THE UNIVERSITY OF READING

Department of Mathematics and Statistics

Theory and Examples of Generalised Prime Systems

Faez Ali AL-Maamori

Thesis submitted for the degree of

Doctor of Philosophy

March 2013

Abstract

A generalised prime system $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, ...$ satisfying $1 < p_1 \leq p_2 \leq ... \leq p_n \leq ...$ and for which $p_n \longrightarrow \infty$ as $n \longrightarrow \infty$. The $\{p_n\}$ called generalised primes (or Beurling primes) with the products $p_1^{a_1}.p_2^{a_2}....p_k^{a_k}$ (where $k \in \mathbb{N}$ and $a_1, a_1, ..., a_k \in \mathbb{N} \cup \{0\}$) forming the generalised integers (or Beurling integers).

In this thesis we study the generalised (or Beurling) prime systems and we examine the behaviour of the generalised prime and integer counting functions $\pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$ and their relation to each other, including the Beurling zeta function $\zeta_{\mathcal{P}}(s)$.

Specifically, we study a problem discussed by Diamond (see [7]) which is to determine the best possible β in $\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-c(\log x)^{\beta}})$, for some $\rho > 0$, given that $\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^{\alpha}})$, $\alpha \in (0, 1)$. We obtain the result that $\beta \leq \alpha$.

We study the connection between the asymptotic behaviour (as $x \to \infty$) of the g-integer counting function $\mathcal{N}_{\mathcal{P}}(x)$ (or rather of $\mathcal{N}_{\mathcal{P}}(x) - ax$) and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$). We show in the first section how assumptions on the growth of $\zeta_{\mathcal{P}}(s)$ imply estimates on the error term of $\mathcal{N}_{\mathcal{P}}(x)$, while in the second half we find the region where $\zeta_{\mathcal{P}}(\sigma + it) = O(t^c)$, for some $c > 0$, if we assume that we have a bound for the error term of $\mathcal{N}_{\mathcal{P}}(x)$.

Finally we apply these results to find O and Ω results for a specific example.

Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Faez Ali AL-Maamori

Acknowledgements

My first thanks go to my supervisor, Dr. Titus Hilberdink for all the guidance and support he has provided me over the course of this PhD. I am grateful for all his efforts and for helping me to constantly challenge and improve myself. On a more personal level, I would like to thank my wife Manal Kadhim. She has been tireless in her support and kindness throughout my research and the preparation of this thesis. Many thanks to the Iraqi government for the funding I have received. Without this funding this research would not have been possible. A special thank you goes to my friends Phillip Wilson and John Young for their support during my language course. I would also like to thank all members of the Department of Mathematics and Statistic for their lovely and friendly dealing. Finally, I would also like to thank my colleagues (Stephen Haben, Alexander Moodey, Tamsin Lee, Sanita Trancinha, Polly Smith, Simon Gaulter, Nick Bird, Dale Partridge, Ross Bullivant, Niall Arthurs, Samuel Groth, Charlotta Howarth, Natalie Lowery, Adam El-Said and David Gilbert. They have all been great mates and as long as I have not been alone at the office, we have always had interesting discussions. Therefore I am grateful for the time we shared. I have truly enjoyed the friendly atmosphere among the PhD colleagues at the department and I will miss you all.

Faez Ali AL-Maamori

Contents

Chapter 1

Introduction

In the late nineteenth century, Weber (see [30]) defined $N(x)$ to be a number of the integral ideals in a fixed algebraic number field $\mathcal{F}$ with the norm not exceeding x and proved that $N(x) = ax + O(x^\theta)$, as $x \to \infty$ for some $a > 0$ and $\theta < 1$. Early in the twentieth century, Landau (see [22]) used Weber's result and the multiplicative structure to prove the Prime Ideal Theorem, which asserts that the number of the distinct prime ideals of the ring of integers in an algebraic number field $\mathcal{F}$ with the norm not exceeding x is asymptotic to $\frac{x}{\log x}$, as x tends to infinity. His result showed that the only 'additive' result needed was Weber's.

