The Application of Optimal Control Theory

to Life Cycle Strategies.

Lucinda Mary Whitfield

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Abstract

The aim of this work, is to develop a numerical model, that could be used to in the calculation of optimal life cycle strategies of given organisms. The theory used from [6] assumes that the organisims have either a two phase 'bang-bang' life cycle strategy or a three phase life cycle strategy with the second phase being a singular arc.

Contents

Chapter 1

Introduction

The aim of this work is to develop a numerical method, which will find the optimal allocation of resources between growth and reproduction for a given organism and, hence, an organism's optimal life cycle strategy. A brief introduction to optimal control theory and life cycle strategies is given here. In addition an overview of the project's content and it's organisation is included.

1.1 What is Optimal Control Theory?

Optimal control theory arises from the consideration of physical systems, which are required to achieve a definite objective as "cheaply" as possible. The translation of the design objectives into a mathematical model gives rise to what is known as the control problem. The essential elements of a control problem are:-

- The system which is to be "controlled"
- A system objective
- A set of admissible "controls" (inputs)

- A performance functional which measures the effectiveness of a given "control action"

The system objective is a given state or set of states which may vary with time. Restrictions or constraints, as they are normally called, are placed on the set of controls (inputs) to the system; controls satisfying these constraints are said to belong to the set of admissible controls. A formal definition of the optimal control problem and associated theory of Pontryagin's principle is looked at in chapter two.

1.2 Life Cycle Strategies

An organism's life cycle strategy is determined by the way in which it allocates energy between growth and reproduction. There are just two main ways in which organisms allocate energy. These are:-

- Determinate growth results in a bang-bang allocation strategy. All energy is initially allocated to growth until maturity, when maturity is reached all energy is switched to reproduction.

- Indeterminate growth results in a split energy allocation. This allows both growth and reproduction to take place simultaneously.

The general analytical model of the resource allocation problem developed in [6] considers the process of resource allocation as an optimal control problem. It uses Pontryagin's maximum principle to find the optimal allocation of energy between growth and reproduction for a given organism.

The idea of finding the optimal allocation of resources between growth and reproduction is the motivation behind this work. The aim is to develop a numerical method for the optimisation problem.

The work begins by looking at the optimal control problem and Pontryagin's Maximum principle before moving on to look at the biological problem and it's analytical solution in greater detail. The numerical method for this problem is then developed in chapter four. The method begins by solving the state and adjoint equations for an arbitrary control u, before looking at ways of making the control u optimal. The model developed is assessed by using trial problems to which analytical solutions can be found.

Chapter 2

Optimal Control Theory

The optimal control problem can be formulated as the finding of the control variables $u_j(t)$ (j=1,...n) and state variables $x_i(t)$ (i=1,...m) satisfying the differential equation

$$\dot{x}_i = f_i(x, u, t),$$

with end conditions $x(t_0) = x_0$, t_0 specified, such that a particular control vector $u = (u_1, ...u_n)^T$ and state vector $x = (x_1, ...x_m)^T$ minimises or maximises the cost functional.

$$J = \phi(x(T), T) + \int_{t_0}^{T} F(x, u, t)dt.$$

The admissible controls for this problem are in fact piecewise continuous functions with a finite number of jump discontinuities on the interval $[t_0, T]$. The functions F and f are continuously differentiable with respect to x, but only satisfy a Lipshitz condition with respect to u.

2.1 Pontryagin's Principle

Pontryagin's principle was developed to deal with control problems where the variables were subject to magnitude constraints of the form:- $| u_i(t) | \leq k_i$. This implies that the set of final states which can be achieved is limited. The assumptions made about u, F and f when defining the optimal control problem are assumed to hold for the u, F and f of Pontryagin's Principle.

<u>THEOREM</u> 2.1 (Pontryagins Principle) *Necessary conditions for a control $u^* \in U$, the admissible control region, to minimise*

$$J(u) = \phi(x(t), t) + \int_0^T F(x, u, t)dt \tag{2.1}$$

subject to

$$\dot{x} = f(x, u, t) \quad x(0) = x_0 \tag{2.2}$$

are that there exists a continuous vector function $\lambda(t)$ that satisfies

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \qquad (adjoint\ \ equation) \tag{2.3}$$

$$\lambda(T) = \left.\frac{\partial \phi}{\partial x}\right|_T \qquad (transversality\ \ condition) \tag{2.4}$$

and that

$$H(x^*, u, \lambda^*, t) \geq H(x^*, u^*, \lambda^*, t) \tag{2.5}$$

for all admissible controls u satisfying the constraints, where H is the Hamiltonian function defined as

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda^T f(x, u, t). \tag{2.6}$$

u^* is assumed to piecewise continuous, and satisfies the constraints.

In time dependent problems Pontryagins principle can be used to determine either a minimum or a maximum of the functional J(u).

Example

Consider the angular motion of a ship which is given by

$$\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} = u, \tag{2.7}$$

where the rudder setting u is subject to $|u| \leq 1$. Find u to minimise the time required to change course from $\phi = 1$, $\frac{\partial\phi}{\partial t} = 0$ when $t = 0$ to $\phi = 0$, $\frac{\partial\phi}{\partial t} = 0$ when $t = T$. The first order equations are formed by taking $x_1 = \phi$, $x_2 = \dot{\phi}$. Then

$$\dot{\phi} = \dot{x_1} = x_2, \quad x_1(0) = 1, \quad x_1(T) = 0 \tag{2.8}$$

$$\ddot{\phi} = \dot{x_2} = u - x_2, \quad x_2(0) = 0, \quad x_2(T) = 0. \tag{2.9}$$

Hence we wish to

$$\min_u \int_0^T 1 \, dt \tag{2.10}$$

subject to equations (2.8) and (2.9). We now form the Hamiltonian function

$$H(x_1, x_2, u, t, \lambda_1, \lambda_2) = 1 + \lambda_1(x_2) + \lambda_2(u - x_2). \tag{2.11}$$

The adjoint equations can be found to be

$$\dot{\lambda_1} = -\frac{\partial H}{\partial x_1} = 0 \tag{2.12}$$

$$\dot{\lambda_2} = -\frac{\partial H}{\partial x_2} = \lambda_2 - \lambda_1. \tag{2.13}$$

These can be solved to give

$$\lambda_1 = A \tag{2.14}$$

$$\lambda_2 = A + Be^t, \tag{2.15}$$

where A and B are constants to be determined. Applying Pontryagins Principle we get

$$1 + \lambda_1^* x_2^* + \lambda_2^*(u^* - x_2^*) \leq 1 + \lambda_1^* x_2^* + \lambda_2^*(u - x_2^*) \tag{2.16}$$

$$\lambda_2^*(u^* - u) \leq 0. \tag{2.17}$$

Hence

$$u^* = -1 \quad if \quad \lambda_2 > 0 \tag{2.18}$$

$$u^* = 1 \quad if \quad \lambda_2 < 0. \tag{2.19}$$

From equation (2.15) we can see that λ_2 can change sign at most once. The equations for x_1 and x_2 are now found to be

$$x_1 = ut - Ce^{-t} + D \tag{2.20}$$

$$x_2 = u + Ce^{-t}, \tag{2.21}$$

where C and D are constants to be found. There are two possible cases to consider: $u = 1$ at $t = 0$, $u = -1$ at $t = T$, and $u = -1$ at $t = 0$, $u = 1$ at $t = T$. From equations (2.15), (2.18) and (2.19) it can be determined that $u = 1$ at $t = 0$ is the correct case, x_1 and x_2 can now be found in terms of known values. Hence

$$x_1 = t + e^{-t} \quad for \quad t \leq t_s$$

$$x_1 = -t - e^{T-t} + T + 1 \quad for \quad t \geq t_s \tag{2.22}$$

$$x_2 = 1 - e^{-t} \quad for \quad t \leq t_s$$

$$x_2 = -1 + e^{T-t} \quad for \quad t \geq t_s, \tag{2.23}$$

from which we can determine the switching time t_s by equating the pairs of equations (2.22) and (2.23) for x_1 and x_2 at the switching time t_s, giving $t_s =$

$\frac{1}{2}(T-1)$. All that remains is to calculate the final time T. This is straight forward and gives $T = 2cosh^{-1}(e^{-\frac{1}{2}})$.

