Cartan factors

(a) $K(A) \cong \left(\sum K(C_i) \right)_0$ [12, Lemma 3.3]

and thus

(b) $K(A)^{**} \cong \left(\sum K(C_i)^{**} \right)_\infty = \left(\sum C_i \right)_\infty$.

5.2.2 Let A be a JB*-triple. Then A is said to be a *weakly compact* JB*-triple [12] if for all $x \in A$, the conjugate linear operator

$$Q_x : A \longrightarrow A$$

is a weakly compact operator (that is, $Q_x(a_n)$ has a weakly convergent subsequence whenever (a_n) is a bounded sequence in A). An array of characterisations is given in [12, Theorem 3.4], including the fact that A is a weakly compact JB*-triple if and only if

$$D(x, x) : A \longrightarrow A$$

is weakly compact for all $x \in A$. For later use, other characterisations are listed in the following statement.

Theorem 5.2.3 [12, Theorem 3.4]

The following conditions are equivalent for a JB*-triple A.

(a) *A is weakly compact.*

(b) *A is an ideal of A^{**}.*

(c) *$K(A) = A$.*

(d) *A contains all minimal tripotents of A^{**} (that is, $K(A) = K(A^{**})$).*

Therefore a JB*-triple A is weakly compact if and only if

$$A = \left(\sum E_i \right)_0,$$

is a c_0-sum of elementary ideals E_i of A. It is clear from the definition that every JB*-subtriple of a weakly compact JB*-triple is a weakly compact JB*-triple. On the other hand, it is of course possible for a weakly compact

JB*-triple to be a JB*-subtriple of a JB*-triple with no minimal tripotents. (For example, take a tripotent u in a continuous JBW*-triple B and put $A = \mathbb{C}u$.)

If A is a JB*-algebra, the condition (b) of Theorem 5.2.3 is equivalent to A_{sa} being an ideal of A_{sa}^{**}. Therefore, [9, Theorem 3.3] yields the next statement.

Lemma 5.2.4

A JB-algebra A is weakly compact if and only if every hereditary JB*-subalgebra of A is of the form $\{e \circ A \circ e\}$ for some projection e of A.*

We shall proceed to prove the further characterisation that weakly compact JB*-triples are precisely the JB*-triples for which every norm closed inner ideal of A is the image of a contractive projection on A.

Lemma 5.2.5

Let $P : A \to A$ be a structural projection on a JB-triple A. Then $P(K(A)) \subset K(A)$.*

Proof

It is sufficient to show that $P(u) \in K(A)$ for each minimal tripotent u of A. Let u be a minimal tripotent of A. Since P is a structural projection (see section 1.8),

$$\{P(u)AP(u)\} \ = \ P\{uP(A)u\} \ = \ \mathbb{C}P(u).$$

Therefore, $P(u)$ is a scalar multiple of a minimal tripotent of A. Hence, $P(u)$ belongs to $K(A)$, as required. $\qquad\square$

Proposition 5.2.6

Let A be a weakly compact JB-triple. Then every norm closed inner ideal of A is the image of a structural projection on A.*

Proof

Let I be a norm closed inner ideal of A. Then I^{**} is a weak* closed inner ideal of A^{**}. Thus, by Theorem 1.8.6, there is a surjective weak* continuous structural projection

$$P : A^{**} \longrightarrow I^{**}.$$

By (c) and (d) of Theorem 5.2.3, $A = K(A^{**})$. Therefore

$$P(A) \subset A,$$

by Lemma 5.2.5. Therefore, P restricts to a structural projection on A and

$$P(A) \subset A \cap I^{**} = I.$$

On the other hand, since P is weak* continuous we have

$$I^{**} = P(A^{**}) \subset P(A)^{**} \subset I^{**}.$$

Therefore, $P(A)^{**} = I^{**}$. Hence, $P(A) = I$. $\qquad\square$

We shall now establish the characterisation mentioned above.

Theorem 5.2.7

Let A be a JB*-triple. Then the following are equivalent.

(a) *Every norm closed inner ideal of A is the image of a contractive projection on A.*

(b) *Every norm closed inner ideal of A is the image of a structural projection on A.*

(c) *A is a weakly compact JB*-triple.*

Proof

The equivalence of (a) and (b) is immediate from Corollary 2.4.9, and the implication (c)$\Rightarrow$(b) was proved in Proposition 5.2.6. It remains to prove that (b)$\Rightarrow$(c).