Developing Landau's idea, Arne Beurling in 1930s introduced generalised (or Beurling) primes. In his definition, from any sequence of reals $\mathcal{P} = \{p_1, p_2, p_3 ...\}$ satisfying

$$1 < p_1 \leq p_2 \leq ... \leq p_n \leq ..., \text{ and } p_n \to \infty \text{ as } n \to \infty$$

called 'generalised primes', can be formed the sequence of generalised (or Beurling) integers $\mathcal{N}$ formed by the products of the form $\prod_{i=1}^{k} p_i^{a_i}$, where $k \in \mathbb{N}$ and $a_i \in \mathbb{N} \cup \{0\}$.

In this sense, Beurling generalises the notion of prime numbers and the natural numbers obtained from them. The generalised primes need not be actual primes, nor even integers and the generalised integers need not to be a uniquely factorisable. Therefore, $\mathcal{P}$ and $\mathcal{N}$ are not sets in general, but multisets where elements can occur with a certain multiplicity. Beurling defined $\pi_{\mathcal{P}}(x)$ to be the counting function of g-primes less than or equal to x and $\mathcal{N}_{\mathcal{P}}(x)$ to be the counting function of g-integers less than or equal to x (counting multiplicities). Beurling was interested to see under which conditions on $\mathcal{N}$ and the multiplicative structure, a Prime Number Theorem holds.

In 1937, Beurling proved (see [6]) that if $\mathcal{N}_{\mathcal{P}}(x) = ax + O\big(\frac{x}{(\log x)^{\gamma}}\big)$ for some $a > 0$ and γ strictly greater than $\frac{3}{2}$, then $\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$. This is called the *Beurling Prime Number Theorem.*

Such systems along with the generalised zeta function $\zeta_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}_{\mathcal{P}}} n^{-s}$ have been studied by numerous authors since then (see in particular the monograph by Bateman and Diamond [5], and papers by Diamond [9], [7], [10], [8], Hall [13], Malliavin [24] and Nyman [25] and more recently Kahane [20], Lagarias [21]). It continues to be an active subject to this day.

Much of the research on this subject has been about connecting the asymptotic behaviour of the g-prime and g-integer counting functions,

$$\pi_{\mathcal{P}}(x) = \sum_{p\leq x, p\in\mathcal{P}} 1, \quad \mathcal{N}_{\mathcal{P}}(x) = \sum_{n\leq x, n\in\mathcal{N}} 1.$$

Although Beurling answered the question of when the Prime Number Theorem holds, there are many more questions regarding these systems. For example, one can look to the corresponding errors ($\pi_{\mathcal{P}}(x) - \mathrm{li}(x)$ and $\mathcal{N}_{\mathcal{P}}(x) - ax$) and how they relate to each other.

Outline of thesis

The main aim of this thesis is to advance some new techniques and to give suitable examples to highlight interactions between $\pi_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{P}}$ and $\zeta_{\mathcal{P}}$. Specifically, we study a problem discussed by Diamond in Theorem 3.3b (see [7]). He proved that

$$\pi_{\mathcal{P}}(x) - \mathrm{li}(x) = O(xe^{-(\log x)^{\alpha}}) \text{ for some } \quad \alpha \in (0, 1), \quad (1)$$

implies

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-c(\log x)^{\beta}}) \text{ for some } \quad \rho, \beta > 0. \quad (2)$$

The problem is to determine the best possible (i.e. largest possible) β, given α.

Furthermore, we investigate the connection between the size of the Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$) and the error term of $\mathcal{N}_{\mathcal{P}}(x)$. As part of this investigation, if we assume that $\zeta_{\mathcal{P}}(s)$ has polynomial growth in a region near $\sigma = 1$, what can be said about the behavior of $\mathcal{N}_{\mathcal{P}}(x)$ (as $x \to \infty$) and vice versa?

Now we give a brief outline of thesis structure.

In Chapter 2 we introduce in the first part relevant concepts and known results which we require in Chapters 3-6 such as the Riemann-Stieltjes integral and Riemann-Stieltjes convolution. In the second part we present known lower and upper bounds for the Riemann-zeta function $\zeta(s)$ in the strip $0 \leq \Re s \leq 1$. Additionally, we mention upper bounds for $\zeta(s)$ which are conditional on the truth of the unproved Riemann hypothesis.