2.2 Singular Intervals

In the special case where the Hamiltonian H is linear in the control vector u, it is possible for the coefficient of the linear term to vanish over a finite interval of time. This gives rise to a singular interval and Pontryagin's principle gives no information about the control variable u. During this phase an additional condition is needed. The additional condition is a condition for local optimality, which holds through out the singular interval, and is the second order Clebsch-Legendre condition stated here without proof. The required condition for a maximum is

$$\frac{\partial}{\partial u}\left[\left(\frac{d}{dt}\right)\left(\frac{dH}{du}\right)\right] > 0 \tag{2.24}$$

For proof of this see [2], examples can also be found in this reference.

Example

Consider the control strategy that causes the response of the system

$$\dot{x_1}(t) = x_2(t), \qquad \dot{x_2}(t) = u(t) \tag{2.25}$$

subject to $x_1(0) = \alpha$, $x_2(0) = \beta$, to minimise

$$J = \frac{1}{2}\int_0^T \left(x_1^2(t) + x_2^2(t)\right)\, dt. \tag{2.26}$$

The final time T and the final states are free, and the controls are constrained by the inequality $\mid u(t) \mid \leq 1$. We now form the Hamiltonian function

$$H(x_1, x_2, u, t, \lambda_1, \lambda_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda_1 x_2 + \lambda_2 u. \tag{2.27}$$

Applying Pontryagin's principle,

$$\frac{1}{2}x_1^{*^2} + \frac{1}{2}x_2^{*^2} + \lambda_1^* x_2^* + \lambda_2^* u^* \le \frac{1}{2}x_1^{*^2} + \frac{1}{2}x_2^{*^2} + \lambda_1^* x_2^* + \lambda_2^* u, \tag{2.28}$$

$$\lambda_2^* (u^* - u) \le 0. \tag{2.29}$$

Hence

$$u^* = 1 \quad if \quad \lambda_2 < 0$$

$$u^* = -1 \quad if \quad \lambda_2 > 0 \tag{2.30}$$

Consider the existence of a singular interval $[t_1, t_2]$, in which case $\lambda_2^* = 0$ for all $t\varepsilon [t_1, t_2]$. Thus, for $t\varepsilon [t_1, t_2]$,

$$\dot{\lambda}_2 = \ddot{\lambda}_2 = ... = 0, \quad for \quad t\varepsilon [t_1, t_2]. \tag{2.31}$$

The adjoint equations are found to be:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -x_1 \tag{2.32}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -x_2 - \lambda_1, \tag{2.33}$$

and hence, for $t\varepsilon [t_1, t_2]$, $\lambda_1 = -x_2$. Since the final time $t = T$ is free and t does not appear explicitly in the Hamiltonian, $[H]_{u=u^*} = 0$, therefore

$$\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda_1 x_2 + \lambda_2 u = 0 \quad for \quad t\varepsilon [0, T]. \tag{2.34}$$

Therefore, for $t\varepsilon [t_1, t_2]$,

$$0 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda_1 x_2, \quad since \quad \lambda_2 = 0,$$

$$0 = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2. \tag{2.35}$$

Thus, equation (2.34) inplies, for $t\varepsilon [t_1, t_2]$,

$$x_1 + x_2 = 0 \tag{2.36}$$

or

$$x_1 - x_2 = 0. \tag{2.37}$$

Differentiating equation (2.36) and substituting in the state equation gives

$$\dot{x_2} = -x_2, \tag{2.38}$$

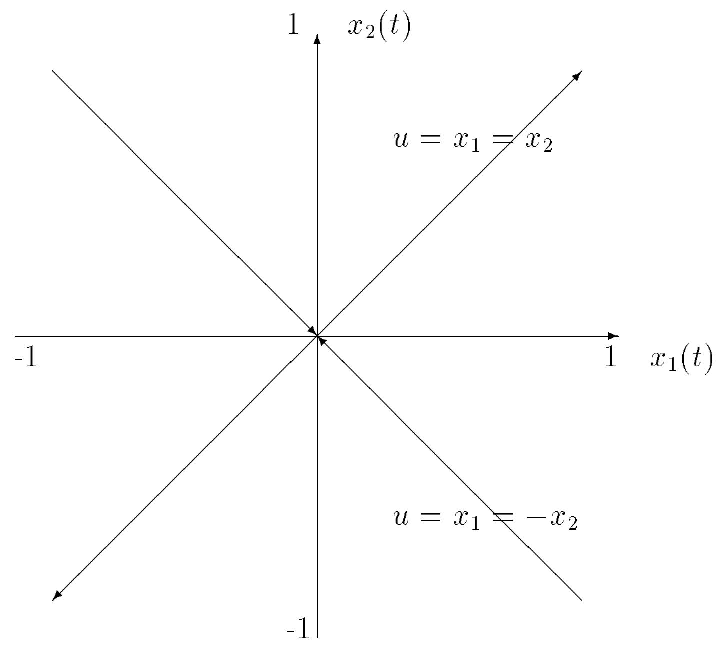
which implies

$$u^*(t) = -x_2^*(t), \quad for \quad all \quad t \varepsilon [t_1, t_2], \tag{2.39}$$

and, therefore, $x_1^*(t)$, $x_2^*(t) \varepsilon [-1, 1]$. Similarly, differentiating equation (2.37), it can be shown that

$$u^*(t) = x_2^*(t), \quad for \quad all \quad t \varepsilon [t_1, t_2], \tag{2.40}$$

and $x_1^*(t)$, $x_2^*(t) \varepsilon [-1, 1]$. Equations (2.36) and (2.37) define the locus of a point in the state plane (x_1, x_2) where singular intervalss may exsist. Since the system



moves away from the origin on the line $x_1 = x_2$, this segment cannot form part of an optimal trajectory. Suppose $\lambda_2^*(t) \neq 0$, then

$$u^* = 1 \quad if \quad \lambda_2 < 0$$

$$u^* = -1 \quad if \quad \lambda_2 > 0. \tag{2.41}$$

Hence,

$$x_2^*(t) = \overset{+}{-}\, t + C_1 \tag{2.42}$$

and therefore

$$x_1^*(t) = \overset{+}{-}\, \frac{t^2}{2} + C_1 t + C_2. \tag{2.43}$$

Where C_1 and C_2 are constants to be determined. Since $x(0)$ lies in the first quadrant of the state plane (x_1, x_2), the optimal control should be $u^* = -1$ initially. Consider now the possibility of a singular interval. If u^* switches from -1 to 1 at some time t_1, then $\lambda_2(t_1) = 0$. Since $\lambda_2 > 0$ for $< t_1$ and $\lambda_2 < 0$ for $t > t_1$,

$$\dot{\lambda}_2 < 0. \tag{2.44}$$

However, $[H]_{u=u^*} = 0$ and $\lambda_2 = 0$, which implies

$$\lambda_1(t_1) = -\frac{x_1^2(t_1) + x_2^2(t_1)}{2x_2(t_1)}. \tag{2.45}$$

Substituting for $\lambda_1(t_1)$ in the adjoint equation gives

$$\dot{\lambda_2(t_1)} = \frac{[x_1(t_1) + x_2(t_1)]\,[x_1(t_1) - x_2(t_1)]}{2x_2(t_1)}. \tag{2.46}$$

Thus, for x in the fourth quadrant of the (x_1, x_2) plane,

$$\frac{x_1(t_1) - x_2(t_1)}{x_2(t_1)} < 0 \tag{2.47}$$

and, hence,

$$\dot{\lambda_2(t_1)} < 0, \quad for \quad x_1 + x_2 > 0, \tag{2.48}$$

whilst

$$\dot{\lambda_2(t_1)} > 0, \quad for \quad x_1 + x_2 < 0. \tag{2.49}$$

Comparing with equation (2.44), it follows that switching is only allowed in the case when $x_1 + x_2 > 0$. There are three possible cases:

1. $\alpha \leq -\frac{1}{2}\beta^2 + \frac{3}{2}$

2. $-\frac{1}{2}\beta^2 + \frac{3}{2} < \alpha < -\frac{1}{2}\beta^2 + 4$

3. $\alpha \geq -\frac{1}{2}\beta^2 + 4$

In case 1, if the initial trajectory segment intersects the line $x_1 + x_2 = 0$, it is not allowed to cross the singular line as switching cannot occur for $x_1 + x_2 < 0$. Hence, the trajectory continues along the singular line until the origin is reached.