Suppose that the condition (b) holds. Let $x \in A$. Let B denote the norm closed inner ideal $A(x)$ of A generated by x. By Theorem 1.7.1 we may regard B as a JB*-algebra with $x \in B_+$. Let I be a an hereditary JB*-subalgebra of B. For some projection e of B^{**} we have

$$I^{**} \ = \ \{e \circ B^{**} \circ e\} \ = \ P_2(e)(B^{**}) \ = \ P_2(e)(A^{**}).$$

Since I is a norm closed inner ideal of A, by assumption there exists a surjective structural projection

$$P : A \longrightarrow I.$$

Since both

$$P^{**} : A^{**} \longrightarrow I^{**} \quad \text{and} \quad P_2(e) : A^{**} \longrightarrow I^{**}$$

are surjective structural projections (see 1.8.5), they are equal by uniqueness (Theorem 1.8.6). Hence, P and $P_2(e)$ agree on A. Therefore,

$$I \ = \ P(A) \ = \ P_2(e)(B) \ = \ \{e \circ B \circ e\}.$$

Therefore, $A(x)$ is weakly compact by Lemma 5.2.4 and so is contained in $K(A)$, since minimal tripotents of $A(x)$ are minimal in A. Hence, $K(A)$ contains all elements of A, which is therefore weakly compact by Theorem 5.2.3 (a)$\Leftrightarrow$(c). $\qquad\qquad \square$

Concerning the extreme extension property involving weakly compact JB*-triples, the essential principles are already laid down in Chapters 3 and 4.

Proposition 5.2.8

Let A be a weakly compact JB*-subtriple of a JB*-triple B. Then A has the Cartan extension property in B if and only if A is a c_0-sum of orthogonal norm closed inner ideals of B.

Proof

Let A have the Cartan extension property in B. If A is an elementary JB*-triple then A^{**} is a Cartan factor, so that

$$\mathcal{C}_e(A_1^*) \;=\; S(A_1^*).$$

In which case, by Theorem 4.4.5(b)$\Rightarrow$(a), A has the extension property in B so that A is an inner ideal of B by Theorem 1.8.3. In general, A has the form (see 5.2.1)

$$A \;=\; \left(\sum J_i\right)_0,$$

where the (J_i) are orthogonal norm closed elementary ideals of A. By the above each J_i is an inner ideal of B since each must have the Cartan extension property in B.

Conversely, suppose that

$$A \;=\; \left(\sum J_i\right)_0,$$

where the J_i are orthogonal norm closed inner ideals of B. Then

$$A^{**} \;=\; \left(\sum J_i^{**}\right)_\infty,$$

is an ℓ_∞-sum of weak* closed inner ideals of B_{at}^{**}. Hence, A has the Cartan extension property in B, by Theorem 4.4.5(b)$\Leftrightarrow$(c). $\qquad\square$

Proposition 5.2.9

Let A be a weakly compact JB*-subtriple of a JB*-triple B. Then the following are equivalent.

  (a) *A has the extreme extension property in B.*

  (b) *Every minimal tripotent of A is a minimal tripotent of B.*

  (c) *A has the extreme extension property in $K(B)$.*

Proof

  (a)$\Rightarrow$(b) By Theorem 4.3.1(a)$\Rightarrow$(e), the condition (a) implies that the minimal tripotents of A^{**} are minimal in B^{**}. But all minimal tripotents of A^{**} are contained in A, by Theorem 5.2.3(d).

  (b)$\Rightarrow$(c) Since $A = K(A)$, by Theorem 5.2.3(c), this is immediate from the definition of $K(B)$.

  (c)$\Rightarrow$(a) This follows from the fact that $K(B)$ is a norm closed ideal of B (and so has the extension property in B). $\qquad\qquad$ $\square$

The next statement is an automatic consequence of the fact, as follows from 5.2.1(b), that A^{**} is an atomic JBW*-triple when A is a weakly compact JB*-triple.

Proposition 5.2.10

Let A be a weakly compact JB*-subtriple of a JB*-triple B. Then A has the atomic extension property in B if and only if A is an inner ideal of B.

5.2.11 Making a closer inspection of the situation where A is a weakly compact JB*-subtriple of a JB*-triple B such that A has the extreme extension property in B, by reducing to $K(B)$, it follows from Proposition 5.2.9 that it may as well be supposed that B is a weakly compact JB*-triple

and thus, with (E_i) and (D_j) being the families of mutually orthogonal elementary ideals of A and B, respectively, we have

(a) $A = (\sum E_i)_0$ and $B = (\sum D_j)_0$.