In Chapter 3 we give background to Beurling (or generalised) prime systems and the associated Beurling zeta function and put the theory in its historical context. First, we introduce discrete g-prime systems with some examples, while in the second section we will define the 'continuous' version of g-prime systems with some examples. Here $\pi_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{P}}$ are still increasing functions but need not be step functions. Now $\zeta_{\mathcal{P}}$ is defined as the *Mellin transform* of $\mathcal{N}_{\mathcal{P}}$. That is,

$$\zeta_{\mathcal{P}}(s) = \int_{1-}^{\infty} x^{-s} d\mathcal{N}_{\mathcal{P}}(x).$$

In the last part, we list relevant known results which are necessary for our work.

In Chapter 4 we introduce Diamond's problem (as mentioned above). Diamond in 1970, proved that given (1), (2) holds with $\beta = \frac{\alpha}{1+\alpha}$ (see [8]). In 1998, Balanzario [3] proved (by giving a concrete continuous example) that there exists a continuous g-prime system for which (1) holds and

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + \Omega_{\pm}(x e^{-c(\log x)^{\beta}}), \qquad (3)$$

holds for some positive constants ρ and c with $\alpha = \beta = \frac{1}{2}$. Thus optimal β lies between $\frac{1}{3}$ and $\frac{1}{2}$ (in case $\alpha = \frac{1}{2}$).

In the first section of this chapter, we generalise Balanzario's result by adapting his method to show that for any $0 < \alpha < 1$ there is a continuous g-prime system for which (1) and (3) hold with $\beta = \alpha$. Thus we cannot (in general) make $\beta > \alpha$.

In the second half of this Chapter we use the method developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a *discrete* system of Beurling primes satisfying (1) and (3) which is similar for the continuous system as in first section.

Finding discrete examples is typically more challenging since one cannot control the various growth rates (of $\pi_{\mathcal{P}}(x)$, $\mathcal{N}_{\mathcal{P}}(x)$ and $\zeta_{\mathcal{P}}(s)$) so easily.

In Chapter 5, we study the connection between the asymptotic behaviour (as $x \to \infty$) of the g-integer counting function $\mathcal{N}_{\mathcal{P}}(x)$ (or rather of $\mathcal{N}_{\mathcal{P}}(x) - ax$) and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$). Using just analysis, we show in the first section how assumptions on the growth of $\zeta_{\mathcal{P}}(s)$ imply estimates on the error term of $\mathcal{N}_{\mathcal{P}}(x)$. In the second half we find the region where $\zeta_{\mathcal{P}}(\sigma + it) = O(t^c)$, for some $c > 0$, if we assume that we have a bound for the error term of $\mathcal{N}_{\mathcal{P}}(x)$. This implication is more challenging if $\mathcal{N}_{\mathcal{P}}(x) - ax$ is $\Omega(x^{1-\epsilon})$ $\forall \epsilon > 0$, since we do not automatically have the analytic continuation of $\zeta_{\mathcal{P}}(s)$ for $\Re s < 1$, and perhaps constitutes the deepest result of this thesis.

In Chapter 6, we study a particular example to which the Theorems of Chapter 5 can be applied. The example gives very precise knowledge about the asymptotic behaviour of Beurling counting function of primes, $\psi_{\mathcal{P}}(x)$. It was initially hoped that this could provide a useful example for Diamond's problem in the case $\alpha = 1$. In this example the Beurling zeta function $\zeta_{\mathcal{P}}(s)$ is directly connected to the Riemann zeta function. This gives improved lower and upper bounds to the error term of $\mathcal{N}_{\mathcal{P}}(x)$ as well as conditional upper bound with the truth of the unproved Riemann Hypothesis.

We finish this Chapter by showing that with this example we have a g-prime system.

Chapter 2

Preliminary concepts

In this chapter we will give details of some relevant concepts and known results which we shall need in Chapters 3-6. In particular, for the definitions of generalised prime systems (especially the continuous version) we need the Riemann-Stieltjes integral and Riemann-Stieltjes convolution.

In the second half of this chapter we summarize some (relevant) results about the Riemann-Zeta function. In particular, we will give a brief survey of some of the known lower bounds for the Riemann-Zeta function in the critical strip $0 < \sigma < 1$. We consider also the upper bounds for the Riemann-Zeta function which are unconditional bounds in that strip and those which are conditional on the unproved Riemann Hypothesis.

We begin with the Riemann-Stieltjes integral.