In case 2, switching is allowed so that a trajectory can follow a parabola $x_1 = \frac{1}{2}x_2^2 + C$ (with $0 < C \leq \frac{1}{2}$) until it reaches the singular line and then travels along the singular line.

In case 3, as in case 2, or switching so that a trajectory follows the parabola $x_1 = \frac{1}{2}x_2^2$ until it reaches the state origin.

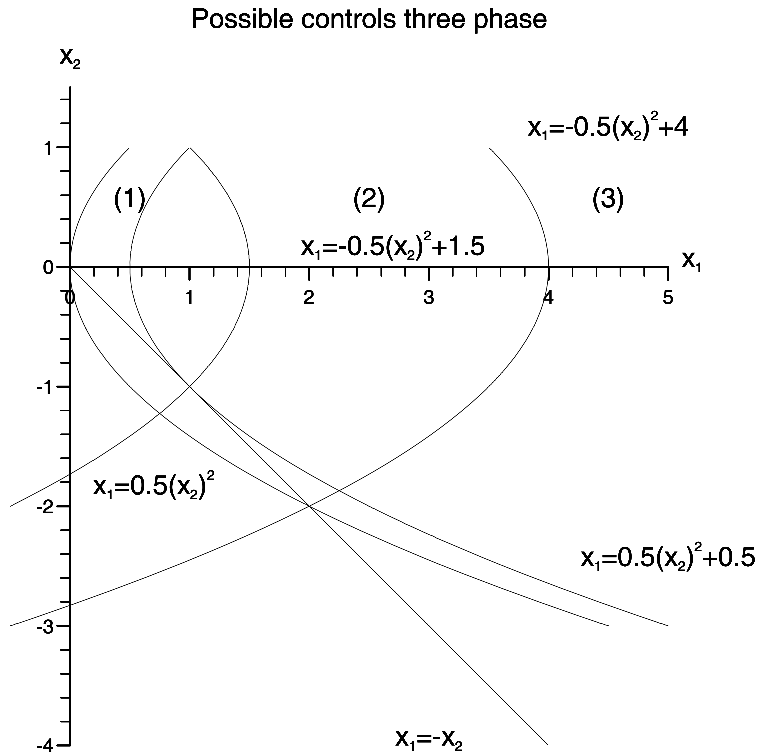


Figure 2.1: Demonstrating control solutions 1,2,and 3

Chapter 3

Biological Problem

3.1 Notation

t = age

T = final time

$w(t)$ = body weight, $w(0)$ = initial weight

$P(w)$ = total resources available for allocation, $p \geq 0$

$w0$ = energy used in producing one off-spring, $w0 = \frac{w(0)}{k}$, k a given constant

$u(t)$ = control variable

$b(u, w)$ = birth rate

$\mu(w)$ = mortality rate

$l(t) = e^{-\int_0^t \mu(x)dx}$ survivorship from birth to age t

$L(t) = e^{-rt}l(t) \quad L(0) = 1$

The combining of e^{-rt} and $l(t)$ into one 'state' variable is possible as the rate of increase of the population is mathematically equivalent to adding a constant term r to the mortality rate μ.

3.2 Biological Problem

The biological problem of resource allocation is considered as an optimal control problem , where fitness is maxismised by the choice of the control variable. Fitness of a life cycle strategy is measured by it's rate of increase r, defined by

$$1 = \int_0^T e^{-rt} l(t) b(t)\, dt, \tag{3.1}$$

where T is the maximum age of reproduction, [4]. A strategy which maximises fitness is used as this is the most probable result of evolution.

The resources $P(w)$ available to a given organism of size w are always subject to the competing demands of growth and reproduction. Since $L(t) = e^{-rt}l(t)$, it can be thought of as a single factor which weights reproduction in equation (3.1) and decreases with time at the rate $r + \mu(w)$. Reproduction decreases in value with time due to the population increasing at a rate of r and the decrease in survival probability by a rate of $\mu(w)$. u denotes the proportion of resources directed to reproduction, the remainder of the resources are directed to growth. Using the model of growth and reproduction developed in [6] we have an optimal control problem in which we choose u to maximise r subject to the following constraints.

$$\dot{w} = (1 - u(t))P(w), \quad w(0) = specified \tag{3.2}$$

$$\dot{L} = -(r + \mu(w))L, \quad L(0) = 1 \tag{3.3}$$

$$\dot{\theta} = \frac{uP(w)L}{w0}, \quad \theta(0) = 0 \quad \theta(T) = 1 \tag{3.4}$$

$$\dot{r} = 0 \tag{3.5}$$

with $0 \leq u \leq 1$. The dot $(\dot{})$ denotes differentiation with respect to time. Equation

(3.1) can now be rewritten as

$$\int_0^T \frac{uP(w)}{w0} L dt = 1. \tag{3.6}$$

When maximising r directly using Pontryagin's principle we need to define the Hamltonian H and from this we get the equations for the adjoint variables by differentiating with respect to the state variables. H is defined as

$$H = \lambda_0 \frac{u(t)L(t)P(w)}{w0} + \lambda_1(1 - u(t))P(w) - \lambda_2(r + \mu(w))L(t). \tag{3.7}$$

λ_3 does not appear in the hamiltonian as $\dot{r} = 0$. The adjoint equations are given by

$$\dot{\lambda_0} = 0, \tag{3.8}$$

$$\dot{\lambda_1} = \lambda_2 m' L - \frac{u\lambda_0 LP'}{w0} - (1 - u(t))\lambda_1 P', \quad \lambda_1(T) = 0 \tag{3.9}$$

$$\dot{\lambda_2} = \lambda_2 m(w) - \frac{u\lambda_0 P(w)}{w0}, \quad \lambda_2(T) = 0 \tag{3.10}$$

$$\dot{\lambda_3} = \lambda_2 L(t), \quad \lambda_3(0) = 0, \quad \lambda_3(T) = 1. \tag{3.11}$$

We note that $m(w) = r + \mu(w)$ and $m'(w) = \mu'(w)$ where (') denotes differentiation with respect to w.

3.3 Two phase solution

In a two phase solution the control u switches instantly from zero to one. The switching point can be found by setting $\frac{\partial H}{\partial u} = 0$, which in this case gives

$$\frac{\partial H}{\partial u} = \frac{\lambda_0 LP}{w0} - \lambda_1 P = 0. \tag{3.12}$$

The control u is dependent on the relationship between $\frac{\lambda_0 L}{w0}$, and λ_1 giving

$$u^* = 0 \quad if \quad \frac{\lambda_0 L}{w0} < \lambda_1 \tag{3.13}$$

$$u^* = 1 \quad if \quad \frac{\lambda_0 L}{w0} > \lambda_1. \tag{3.14}$$

(A boundary solution maximises the Hamiltonian.) This occurs due to the inherent properties of the problem, allowing all resources to be allocated to growth initially and all resources to reproduction at $t = T$. This gives the two phase strategy.

3.3.1 Finding a two phase solution

Taking a simple case where the functions $P(w)$, and $\mu(w)$ are both linear the two phase exact solution can be found. Working with the general case where $P(w) = Aw$, $\mu(w) = Bw$ and $w(0) = 0.25$ the equations necessary for solving the linear case are derived.