We note that, for fixed j,

(i) $A \cap D_j = \{0\}$ *if and only if* $E_i \cap D_j = \{0\}$ *for all* i;

(ii) *if* i *is such that* $E_i \cap D_j \neq \{0\}$, *then* $E_i \subset D_j$, *and* E_i *has the extreme extension property in* D_j.

To see (i) note that if $A \cap D_j \neq \{0\}$ then it is a weakly compact ideal of A and therefore contains a minimal tripotent u of A. In that case, u must belong to some E_i, giving $E_i \cap D_j \neq \{0\}$. The converse is trivial.

To see (ii): $E_i \cap D_j$ is a norm closed ideal of the elementary JB*-triple E_i and thus, if it is non-zero, it must be all of E_i, giving $E_i \subset D_j$. In that case it is clear that every minimal tripotent of E_i is minimal in D_j.

Since the D_j for which $A \cap D_j = \{0\}$ make no contribution to the analysis, there is no harm in assuming that

(b) $A = (\sum E_i)_0$, $B = (\sum D_j)_0$, *where each* $E_i \subset D_j$ *for some (unique)* j.

(We remark that A can have many summands and B just one. See 5.2.13.)

The main structural results of Theorems 3.3.11, 3.4.8, 3.5.7 as well as Theorem 4.2.7 and Corollary 4.2.8, are available for a further analysis. We shall treat one example.

If F is a weakly compact JB*-triple, let $F_{\mathbb{R}}$ denote the c_0-sum of all hermitian elementary ideals of F, and let $F_{\mathbb{C}}$ and $F_{\mathbb{H}}$ denote the c_0-sum of all rectangular and of all symplectic elementary ideals of F, respectively. If F has only elementary ideals of this type, we have the ℓ_∞ decomposition

$$F = F_{\mathbb{R}} \oplus F_{\mathbb{C}} \oplus F_{\mathbb{H}}.$$

In Proposition 5.2.12 it is assumed that

> A is a JB*-subtriple of a JB*-triple B, that A and B are weakly compact and that $A \cap D \neq \{0\}$, for every elementary ideal D of B.

Proposition 5.2.12

Suppose that all elementary ideals of A and B are hermitian, rectangular or symplectic and that A has no rank 1 (that is, Hilbert space) elementary ideals. We have

(a) $A_{\mathbb{H}} \subset B_{\mathbb{H}}$ *and $A_{\mathbb{H}}$ is a c_0-sum of norm closed inner ideals of B;*

(b) *if $B_{\mathbb{H}} = \{0\}$, then $A_{\mathbb{C}} \subset B_{\mathbb{C}}$ and $A_{\mathbb{C}}$ is a c_0-sum of norm closed inner ideals of B;*

(c) *if $B_{\mathbb{C}} = B_{\mathbb{H}} = \{0\}$, then $A = A_{\mathbb{R}}$ and A is a c_0-sum of norm closed inner ideals of B.*

Proof

(a) We use Theorem 3.3.11(c) together with Theorem 3.4.8 applied to corresponding elementary ideals of Cartan factors. Thus, if E is an elementary ideal of $A_{\mathbb{H}}$ and D is an elementary ideal of B such that

$$E \subset D$$

then, since E must have the extreme extension property in D, it follows from Theorem 3.3.11(c) that D must be symplectic, so that

$$D \subset B_{\mathbb{H}},$$

and that, using Theorem 3.4.8, E must be an inner ideal of D. The statement (a) is now immediate from the definitions of $A_{\mathbb{H}}$ and $B_{\mathbb{H}}$.

(b) Suppose that $B_{\mathbb{H}} = \{0\}$. We then have

$$A = A_{\mathbb{R}} \oplus A_{\mathbb{C}} \quad \text{and} \quad B = B_{\mathbb{R}} \oplus B_{\mathbb{C}}.$$

As in the proof of part (a), the same combination of Theorems 3.3.11(c) and 3.4.8 shows that every elementary ideal of $A_{\mathbb{C}}$ is an inner ideal of $B_{\mathbb{C}}$.

(c) Since in this case

$$A = A_{\mathbb{R}} \quad \text{and} \quad B = B_{\mathbb{R}}$$

the result follows from Theorem 3.4.8 in a similar manner to that above. $\qquad\square$