2.1 Riemann-Stieltjes integral

Let f and α be bounded (real or complex) functions on $[a, b]$. Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ and let $t_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \cdots, n$. We define a Riemann-Stieltjes sum of f with respect to α as

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\ \big(\alpha(x_k) - \alpha(x_{k-1})\big).$$

Definition 1. *A function f is Riemann Integrable with respect to α on $[a, b]$, if there exists $r \in \mathbb{R}$ having the following property: For every $\epsilon > 0$, there exists a* *partition P_ϵ of $[a,b]$ such that for every partition P finer than P_ϵ and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P,f,\alpha) - r| < \epsilon$. As such, we say the Riemann-Stieltjes integral $\int_a^b f(x)\ d\alpha(x)$ exists and equals r.*

We need the following theorems in Chapter 3.

Theorem 2.1. *If $f \in R(\alpha)$ on $[a,b]$, that is, f is Riemann Integrable with respect to α on $[a,b]$, then $\alpha \in R(f)$ on $[a,b]$ and we have*

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x).$$

Theorem 2.2. *Assume $f \in R(\alpha)$ on $[a,b]$ and assume that α has a continuous derivative $\alpha^{'}$ on $[a,b]$. Then the Riemann Integral $\int_a^b f(x)\ \alpha^{'}(x)\ dx$ exists and we have*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha^{'}(x)\, dx.$$

Proof. See Theorem 7.6. and Theorem 7.8. in [1]. □

We shall need the notion of bounded variation.

Definition 2. *The function $\alpha : [a,b] \longrightarrow \mathbb{C}$ is said to be of bounded variation on $[a,b]$ if and only if there is a constant $M > 0$ such that*

$$\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \leq M,$$

for all partitions $P = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a,b]$. As such the total variation of α on $[a,b]$ is defined to be

$$V_\alpha(a,b) = \sup_{\mathcal{P}} \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})|\,,$$

where the supremum runs over the set $\mathcal{P}$ of all partitions of $[a,b]$.

The function $\alpha : [a,\infty) \longrightarrow \mathbb{C}$ is said to be *locally of bounded variation* if the variation of α on each compact subinterval $[b,c] \subset (a,\infty)$ is finite.

Now, we are able to introduce the Riemann-Stieltjes convolution. Let S denote the space of all functions $f : \mathbb{R} \longmapsto \mathbb{C}$ such that f is right-continuous and of local bounded variation with $f(x) = 0$, $\forall x \in (-\infty, 1)$. Let $S^+ \subseteq S$ such that for any $f \in S^+$, f is an increasing function. For $a \in \mathbb{R}$, let $S_a = \{f \in S : f(1) = a\}$ and $S_a^+ = S_a \cap S^+$.

Definition 3. *For any $f, g \in S$, we define the convolution (or Riemann-Stieltjes convolution) by*

$$(f * g)(x) = \int_{1-}^{x} f\left(\frac{x}{t}\right) dg(t).$$

We note that $(S, *)$ is a commutative semigroup and the identity (with respect to $*$) is $i(x) = 1$ for $x \geq 1$ and zero otherwise.

We require the following properties from the literature which are necessary for this work:

1. If f or g is continuous on $\mathbb{R}$, then $f * g$ is continuous.

2. For $f \in S_1$, there exists $g \in S_0$ such that $f = \exp_* g$. That is,

$$f = \sum_{n=0}^{\infty} \frac{g^{*n}}{n!},$$

   where $g^{*n} = g * g^{*(n-1)}$ and $g^{*0} = i$. The above series converges in S (see section 2.1. in [7]).

3. $f = \exp_* g$ if and only if $f * g_L = f_L$, where $f_L \in S$ defined for $x \geq 1$ by $f_L(x) = \int_1^x \log t \; df(t)$.

4. For $f, g \in S$ define the *Mellin transform* of f by

$$\hat{f}(s) = \int_{1-}^{\infty} x^{-s} df(x).$$

   We note that $\widehat{f * g} = \hat{f}\hat{g}$ and $\widehat{\exp_* f} = \exp \hat{f}$ whenever the transforms converge absolutely.

Further details of the above properties are in [15].

2.2 The Riemann zeta function

We will move our attention to the Riemann zeta function which we need for later chapters. In particular, we shall give a brief survey of some of the known results for the order of the Riemann Zeta function in the critical strip $0 < \sigma < 1$. We consider both unconditional results and those results conditional upon the Riemann hypothesis.