Since the solution is two phase, there is only one switching point, which will be denoted with a subscript of 1. During the initial phase $u = 0$ and so from equation (3.2) we have $\frac{dw}{dt} = P(w)$ and hence we can find t_1 in terms of w_1

$$t_1 = \int_{w(0)}^{w_1} \frac{1}{P(w)}\, dw. \tag{3.15}$$

Therefore

$$w_1 = 0.25e^{At_1} \tag{3.16}$$

and from equation (3.3) L can be found to be

$$L(t) = e^{-\int_{w(0)}^{w_1} \frac{r+\mu(w)}{P(W)}\, dw}.$$

Hence L_1, is given by

$$L_1 = e^{-\int_{w(0)}^{w_1} \frac{r+Bw}{Aw}\, dw}. \tag{3.17}$$

Now using equation (3.6) we get

$$1 = \int_{t_1}^{T} \frac{P(w_1)L(t)}{w0}\,dt = \frac{P(w_1)L_1}{w0}\int_{t_1}^{T} e^{-(r+\mu(w_1))(t-t_1)}\,dt =$$

$$\frac{P(w_1)L_1}{w0(r+\mu(w_1))}\left(1 - e^{-(r+\mu(w_1))(T-t_1)}\right). \qquad (3.18)$$

This equation gives a relationship between r and w_1. This can be shown graphically and the optimal strategy determined from this by observation. This reduces the optimal control problem to a simple static problem of finding w_1 to maximise r. A necessary condition for optimality is that

$$\frac{dr}{dw_1} = \frac{\partial f}{\partial w_1} / \frac{\partial f}{\partial r} = 0. \qquad (3.19)$$

Since we assume $\frac{\partial f}{\partial r}$ does not equal zero, we require $\frac{\partial f}{\partial w_1}$ to be equal to zero. Hence from equation (3.18) we obtain

$$m_1^{'} \frac{P(w_1)}{m_1}(T-t_1)e^{-m_1(T-t_1)} + \left(\frac{P(w_1)}{m_1}\right)^{'}\left(1 - e^{-m_1(T-t_1)}\right) - 1 = 0 \qquad (3.20)$$

where $m_1 = r + \mu(w_1)$ and $m_1^{'} = \mu^{'}$. We now have three equations (3.16, 3.18, 3.20) and three unknowns r, w_1 and t_1 from which the exact solution can be determined.

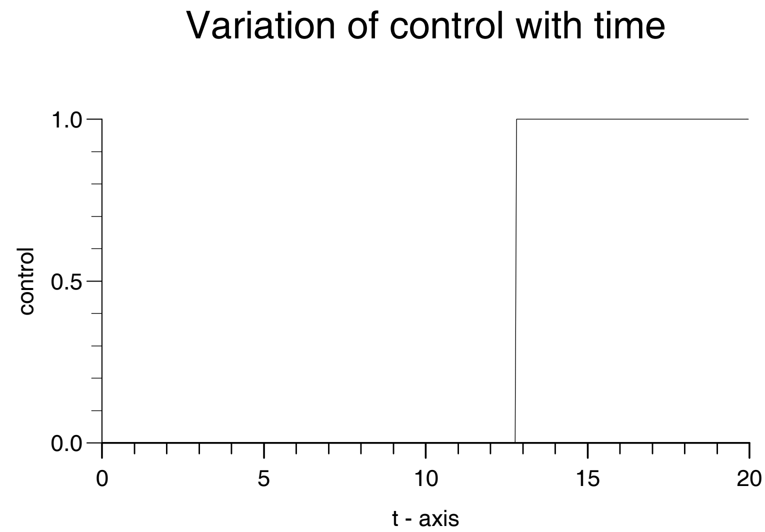


Figure 3.1: Two phase control

Example-linear

The functions are given as

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100$

$w0 = \frac{w(0)}{k}$

which on substitution into equations (3.16, 3.18, 3.20) give the following equations

$$w_1 = 0.25e^{0.0702t_1}, \tag{3.21}$$

$$w0 = \frac{0.0702w_1}{r + 0.01w_1}\left(e^{-\int_{w(0)}^{w_1}\frac{r+0.1w_1}{0.0702w_1}dw}\right)\left(1 - e^{-(r+0.01w_1)(100-t_1)}\right), \tag{3.22}$$

$$0.1\left(\frac{0.0702w_1}{r + 0.1w_1}\right)(100 - t_1)\,e^{-(r+0.01w_1)(100-t_1)} +$$

$$\left(\frac{P(w_1)}{r + 0.1w_1}\right)'\left(1 - e^{-(r+0.1w_1)(100-t_1)}\right) - 1 = 0. \tag{3.23}$$

One possible solution satisfying all the equations is given by $r = 0$, $w_1 = 2.29$ and $t_1 = 31.6$. This satisfies the necessary conditions for a stationary value of the functional.

3.4 Three phase solution

The three phase solution occurs when $\frac{\lambda_0 L}{w0} = \lambda_1$ for a non-zero length of time. It is this property which gives rise to the singular arc. The specification of the problem implies that u must be zero initially and one finally, so the singular arc must occur in the middle of the life history. From [6] we have that the necessary conditions for a singular arc are dependent on whether mortality and growth are both increasing or decreasing with age.

1. Case 1 (mortality and production increasing)

   - $\mu'(w_2) > 0$
   - $P'(w_2) > r + \mu(w_2)$
   - $\left(\frac{P(w_2)}{r+\mu(w_2)}\right)' < 1$
   - $\left(\frac{P'-m}{m'}\right)' < 1$ (Clebsch Legendre condition)

2. Case 2 (mortality and production decreasing)

   - $\mu'(w_2) < 0$
   - $\left(\frac{P(w_2)}{r+\mu(w_2)}\right)' > 1$
   - $\left(\frac{P'-(r+\mu)}{\mu'}\right)' > 1$ (Clebsch Legendre condition)

Note The dash ($'$) denotes differentiation with respect to w_2 which is the weight at the second switching point.

3.4.1 Finding a three phase solution

A general three phase solution is looked at, in the simple case where the functions $P(w)$ and $\mu(w)$ are any general linear function. The first and second switching points are denoted by the subscripts 1 and 2 respectively. Equation (3.16) relating w_1 and t_1 remains valid as does equation (3.17) relating w_1 and L_1. In the three phase optimal control problem, it is assumed that $\lambda_1 = \frac{\lambda_0 L}{w0}$ for a non- zero period of time. During this phase $\frac{\partial H}{\partial u} = 0$ and the trajectory is a singular arc. Since $\lambda_1 = \frac{\lambda_0 L}{w0}$ during the singular arc we can differentiate to get

$$\dot{\lambda_1} = \frac{\lambda_0 \dot{L}}{w0}. \tag{3.24}$$

Substituting equation (3.9) into this equation and using equation (3.3) as well as the relationship $\lambda_1 = \frac{\lambda_0 L}{w0}$ gives

$$\lambda_2 = \frac{\lambda_0 (P' - m(w))}{w0 m'}, \tag{3.25}$$

where $(')$ denotes differentiation with respect to w. λ_2 can now be found during the final phase from equation (3.10) giving

$$\lambda_2 = \left(1 - e^{-m(w_2)(T-t_2)}\right) \frac{\lambda_0 P'(w_2)}{w0 m(w_2)}, \tag{3.26}$$

which on substitution into equation (3.9) gives

$$\frac{\lambda_1 w0}{\lambda_0} = m_2' \frac{P(w_2)}{m_2} L(T)(T-t) + \left(\frac{P(w_2)}{m_2}\right)' [L(t) - L(T)]. \tag{3.27}$$

At the start of the final phase $\frac{\lambda_1 w0}{\lambda_0} = L(T) e^{m(w_2)(T-t_2)}$ which on substituting into equation (3.27) gives the criterion equation

$$m_2' \frac{P(w_2)}{m_2} (T-t_2) e^{-m_2(T-t_2)} + \left(\frac{P(w_2)}{m_2}\right)' \left(1 - e^{-m_2(T-t_2)}\right) - 1 = 0 \tag{3.28}$$

where $m_2 = r + \mu(w_2)$. The next step is to equate equations (3.25) and (3.26), the trajectories of λ_2 during the singular arc and final phase respectively at $t = t_2$. These give a new equation between T and t_2 as

$$T - t_2 = \frac{1}{P(w_2)} \frac{P'(w_2)}{m'(w_2)}. \tag{3.29}$$

Before the singular arc analogue of equation (3.18) can be defined it is necessary to define the optimal control u during the singular arc. This is defined as

$$u = \frac{\left(\frac{P'(w)-m(w)}{m'(w)}\right)' - \frac{m(w)}{P(w)} \frac{P'(w)-m(w)}{m'(w)}}{\left(\frac{P'(w)-m(w)}{m'(w)}\right)' - 1}, \tag{3.30}$$

which on substitution into equation (3.2) gives

$$\dot{w} = \frac{\left(\frac{P'(w)-m(w)}{m'(w)}\right) m(w) - P}{\left(\frac{P'(w)-m(w)}{m'(w)}\right)' - 1} = G(w, r). \tag{3.31}$$