5.2.13 Examples

We note that in 5.2.11 (or 5.2.12) it is possible for A to have the unique extension property in B, where B is itself an elementary JB*-triple and A is not elementary. For instance, this is the case when

$$A = \left(\sum \mathbb{C} u_i\right)_0 \quad \text{and} \quad B = K(C),$$

where C is a Cartan factor of rank $\geqslant 2$ and (u_i) is a family of orthogonal minimal tripotents of cardinality $\geqslant 2$ in C. For an example, where all factors are infinite dimensional and non-abelian consider a type I_∞ JW*-algebra factor M. Let $(e_i)_{i\in I}$ be an (infinite) orthogonal family of minimal projections in M such that

$$\sum_I e_i = 1.$$

Choose an infinite sequence (S_n) of infinite disjoint subsets of I such that

$$I = \bigcup_{n=1}^{\infty} S_n.$$

For each n, put

$$f_n = \sum_{S_n} e_i.$$

Then (f_n) is an orthogonal sequence of infinite rank projections in M, and $f_n \notin K(M)$ for all n. For each n, consider the type I_∞ factor

$$M_n = f_n M f_n.$$

Then

$$K(M_n) = f_n K(M) f_n$$

is a norm closed inner ideal of $K(M)$ for each n, and with

$$A = \left(\sum K(M_n) \right)_0 \quad \text{and} \quad B = K(M),$$

A has the extreme extension property in B.

5.3 Weak and Weak* Sequential Convergence and Unique Extensions

In this section we consider certain Banach space properties involving weak sequential convergence and weak* sequential convergence on the unit spheres of dual balls of JB*-triples and their restriction to sets of dual ball extreme points.

5.3.1 Definitions Let X be a Banach space. We continue to write

$$S(X) = \{x \in X : \|x\| = 1\}.$$

(a) X is said to have the *Kadec-Klee property* if weak sequential convergence in $S(X_1)$ implies norm convergence (that is, if (x_n) is a sequence in $S(X_1)$ and $x \in S(X_1)$ such that $x_n \to x$ in the weak topology then $\|x_n - x\| \to 0$);

(b) X^* is said to have the *weak* Kadec-Klee property* if weak* sequential convergence in $S(X_1)$ implies norm convergence (with the obvious meaning corresponding to that in parenthesis above).

In the context of JB*-triples and their dual spaces these properties have been studied in [1] and [15] from which the following is extracted for later use and for comparison with theory developed below.

Proposition 5.3.2

Let M be a JBW*-triple and let A be a JB*-triple.

(a) *M_* has the Kadec-Klee property if and only if M is atomic without infinite dimensional spin factor weak* closed ideals.*

(b) *A^* has the Kadec-Klee property if and only if A^{**} is atomic and for each Cartan factor representation, $\pi : A \to C$, C is not an infinite dimensional spin factor.*

(c) *A^* has the weak* Kadec-Klee property if and only if A is a weakly compact JB*-triple without infinite dimensional spin factor elementary ideals.*

(d) *A has the Kadec-Klee property if and only if A is finite dimensional, a Hilbert space or a spin factor.*

Remarks. Let V be an infinite dimensional spin factor. Then V has the Kadec-Klee property, by Proposition 5.3.2(d). On the other hand, V^* does not have the Kadec-Klee property, by Proposition 5.3.2(a) or (b). Note also in this case the weak and weak* topologies on V^* coincide since $V = V^{**}$. For any Banach space X, it is clear that the weak* Kadec-Klee property for X^* implies the Kadec-Klee property. The converse is false. Consider for example the C*-algebra $A = B + \mathbb{C}1$ where B is the algebra of compact

operators in $B(H)$ and 1 is the identity element of $B(H)$, where H is an infinite dimensional Hilbert space. Hence, A^* has the Kadec-Klee property (since $A^{**} \cong B^{**} \oplus_\infty \mathbb{C}$) but A is not weakly compact.

In the sequel we study what amounts to weakening of the Kadec-Klee property and its weak* topological variation replacing $S(A_1^*)$ with the smaller set $\partial_e(A_1^*)$ for JB*-triples A. One benefit is the recovery of infinite dimensional spin factors excluded from the original theory (see Proposition 5.3.2(a), (b), (c)).

Lemma 5.3.3

Let J be a norm closed ideal in a JB-triple A. Let S be the subset of $\partial_e(A_1^*)$,*

$$S \ = \ \{\rho \in \partial_e(A_1^*) : \rho(J) \neq 0\}.$$

Then S is weak closed in $\partial_e(A_1^*)$ if and only if*

$$A \ = \ J \oplus I,$$

for some norm closed ideal I of A.