Hence

$$\int_{t_1}^{t_2} dt = \int_{w_1}^{w_2} \frac{1}{G(w,r)} dw \tag{3.32}$$

Using equation (3.30) to specify u during the singular arc, w_2 can be found and having found w_2 an expression specifying L_2 can be found. From these it is possible to construct the singular arc analogue of equation (3.18), given by

$$1 = \int_{w_1}^{w_2} \frac{(P(w) - G(w,r))\, e^{-\int_{w(0)}^{w_1} \frac{r+\mu(w)}{P(w)} dw - \int_{w_1}^{w_2} \frac{r+\mu(w)}{G(w,r)} dw}}{w0G(w,r)} dw \; +$$

$$\frac{P(w_2) e^{-\int_{w(0)}^{w_1} \frac{r+\mu(w)}{P(w)} dw - \int_{w_1}^{w_2} \frac{r+\mu(w)}{G(w,r)} dw}}{w0(r + \mu(w_2))} \left(1 - e^{-(r+\mu(w_2))(T-t_2)}\right) \tag{3.33}$$

We now have five equations (3.16), (3.28), (3.29), (3.32) and (3.33) and five unknowns r, w_1 w_2, t_1 and t_2 from which the exact solution can be determined.

Variation of control three phase

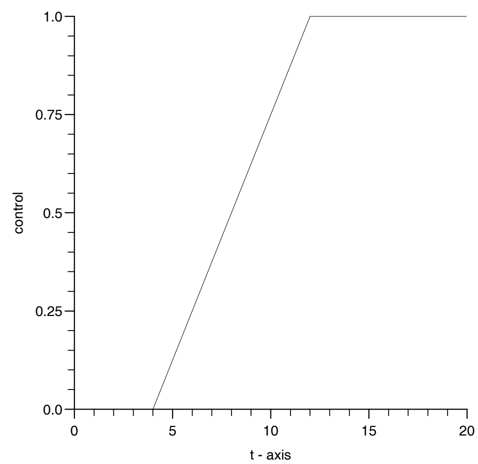


Figure 3.2: Three phase control

Example-linear

The functions are given as

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100$

$w0 = \frac{w(0)}{k}$

which on substitution into equations (3.16), (3.28), (3.29), (3.32), (3.33) give the following equations

$$w_1 = 0.25e^{0.0702t_1}, \tag{3.34}$$

$$0.1\left(\frac{0.0702w_1}{r+0.1w_1}\right)(100-t_1)\, e^{-(r+0.01w_1)(100-t_1)} +$$

$$\left(\frac{P(w_1)}{r+0.1w_1}\right)'\left(1-e^{-(r+0.1w_1)(100-t_1)}\right) - 1 = 0, \tag{3.35}$$

$$100 - t_2 = \frac{1}{0.0702w_2}\frac{0.0702}{0.01} \tag{3.36}$$

$$\int_{t_1}^{t_2} dt = \int_{w_1}^{w_2} \frac{1}{G(w,r)}dw \quad where \quad G(w,r) = 0.005w \tag{3.37}$$

$$1 = \int_{w_1}^{w_2} \frac{(0.0702w - 0.005w)\, e^{-\int_{w(0)}^{w_1} \frac{r+0.01w}{0.0702w}dw - \int_{w_1}^{w_2} \frac{r+0.01w}{0.005w}dw}}{w0\ 0.005w}dw \ +$$

$$\frac{0.0702we^{-\int_{w(0)}^{w_1} \frac{r+0.01w}{0.0702w}dw - \int_{w_1}^{w_2} \frac{r+0.01w}{G(w,r)}dw}}{w0(r+0.01w_2)}\left(1 - e^{-(r+0.01w_2)(100-t_2)}\right) \tag{3.38}$$

One possible solution, is given by $r = 0$, $w_1 = 1.81$, $w_2 = 2.58$, $t_1 = 28.20$ and $t_1 = 61.28$. This solution satisfies the necessary conditions for a stationary value of the functional.

Chapter 4

Development of Numerical Method

The development of the numerical method begins by deriving numerical schemes for solving the state equations for an arbitary control vector u. The adjoint equations are then solved for this arbitary control vector u using similar numerical methods. Since the equations for the adjoint variables are dependent on the state equations they are solved for the same step size to avoid having to use interpolation. The problem solution obtained at this point will not be an optimal solution. The projected gradient and conditional gradient methods are used to find the optimal solution.

4.1 Solving the state equations

The numerical scheme begins with finding solutions to the state equations, for a given control u. The equations for the state variables are solved first as the adjoint equations are dependent upon them; these equations are solved from $t = 0$

to a final time $t = T$.

We commence with the state equation for w, equation (3.2), which is independent. The remaining state equations are solved simultaneously. The function $P(w)$ given in equation (3.2) is assumed to be problem specific and user defined. A simple trapezium rule discretisation is applied to equation (3.2) for w, giving

$$w_{j+1} = w_j + \frac{h}{2}((1-u_j)P(w_j) + (1-u_{j+1})P(w_{j+1})), \tag{4.1}$$

where h is the step length ($h = T/n$ where T is the final time and n is the number of steps), and j denotes the number of steps taken from $t = 0$.

It can be seen that $P(w_{j+1})$ cannot be evaluated as w_{j+1} is unknown and so this form of simple discretisation cannot be used. Instead it is used to form an iterative method, such that

$$w_{j+1}^{n+1} = w_j + \frac{h}{2}(1-u_j)P(w_j)) + \frac{h}{2}(1-u_{j+1})P(w_{j+1}^n) \tag{4.2}$$

where w_0 is known and where $w_{j+1}^1 = w_j$ at each step.

The iteration process is stopped when $w_{j+1}^{n+1} - w_{j+1}^n \leq tol$ at each step. Having developed a numerical scheme to find w, we must now show that the method converges. It is assumed that P satisfies the Lipschitz condition.

$$\mid P(w) - P(y) \mid \leq A \mid w - y \mid \tag{4.3}$$

Then for convergence it is necessary that

$$\mid w_{j+1} - w_{j+1}^{n+1} \mid \leq K \mid w_{j+1} - w_{j+1}^n \mid \quad K < 1 \tag{4.4}$$

Using equation (4.2) and (4.4), we obtained

$$\mid w_{j+1} - w_{j+1}^{n+1} \mid = \mid \frac{h}{2}(1-u_{j+1}) \mid \mid P(w_{j+1}) - P(w_{j+1}^n) \mid, \tag{4.5}$$

which from equation (4.3) gives

$$| w_{j+1} - w_{j+1}^{n+1} | \leq | \frac{h}{2}(1 - u_{j+1}) | A \mid w_{j+1} - w_{j+1}^{n} \mid . \tag{4.6}$$

Hence for convergence

$$| \frac{h}{2}(1 - u_{j+1}) | A < 1 \tag{4.7}$$

which gives a bound on the size of h as $h \leq \frac{2}{A}$ where A is the Lipshitz constant. The Lipshitz constant A is dependent on the function $P(w)$. If $P(w)$ is a linear function such that $P(w) = \alpha w$, then

$$\begin{aligned} | P(w) - P(y) | &= | \alpha w - \alpha y | \\ &= | \alpha | \, | w - y | \end{aligned} \tag{4.8}$$

which gives the Lipshitz constant A as α. If $P(w)$ is a non-linear function then the Mean value theorem is used to determine A.

$$\begin{aligned} | P(w) - P(y) | &\leq | P^{'}(\zeta)(w - y) | \\ &\leq Max \mid P^{'}(\zeta) \mid \mid w - y \mid, \end{aligned} \tag{4.9}$$

giving a bound on the Lipshitz constant A as A$\leq Max \mid P^{'}(\zeta) \mid$.

We now continue by developing a numerical scheme to solve the remaining state equations, equations (3.3), (3.4) and (3.5) simultaneously. The equations are again discretised using the Trapezium rule such that

$$L_{j+1} = \frac{1 - \frac{h}{2}(r^n + \mu(w_j))}{1 + \frac{h}{2}(r^n + \mu(w_{j+1}))} L_j, \quad L_0 = 1 \tag{4.10}$$

$$\theta_{j+1} = \theta_j + \frac{h}{2w0}(u_j P(w_j) L_j + u_{j+1} P(w_{j+1}) L_{j+1}), \quad \theta_0 = 0. \tag{4.11}$$

Since r is an unknown constant, equations (4.9) and (4.10) can not be solved without a value for r being specified. This is done by starting a sequence of iterations for the value of r with two initial guesses and evaluating L and θ respectively for each guess. If the end condition $\theta(T) = 1$ is not satisfied, a Secant method is used to update the value of r, which is then used to recalculate the value of L and θ until the end condition on θ is satisfied. r is updated by

$$r^{n+1} = r^n - \frac{g(r^n)(r^n - r^{n-1})}{g(r^n) - g(r^{n-1})}, \tag{4.12}$$

stopping only when $r^{n+1} - r^n$ is small enough. The function $g(r^n)$ is defined to force the end condition $\theta(T) = 1$; hence

$$g(r^n) = \theta(T) - 1. \tag{4.13}$$

The solutions found for w, L, θ and r are now used in determining the numerical solutions of the adjoint equations. Due to the interdependence of the equations, the equations for λ_0, λ_2 and λ_3 are solved simultaneously.

4.2 Solving the adjoint equations

The adjoint equations are equations (3.8), (3.9), (3.10) and (3.11). To begin solving the adjoint equations f λ_0, λ_2 and λ_3 the end conditions are used.

The numerical solution is found using the same technique as that used for finding r, L and θ and so only the outline is given here. Two initial estimates of the constant λ_0 are obtained and equation (3.10) for λ_2 and equation (3.11) for λ_3 are solved for these values, with only the conditions at $t = T$ being used. The end condition on λ_3 for $t = 0$ is used in the secant update to give

$$\lambda_0^{n+1} = \lambda_0^n - \frac{\lambda_3^n(0)(\lambda_0^n - \lambda_0^{n-1})}{\lambda_3^n(0) - \lambda_3^{n-1}(0)} \tag{4.14}$$

Equations (3.10) and (3.11) for λ_2 and λ_3 are again solved by using a trapezium discretisation, working backwards in time from the final time $t = T$, so that

$$\lambda_{2_j} = \frac{(1 - \frac{h}{2}(r + \mu(w_{j+1})))\lambda_{2_{j+1}} + \frac{h\lambda_0}{2w0}(u_j P(w_j) + u_{j+1} P(w_{j+1}))}{(1 + \frac{h}{2}(r + \mu(w_j))} \tag{4.15}$$

$$\lambda_{3_j} = \frac{-h}{2}(\lambda_{2_j} L_j + \lambda_{2_{j+1}} L_{j+1}) + \lambda_{3_{j+1}}. \tag{4.16}$$

It is now possible to solve equation (3.9) for λ_1; this is again done with a trapezium discretisation, stepping back in time from $t = T$ to $t = 0$. This gives

$$\lambda_{1_j} = \frac{(1 + \frac{h}{2}(1 - u_{j+1})P'(w_{j+1}))\lambda_{1_{j+1}}}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))}$$

$$-\frac{h}{2}\frac{(\lambda_{2_j}\mu'(w_j)L_j + \lambda_{2_{j+1}}\mu'(w_{j+1})L_{j+1})}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))}$$

$$+\frac{h\lambda_0}{2w0}\frac{(u_j L_j P'(w_j) + u_{j+1} L_{j+1} P'(w_{j+1})}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))} \tag{4.17}$$

from which λ_1 can be found.

4.3 Ways of finding optimal solutions

A numerical method for finding the optimal control u, and hence the optimal solutions to equations (3.2-3.5) and (3.8-3.11) is needed, assuming such an optimal exists. To optimise the value of the control vector u, starting from an arbitrary control vector, two methods have been tried, the projected gradient method, and the conditional gradient method. The basic algorithms were taken from [1] and adapted as necessary to suit this problem.

4.3.1 Projected gradient method

A basic outline to this method is shown in the flowchart in Fig 4.1 The new approximation to the optimal control u^* is chosen such that

$$u_{new} = u_{old} + step\frac{\partial H}{\partial u_{old}}, \tag{4.18}$$

where $step$ is the step size, and $\frac{\partial H}{\partial u_old}$ is specified by equation (3.12). If u_{new} is greater than one then it is set to the maximum value of one; similarly if u_{new} is less than zero then it is set to zero. If the value of r (which we wish to maximise) has not increased, then the step size is halved and the process of finding a new control u is repeated until r has increased in value. The adjoint variables are then evaluated for the new control, as is $\frac{\partial H}{\partial u_{new}}$, which is used to determine if the solution has converged to the optimal control u. A new variable $\tilde{u}$ is evaluated such that $\tilde{u}$ equals zero if $\frac{\partial H}{\partial u_{new}}$ is less than zero and one otherwise, where $\frac{\partial H}{\partial u_{new}}$is evaluated using equation (3.12). Convergence is then measured by the inner product

$$< \frac{\partial H}{\partial u}, (\tilde{u} - u) >, \tag{4.19}$$

which is determined numerically as

$$h\sum_{k=1}^{n} \left.\frac{\partial H}{\partial u}\right|_{u_k} (\tilde{u_k} - u_k). \tag{4.20}$$

This maximises the first variationof the functional over all possible choices of u. If this is sufficiently small then the process is said to have converged and the optimal control u has been determined; otherwise the step size is set to one again and the whole process repeated until the optimal value of u is determined.

4.3.2 Conditional gradient method

Again the basic outline to the method is shown using a flowchart, see Fig 4.2. The idea is to generate a sequence of possible control vectors u, for which the values of the functional r (which we wish to maximise) is non-decreasing. u is an approximation to the optimal control u^*, with solutions r, w, l, and θ to the state equations and solutions λ_0, λ_1, λ_2 and λ_3 to the adjoint equations. Then new approximation to u is made as

$$u_{new} = (1 - s)u_{old} + s\tilde{u}, \tag{4.21}$$

where s = step size and $\tilde{u}$ is obtained as follows

$$\tilde{u} = 1 \quad if \quad \frac{\partial H}{\partial u_{new}} > 0$$

$$\tilde{u} = 0 \quad otherwise,$$

where $\frac{\partial H}{\partial u_{new}}$ is the functional gradient specified by equation (3.12). This selection maximises the first variation $< \frac{\partial H}{\partial u_{new}}, \tilde{u} - u_{new} >$ of the functional over all possible choices of $\tilde{u}$ and is used to determine when the optimal has been found. This is the same convergence criterion as used for the Projected gradient method and is given by equation (4.20).