Proof

Suppose that S is weak* closed in $\partial_e(A_1^*)$. Then by Proposition 4.3.8 and notation we have that, for some norm closed ideal I of A,

$$
\begin{aligned}
\mathrm{Prim}(A) \setminus h(I) \ &= \ \psi_A(\partial_e(A_1^*) \setminus S) \\
&= \ \{\ker \pi_\rho : \rho \in \partial_e(A_1^*) \cap J^\circ\} \\
&= \ h(J).
\end{aligned}
$$

Therefore,

$$\mathrm{Prim}(A) \ = \ h(I) \cup h(J) \ = \ h(I \cap J)$$

which gives $I \cap J = \{0\}$, so that I and J are orthogonal. In addition,

$$h(I + J) \ = \ h(I) \cap h(J) = \emptyset$$

so that

$$A = I + J.$$

Conversely, if $A = I \oplus J$, where I is a norm closed ideal of A, then

$$S = \{\rho \in \partial_e(A_1^*) : \rho(J) \neq 0\} = \partial_e(A_1^*) \cap I^\circ$$

which is weak* closed in $\partial_e(A_1^*)$. □

Lemma 5.3.4

Let J be a norm closed ideal of a JB*-triple A and let $\rho, \tau \in \partial_e(A_1^*)$ such that $\rho(J) = \{0\}$ and $\tau(J) \neq \{0\}$. Then $\|\rho - \tau\| = 2$.

Proof

This follows from 1.10.5 since, by Corollary 1.10.4, $s(\tau) \in J^{**}$ and $s(\rho)$ lies in the orthogonal complement of J^{**} in A^{**}. □

Lemma 5.3.5

Let J be a weak* closed ideal of a JBW*-triple M. Suppose that $\rho \in \partial_e(M_{*,1})$ such that $s(\rho) \in J$, and that (ρ_n) is a sequence in $\partial_e(M_{*,1})$ such that $\rho_n \to \rho$ weakly. Then there exists $N \in \mathbb{N}$ such that $s(\rho_n) \in J$, for all $n \geqslant N$.

Proof

We have

$$\rho_n(s(\rho)) \to \rho(s(\rho)) = 1.$$

Thus, for some $N \in \mathbb{N}$,

$$\rho_n(J) \neq 0 \quad \text{for all } n \geqslant N.$$

Hence, since the $\rho_n \in \partial_e(M_{*,1})$, the result follows. □

In Proposition 5.3.6 it is convenient to make use of the fact that [45] (see also [34, p60]) that a (complex) spin factor V can be constructed from a complex Hilbert space $(H, \|\cdot\|_2)$ and a conjugation $x \mapsto \bar{x}$ on H by defining the triple product and norm given by

$$\begin{aligned}
\{xyz\} &= \langle x, y \rangle z + \langle z, y \rangle x - \langle x, \bar{z} \rangle \bar{y}, \\
\|x\|^2 &= \|x\|_2^2 + \left(\|x\|_2^4 - |\langle x, \bar{x} \rangle|^2 \right)^{1/2}.
\end{aligned}$$

Moreover, $(V, \|\cdot\|)$ and $(H, \|\cdot\|_2)$ are equivalent Banach spaces.

Note that if u is a minimal tripotent of V then

$$\langle u, \bar{u} \rangle = 0 \quad \text{and} \quad \|u\|_2 = \frac{1}{\sqrt{2}}$$

since, for all $x \in V$

$$2\langle u, x \rangle u - \langle u, \bar{u} \rangle \bar{x} = \{uxu\} \in \mathbb{C}u$$

and

$$u = \{uuu\} = 2\langle u, u \rangle u.$$

These notations are retained in the proof below. Proposition 5.3.6 can be compared with [16, Proposition 2] which proves that $\partial_e(V_1^*)$ is weakly sequentially dense in V_1^* whenever V is an infinite dimensional spin factor.

Proposition 5.3.6

Let V be a spin factor. Let $\rho \in \partial_e(V_1^*)$ and let (ρ_n) be sequence in $\partial_e(V_1^*)$ such that $\rho_n \to \rho$ weakly. Then $\|\rho_n - \rho\| \to 0$.

Proof

Let u_n and u be the minimal tripotents of V such that

$$u_n = s(\rho_n) \text{ for each } n, \text{ and } u = s(\rho).$$

Let $x \in V$. We have

$$P_2(u)(x) = \{u\{uxu\}u\} = \{u(2\langle u, x \rangle u)u\} = 2\langle x, u \rangle u.$$