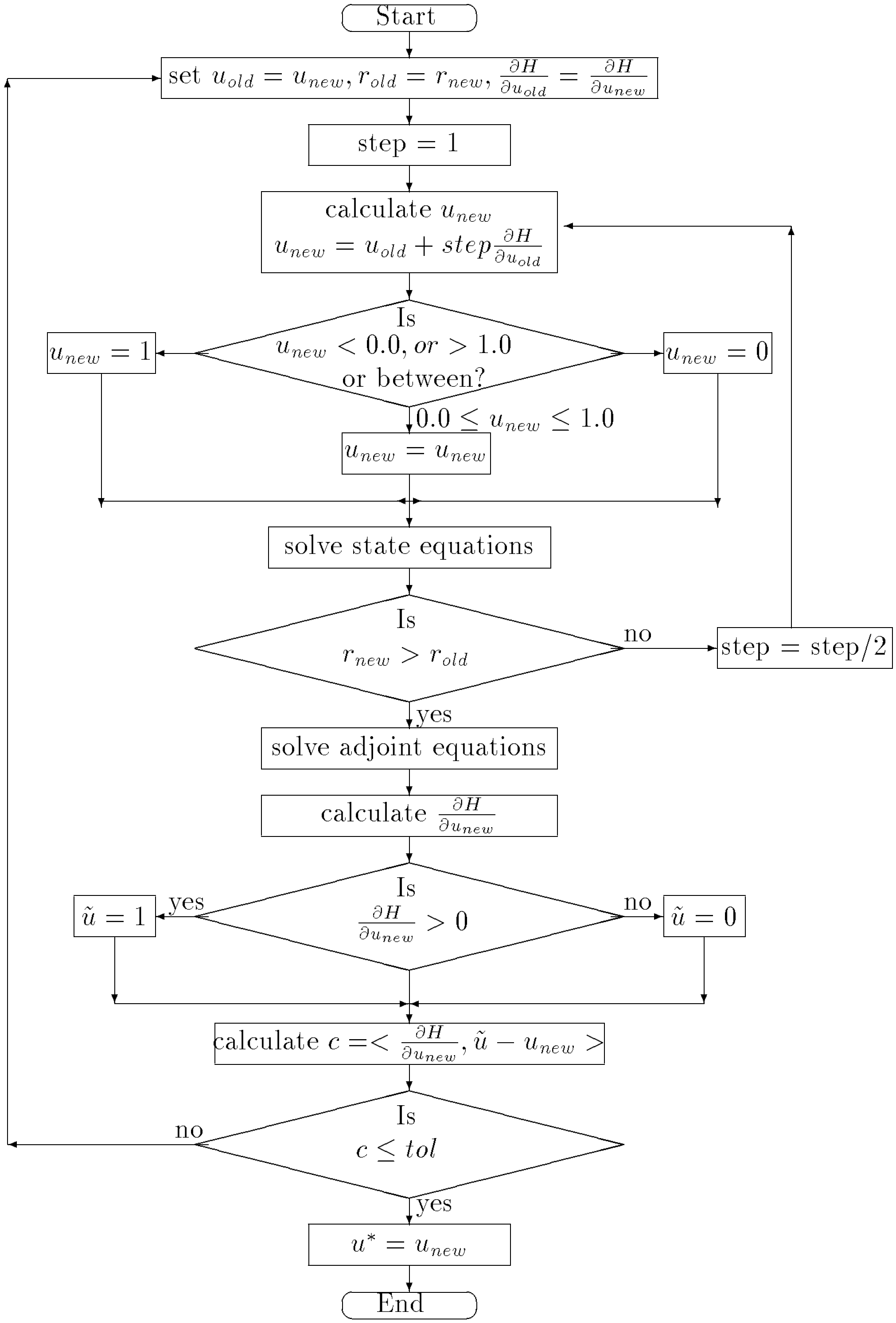


Figure 4.1: Projected Gradient Algorithim

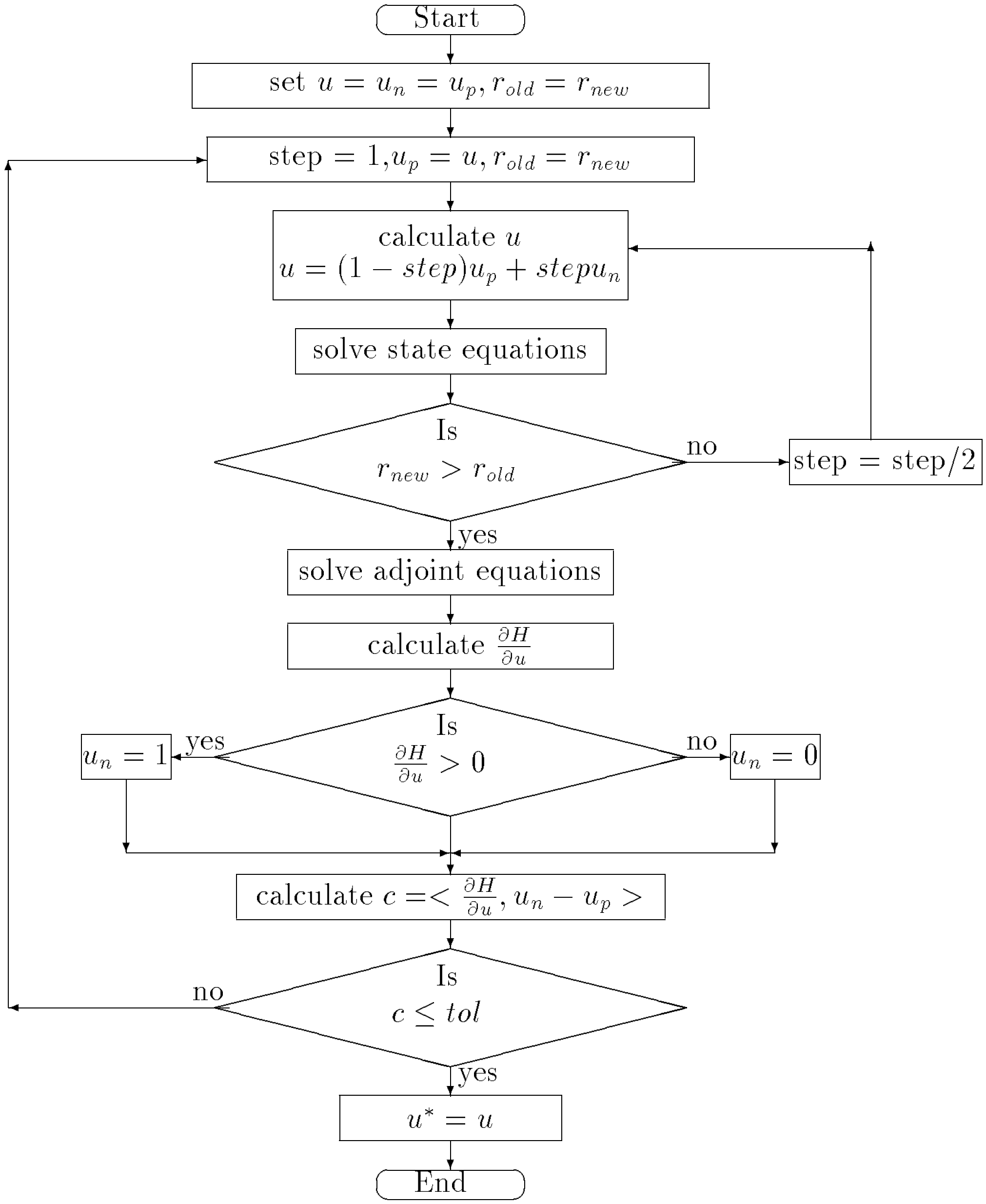


Figure 4.2: Conditional Gradient Algorithim

Chapter 5

Numerical Results

In this chapter the results from a series of test problems are presented. Due to the difficulty in calculating analytical solutions even in the case of linear functions for $P(w)$ and $\mu(w)$ the analytical solutions have only been fully calculated for a linear test problem. Tests are made which look at the sensitivity of the method to the initial approximation of r, the effect on the solution of the initial arbitrary control u and the effect of changing the length of the step taken in the gradient direction on the rate of convergence and accuracy of the solution for both optimisation methods. The efficiency of the two optimisation methods is compared by looking at the total number of inner iterations (number of times r is calculated) and the number of outer iterations (number of times the new control is recalculated if the convergence criterion is not satisfied).

5.1 The Effects of Initial Data

Consideration is given firstly to what initial data the method requires and then to it's possible effect on the methods numerical solutions. The initial data required are

i) $w(0)$, weight of w at $t = 0$

ii) $t = T$ final time

iii) N The number of steps ($h = T/N$)

iv) Two initial approximations to the value of r

v) Two initial approximations to the value of λ_0

vi) The switching point (zero to one) for the arbitrary initial control

which is not dependent on the optimisation method used. Points iv and vi are of primary interest.

5.2 Testing Optimality

The numerical method developed solves eight first order differential equations which are related in such away that seven of the eight differential equations are effected either directly or indirectly by the value of r. This makes the method sensitive to the value of r to the extent that if the initial approximation is not close enough to r the method provides a false solution. This can be detected by using the condition $\lambda_0 = \lambda_2(0)$ obtained by equating λ_0 times equation (3.6) and rearranging and integrating L times equation (3.10) giving

$$\lambda_0 = \lambda_0 \int_0^T \frac{uP(w)L}{w0} dt = \int_0^T \lambda_2 mL - \lambda_2' L \; dt. \tag{5.1}$$

Integrating by parts gives

$$\lambda_0 = \int_0^T \lambda_2 mL + \lambda_2 L' dt - \lambda_2 L|_0^T . \tag{5.2}$$

Using equation(3.3) gives

$$\lambda_0 = \int_0^T \lambda_2 mL - \lambda_2 mLdt + \lambda_2(0). \tag{5.3}$$

Hence $\lambda_0 = \lambda_2(0)$. This condition is only satisfied when the solution is an optimal solution. Since r is calculated using the Secant method the initial approximations must be easonably close to r. This will hopefully not present a problem when dealing with real data.

5.3 Convergence

In this section we look at the convergence criteria used throughout the numerical method and the number of steps N. The convergence criteria used in evaluating the state equations begins by looking at that used in the evaluation of w. The convergence of w is pointwise (when two succesive approximations to w_j are within the given tolerance then the solution at the point w_j is accepted.) The tolerance used is 10^{-6}. The convergence criteria for the remaining state equations r, L, and θ is dependant on the convergence of the secant iteration for r this uses a tolerance of 10^{-6} again. The adjoint equations for λ_0, λ_2 and λ_3 depend on the convergence of the secant iteration for λ_0 which again has a tolerance of 10^{-6}. The solution of λ_1, a backward time steping scheme using a trapezium rule discretisation, convergences with order (h^2).

The control u calculated from both optimisation methods uses a convergence criteria, dependent on the inner product specified by equation (4.19). The magnitude of this inner product dictates the convergence of the control u. The method requires the inner product to be less than 0.005 for convergence.

Table 5.1 shows the variation in w_1 at the first switching point and the variation in w_2 at the second switching point, as the number of steps N is increased. The solutions can be seen to converge to two decimal places when $N = 200$.

N	w_1	w_2	t_1	t_2
500	2.0395	2.0413	30.0	74.6000
1000	2.0466	2.0484	30.0	74.4000
1500	2.0490	2.0508	30.0	74.2667
2000	2.0502	2.0520	30.0	74.2500
2500	2.0509	2.0527	30.0	74.2800
3000	2.0514	2.0532	30.0	74.2667

Table 5.1: The effect of N on convergence of the Projected gradient method

5.4 Moving the Initial Switching Point

The switching point is the time t_s at which the arbitrary initial control u switches from zero to one. Hence

$$u = 0 \quad if \quad t < t_s$$
$$u = 1 \quad if \quad t > t_s. \tag{5.4}$$

The effect of the position of t_s on the solutions found using both the Projected and Conditional gradient methods is considered by looking at the changes in r and λ_0 as well as the number of iterations necessary for the methods to converge.

The problem used for this evaluation is the linear test problem specified by the functions $P(w)$ and $\mu(w)$, the initial value $w(0)$, a constant k and the final time $t = T$, given by

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100.$

The analytical solution has already been calculated (chapter 3) for this problem with both two and three phase solutions being found.

Projected Gradient Method

u switch point	no of iterations	r	$\lambda_0 = \lambda_2(0)$
t= 5	266	-0.0009	yes 0.0163
t=10	263	-0.0007	yes 0.0166
t=15	254	-0.0005	yes 0.0169
t=20	224	-0.0004	yes 0.0172
t=25	145	-0.0002	yes 0.0175
t=30	1	0.0	yes 0.0176
t=35	19	-0.0002	yes 0.0173
t=40	104	-0.0003	yes 0.0170

Table 5.2: The effect of moving the switching point for the Projected gradient method, $N = 2000$

The results in table 5.2 show that moving the switching point (t_s) has a quite dramatic effect on the number of iterations required for the method to converge. In all cases where a three phase solution is optimal, the projected gradient method will find this solution and has done so for this example. From chapter three we know that the first switching point is at $t = 28.20$ for this problem, and so the rapid convergence of the method when $t_s = 30$ is only to be expected. The changes in the values of r and λ_0 are less marked and indicate the method converging to a stable solution at $r = 0$. The solutions are identical to two decimal places which is all that can be expected when $N = 2000$. This variation in the results is effected by the tolerance used for calculating the control variable although making this stricter to reduce the variation in the solution is not viable due to the resulting large increase in the number of iterations.

Conditional Gradient Method

u switch point	no iterations	r	$\lambda_0 = \lambda_2(0)$
t=30	1	-0.0001	yes 0.0176
t=32.5	1	0.0000	yes 0.0174

Table 5.3: The effect of moving the switching point for the Conditional gradient method $N = 2000$

The conditional gradient method had considerable difficulty in solving this problem, with solutions only being achieved for the two switching times in table 5.3. The solutions generated were two-phase 'bang-bang' type solutions. From the analytical solution in chapter three we know that the actual switching point for the two phase solution is $t_s = 31.60$, so the method's rapid convergence at the above switching times, which are very close to the exact solution, is not surprising. It is however disappointing that the method fails to produce a solution if the switching point is moved elsewhere. The reason for this needs further investigation to determine it's cause, which would appear to be either the method's inability to cope with problems that have a possible three phase solution containing a singular arc, which would render the method almost useless, or the magnitude of T causing some type of error propogation..

5.5 Changing the Step Length in the Gradient Direction

The step length in the gradient direction directly concerns the optimisation methods. The aim is to try and improve the rate of convergence without loosing accuracy. The projected gradient method only was considered in detail. When using the original step length of one the method appeared to be unusable, requiring over eight hundred iterations to converge to a solution when the switching point t_s was not very close to the exact solution. This caused errors to effect the solution resulting in a sub-optimal solution being found. Increases in the step length reduced the number of iterations required for convergence and the results are shown in table 5.4 for a variety os switching times as it would be most unusual for the initial guess to be almost identical to the exact solution.

It appears from the results in table 5.4 that very large increases in step length are possible without the solution being effected, while reducing the number of iterations required for convergence. However caution must be advised, as the step length possible is dependent on the size of the functional gradient, which for this example is very small. A step length of four would, however, be a good starting point for most problems. The development of a scaling method to determine the length of the step size based on the magnitude of the functional gradient would be a useful addition to the numerical method and is an area of future work.

u switch point	no iterations	step length	r	$\lambda_0 = \lambda_2(0)$
t=10	263	4	-0.0007	yes 0.0166
t=20	224	4	-0.0004	yes 0.0172
t=30	1	4	0.0	yes 0.0176
t=40	104	4	-0.0003	yes 0.0170
t=10	131	8	-0.0007	yes 0.0166
t=20	112	8	-0.0004	yes 0.0172
t=30	1	8	0.0	yes 0.0176
t=40	51	8	-0.0003	yes 0.0170
t=10	65	16	-0.0007	yes 0.0166
t=20	55	16	-0.0004	yes 0.0172
t=30	1	16	0.0	yes 0.0176
t=40	25	16	-0.0003	yes 0.0170
t=10	32	32	-0.0007	yes 0.0166
t=20	27	32	-0.0004	yes 0.0172
t=30	1	32	0.0	yes 0.0176
t=40	12	32	-0.0003	yes 0.0170

Table 5.4: Projected Gradient method variable initial u and step length in the gradient direction

Chapter 6

Conclusions

This project set out to develop a numerical method that was capable of solving the problem of finding optimal life cycle strategies for a given organism using the model developed in [6]. The numerical method developed here provides a solution to this problem.

The basic numerical method for solving differential equations (3.2-3.5) and (3.8-3.11) is simple in concept and design and could easily be extended to deal with further constraints in a more complex model. The two optimisation methods considered, the projected gradient and conditional gradient, have individual problems which need further consideration. The projected gradient method is very slow to converge if the arbitrary initial control has it's switching point away from the exact switching point. The conditional gradient method does not cope very well with problems whose solution contains a singular arc, either failing to converge or finding an alternative two phase solution if the switching point is chosen to be in almost the exact position.

The linear problem looked at in chapter five converges to a slightly different solution for each switching point specified using the projected gradient method with a maximum solution obtaining the value of $r = 0$ when the switching point is chosen to be almost the exact solution. There are a number of possible reasons why this may happen, it could be an inherent property of the problem caused by the exact solution containing an almost flat basin around the minimum causing the numerical method to converge to a series of sub-optimal solutions that are very close to the optimal as can be observed in table 5.2 and table 5.4. The most likely alternative is that the errors are a result of the convergence criteria not being strict enough on the outer iteration loop. The reason for not making these stricter lies in the number of iterations necessary for convergence being excessive if the tolerance used is less than 0.005.

It would be interesting to test the method with some real data and compare the strategies predicted by the numerical method and the observed strategies used by the organism. This would not only provide a means of validating the numerical method fully, but would also validate thoroghly the model developed in [6].

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