Thus, since $\rho(x)u = P_2(u)(x)$, by Theorem 1.10.2,

$$\rho(x)u \;=\; 2\langle x, u\rangle u \quad\text{and, similarly,}\quad \rho_n(x)u_n \;=\; 2\langle x, u_n\rangle u_n \qquad (\dagger)$$

for each n. Therefore, for each n,

$$|(\rho_n - \rho)(x)| \;=\; 2|\langle x, u_n - u\rangle| \;\leqslant\; 2\|x\|_2\|u_n - u\|_2 \;\leqslant\; 2\|x\|\|u_n - u\|_2$$

and so

$$\|\rho_n - \rho\| \;\leqslant\; 2\|u_n - u\|_2 \qquad (*)$$

By $(\dagger)$, since $\rho_n \to \rho$ weakly we see that $\langle u_n, x\rangle \to \langle u, x\rangle$ and thus that $u_n \to u$ weakly in H. Hence, since H has the Kadec-Klee property and $\|u_n\| = \|u\|$ $(= 1/\sqrt{2})$ for each n, we have that

$$\|u_n - u\|_2 \to 0.$$

The inequality $(*)$ now implies that $\|\rho_n - \rho\| \to 0$. $\qquad\square$

We next extend Proposition 5.3.6 to all JB*-triples. This shows that the dual space of every JB*-triple satisfies the 'extreme Kadec-Klee' property.

Theorem 5.3.7

Let A be a JB-triple. Let $\rho \in \partial_e(A_1^*)$ and let (ρ_n) be a sequence in $\partial_e(A_1^*)$ such that $\rho_n \to \rho$ weakly. Then $\|\rho_n - \rho\| \to 0$.*

Proof

Let M denote A_ρ^{**}, the weak* closed ideal of A^{**} generated by $s(\rho)$. Via the usual identification we can take $\rho \in \partial_e(M_{*,1})$. Further, by Lemma 5.3.5, we may suppose that $s(\rho_n) \in M$ and therefore that $\rho_n \in \partial_e(M_{*,1})$, for all n. Accordingly, since $\rho_n \to \rho$ in the $\sigma(A^*, A^{**})$ topology, we have that

$$\rho_n \to \rho \text{ weakly in } \partial_e(M_{*,1}).$$

If M is a spin factor then Proposition 5.3.6 implies that

$$\|\rho_n - \rho\| \to 0.$$

If M is not a spin factor the same conclusion is deduced because by Lemma 5.3.2(a), M_* has the Kadec-Klee property. $\qquad\square$

We shall now investigate weak* sequential convergence in $\partial_e(A_1^*)$ when A is a JB*-triple. Our aim is to show that when A is *separable* and the weak* topology coincides with the norm topology on $\partial_e(A_1^*)$ then A must be weakly compact. In order to achieve this aim we shall need to draw conclusions in the case when A need not be separable, as is nearly always the situation when A is a JBW*-triple. In fact, as shown below, weak* sequential convergence in $\partial_e(A_1^*)$ always implies norm convergence whenever A is a JBW*-triple. It is deduced from Theorem 5.3.7 and the following deep result implicit in [19].

Theorem 5.3.8 [19]
If M is a JBW-triple, (ρ_n) is a sequence in M^* and $\rho \in M^*$ such that $\rho_n \to \rho$ in the weak* topology, then $\rho_n \to \rho$ weakly.*

In fact, [19] proves the stronger result that every JB*-triple satisfies a certain Banach space property known as Property V. It is well-known [23, p40] that if X is a Banach space such that X^* has Property V, then weak* sequential convergence in X^{**} implies weak convergence. Therefore, for any JBW*-triple $M(= (M_*)^*)$, Theorem 5.3.8 is immediate from [19].

Corollary 5.3.9
Let M be a JBW-triple. Let $\rho \in \partial_e(M_1^*)$ and let (ρ_n) be a sequence in $\partial_e(M_1^*)$ such that $\rho_n \to \rho$ in the weak* topology. Then $\|\rho_n - \rho\| \to 0$.*

Proof

By Theorem 5.3.8, $\rho_n \to \rho$ weakly. Therefore $\|\rho_n - \rho\| \to 0$, by Theorem 5.3.7. □

Proposition 5.3.10

Let A be a JB*-subtriple of a JB*-triple B. Let I be a norm closed inner ideal of A and let J be a norm closed ideal of A. Suppose that A has the extreme extension property in B and that B satisfies the condition:

(†) *if (τ_n) is a sequence in $\partial_e(B_1^*)$ with weak* limit $\tau \in \partial_e(B_1^*)$, then $\|\tau_n - \tau\| \to 0$.*

Then A, I and A/J satisfy the condition (†).

Proof

The unique extension map $\partial_e(A_1^*) \to \partial_e(B_1^*)$ $(\rho \mapsto \bar{\rho})$ is weak* continuous by Proposition 2.4.7. Thus, if (ρ_n) is a sequence in $\partial_e(A_1^*)$ and $\rho \in \partial_e(A_1^*)$ such that $\rho_n \to \rho$ in the weak* topology, then $\bar{\rho}_n \to \bar{\rho}$ in the weak* topology. Hence,

$$\|\rho_n - \rho\| \leqslant \|\bar{\rho}_n - \bar{\rho}\| \to 0,$$

proving the first statement. Since I has the extension property in A, the second statement is immediate from the first statement.

Consider the surjective linear isometry $(A/J)^* \to J^\circ$ $(\rho' \mapsto \rho' \circ \pi)$, where π is the quotient map onto A/J. Let $\rho'_n, \rho \in \partial_e((A/J)_1^*)$ for all $n \in \mathbb{N}$, where ρ' is the weak* limit of ρ'_n. Then

$$\rho'_n \circ \pi \to \rho' \circ \pi \quad \text{in } \partial_e(A_1^*),$$

in the weak* topology. Hence, by the above,

$$\|\rho'_n - \rho'\| = \|\rho'_n \circ \pi - \rho' \circ \pi\| \to 0. \qquad \square$$

We now focus upon separable JB*-triples. The following notations are employed below. If τ is a topology on X^*, where X is a Banach space and $S \subset X^*$, (S, τ) denotes the topological subspace S of X^* with respect to τ. The weak, weak* and norm topologies on X^* are indicated by the symbols w, w^* and $\| \cdot \|$, respectively.

We shall need the following result of [47, Corollary], reproduced in [24, Theorem 10, p161].

Theorem 5.3.11

If X is a separable Banach space such that $\partial_e(X_1^*)$ is norm separable then X^* is norm separable.

Corollary 5.3.12

Let X be a separable Banach space such that whenever (ρ_n) is a sequence in $\partial_e(X_1^*)$ and $\rho \in \partial_e(X_1^*)$ such that

$$\rho_n \to \rho \quad \text{in the weak* topology,}$$

we have

$$\|\rho_n - \rho\| \to 0.$$

Then X^* is norm separable.

Proof

Since X is separable, X_1^* is a compact metrisable, and hence separable, topological space [27, p467]. In particular, $(\partial_e(X_1^*), w^*)$ is metrisable and separable. It follows that the sequential convergence condition imposed in the statement is equivalent to the identity map

$$(\partial_e(X_1^*), w^*) \longrightarrow (\partial_e(X_1^*), \| \cdot \|)$$

being a homeomorphism. Therefore, $(\partial_e(X_1^*), \| \cdot \|)$ is separable. Hence, X^* is norm separable by Theorem 5.3.11. $\qquad \qquad \square$

The structure of JB*-triples A for which A^* is norm separable was determined in [12]. In particular, by [12, Corollary 3.6] and the fact that if M is a closed subspace of a Banach space X such that X^* is norm separable, then $M^*(\cong X^*/M^\circ)$ is also norm separable, we have the following.

Lemma 5.3.13

Let A be a JB-triple such that A^* is norm separable. Then $K(B) \neq \{0\}$ for every non-zero JB*-subtriple B of A.*

Theorem 5.3.14

Let A be a separable JB-triple. Then the following are equivalent.*

(a) *If (ρ_n) is a sequence in $\partial_e(A_1^*)$ with weak* limit $\rho \in \partial_e(A_1^*)$, then*

$$\|\rho_n - \rho\| \to 0.$$

(b) *A is weakly compact.*

(c) *$A = (\sum J_n))_0$, where (J_n) is a sequence (possibly finite) of mutually orthogonal elementary ideals of A.*

Proof

(a)$\Rightarrow$(b) Let (a) hold. Then A^* is norm separable by Corollary 5.3.12. Thus $K(A) \neq \{0\}$, by Lemma 5.3.13. Suppose that $K(A) \neq A$. Consider the subset S of $\partial_e(A_1^*)$,

$$S = \{\rho \in \partial_e(A_1^*) : \rho(K(A)) \neq 0\}.$$

If S is weak* closed in $\partial_e(A_1^*)$ then, by Lemma 5.3.3,

$$A = K(A) \oplus_\infty J$$

for some non-zero norm closed ideal J of A. Since minimal tripotents of J are minimal in A, this implies that $K(J) = \{0\}$. This

contradicts Lemma 5.3.13. Hence, there is no such ideal and therefore S is not weak* closed in $\partial_e(A_1^*)$. Therefore there is a sequence (ρ_n) in S and $\rho \in \partial_e(A_1^*) \cap K(A)^\circ$ such that

$$\rho_n \longrightarrow \rho \text{ in the weak* topology.}$$

But

$$\|\rho_n - \rho\| = 2$$

for all n, by Lemma 5.3.4. This contradicts condition (a). Therefore, $K(A) = A$.

(b)$\Leftrightarrow$(c) The decomposition is countable because A is separable.

(b)$\Rightarrow$(a) Let A be weakly compact. Then A is an ideal of A^{**}, by Theorem 5.2.3. In particular, A has the extension property in A^{**}. Consider the unique extension map

$$\partial_e(A_1^*) \longrightarrow \partial_e(A_1^{***}) \quad (\rho \longmapsto \bar{\rho})$$

and recall that it is weak* continuous, by Proposition 2.4.7. Let

$$\rho_n \to \rho \text{ in } (\partial_e(A_1^*), w^*).$$

Then

$$\bar{\rho}_n \to \bar{\rho} \text{ in } (\partial_e(A_1^{***}), w^*) \quad (\text{here } w^* \equiv \sigma(A^{***}, A^{**})).$$

Now applying Corollary 5.3.9 (with $M = A^{**}$) we have that

$$\|\bar{\rho}_n - \bar{\rho}\| \to 0.$$

Hence,

$$\|\rho_n - \rho\| \to 0. \qquad \square$$

5.4 Separable JB*-Subtriples of JBW*-Triples and Unique Extension Properties

In this final section we consider a separable JB*-subtriple A of a JBW*-triple M and characterise the various unique extension conditions (considered earlier in this thesis) of A in M in terms of the structure of A including that in relation to the structure of M.

Theorem 5.4.1

Let A be a separable JB-subtriple of a JBW*-triple M.*

(a) *A has the extreme extension property in M if and only if A is weakly compact and has the extreme extension property in $K(M)$.*

(b) *A has the Cartan factor extension property in M if and only if A is a c_0-sum of orthogonal norm closed inner ideals of $K(M)$.*

(c) *A has the atomic extension property in M if and only if A is an inner ideal of $K(M)$.*

Proof

(a) Suppose that A has the extreme extension property in M and let

$$\partial_e(A_1^*) \longrightarrow \partial_e(M_1^*) \quad (\rho \longmapsto \bar{\rho})$$

be the unique extension map. By argument rehearsed in the proof of Theorem 5.3.14, if

$$\rho_n \to \rho \quad \text{in } (\partial_e(A_1^*), w^*)$$

then Proposition 2.4.7 implies that

$$\bar{\rho}_n \to \bar{\rho} \quad \text{in } (\partial_e(M_1^*), w^*)$$

so that by Corollary 5.3.9

$$\|\bar{\rho}_n - \bar{\rho}\| \to 0$$

which implies that $\|\rho_n - \rho\| \to 0$ and thus that A is weakly compact, by Theorem 5.3.14. Now Proposition 5.2.9 implies that A has the extreme extension property in $K(M)$.

Conversely, if A has the extreme extension property in $K(M)$ then it has the extreme extension property in M.

(b) Let A have the Cartan extension property in M. As previously observed (see the proof of Theorem 4.4.5 (b)$\Rightarrow$(c)) A must have the extreme extension property in M. Thus A is weakly compact and

$$A \subset K(M).$$

Since A therefore has the Cartan extension property in $K(M)$, A is of the claimed form by Proposition 5.2.8.

Conversely, if A has the stated decomposition, it must be weakly compact, because $K(M)$ is weakly compact, and therefore have the Cartan extension property in $K(M)$, by Proposition 5.2.8

(c) By similar argument, via (a), if A has the atomic extension property in M, then A has the atomic extension property in $K(M)$ and thus is an inner ideal of $K(M)$, using Proposition 5.2.10.

Conversely, any norm closed inner ideal of $K(M)$ is a norm closed inner ideal of M and thus must have the extension property in M and hence the atomic extension property in M, by Theorem 4.5.6. □

Finally, we shall record the 'JB*-algebra state' version of Theorem 5.4.1, the proof of which can now be read from Theorem 5.4.1 and the relevant results of Section 4.6, including Theorem 4.6.8, Theorem 4.6.10, Theorem 4.6.12 and Lemma 4.6.4 and Lemma 4.6.9.

Theorem 5.4.2

Let A be a separable JB*-subalgebra of a JBW*-algebra M.

(a) *A has the pure extension property in M if and only if A is weakly compact and has the pure extension property in $K(M)$.*

(b) *A has the type I factor extension property in M if and only if A is a c_0-sum of orthogonal hereditary JB*-subalgebras of $K(M)$.*

(c) *A has the atomic state extension property in M if and only if A is an hereditary JB*-subalgebra of $K(M)$.*

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