Definition 4. *The Riemann zeta function is defined for* $\Re s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The above series converges absolutely and locally uniformly in the half-plane $\Re s > 1$ and defines a holomorphic function here. Moreover, $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at 1 with residue 1 and is of finite order (i.e. $\zeta(\sigma + it) = O(t^A)$, for some $A > 0$ dependent on σ). The Riemann zeta function $\zeta(s)$ had been studied by Euler (1707-1783) as a function of real variable s. The notion of $\zeta(s)$ as a function of complex variable $s = \sigma + it$, $(\sigma, t \in \mathbb{R})$ is due to B. Riemann (1826-1866). As is well known, there is an intimate connection between the Riemann zeta function and prime numbers. This connection comes from the Euler product representation for the zeta function given as follows:

$$\zeta(s) = \prod_{p\ prime} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This infinite product converges for $\sigma > 1$.

The term ‘critical strip’ refers to the region $\{s \in \mathbb{C} : 0 < \Re s < 1\}$. The location of the zeros of the Riemann zeta function inside the critical strip is a matter of great significance and conjecture. Bernhard Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of the zeros of $\zeta(s)$. He conjectured that all non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$. This is known as the Riemann Hypothesis.

We remark here that information about these zeros is crucial in analytic number theory and the distribution of primes. There are no zeros for $\Re s > 1$ (from the Euler product) nor for $\Re s < 0$ (by the Functional Equation) except for so called 'trivial zeros' at $-2n$ ($n \in \mathbb{N}$). Furthermore, it is well known that no zeros of $\zeta(s)$ lie on either of the lines $\Re s = 1$ and $\Re s = 0$ (see [29]). Note that $\zeta(s)$ is the *Mellin transform* of $[x]$ (see [2]).

Notation

We define the big oh notation O (or $\ll$), little oh notation o, asymptotic equality of functions $\sim$ and Ω notation as follows:

Definition 5. *If $g(x) > 0$ for all $x \geq a$, we write*

$$f(x) = O(g(x)) \quad \textit{or} \ f(x) \ll g(x),$$

to mean that the quotient $\left|\frac{f(x)}{g(x)}\right|$ is bounded for $x \geq a$; that is there exists a constant $M > 0$ such that

$$|f(x)| \leq Mg(x), \ \textit{for all} \quad x \geq a.$$

An equation of the form $f(x) = h(x)+O(g(x))$ means that $f(x)-h(x) = O(g(x))$.

Definition 6. *Let $g(x) > 0$ for all $x \geq a$, then the notation*

$$f(x) = o(g(x)) \quad \textit{as} \ x \to \infty,$$

means that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

An equation of the form $f(x) = h(x)+o(g(x))$ as $x \to \infty$ means that $f(x)-h(x) = o(g(x))$ as $x \to \infty$.

Definition 7. *Let $g(x) > 0$ for all $x \geq a$. If*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$, and write $f(x) \sim g(x)$ as $x \to \infty$.

We define Ω notation as follows:

Definition 8. *Let F, G be functions defined on some interval (a, ∞) with $G \geq 0$. We write*

$$F(t) = \Omega(G(t)),$$

to mean the negation of the $F(t) = o(G(t))$. That is, there exist a constant $c > 0$ such that $|F(t)| \geq cG(t)$ for some arbitrarily large values of t.

Further, we write $F(t) = \Omega_+(G(t))$ and $F(t) = \Omega_-(G(t))$ if there exist a constant $c > 0$ such that $F(t) \geq cG(t)$ and $F(t) \leq -cG(t)$ hold respectively for some arbitrarily large values of t.

We write $F(t) = \Omega_\pm(G(t))$ if both $F(t) = \Omega_+(G(t))$ and $F(t) = \Omega_-(G(t))$ hold.

Lower bounds for $\zeta(s)$ in the critical strip

Now, we give some lower bounds for the Riemann-Zeta function in the strip $\frac{1}{2} \leq \sigma \leq 1$. The following lower bounds are taken from the literature.

Theorem 2.3. *For any fixed $c > 1$, $\log^c T \leq Y \leq T, T \geq T_0$*

$$\max_{T \leq t \leq T+Y} \left|\zeta(\frac{1}{2} + it)\right| \geq \exp\left\{A_1 \left(\frac{\log Y}{\log\log Y}\right)^{\frac{1}{2}}\right\}$$

$$\max_{T \leq t \leq T+Y} |\zeta(\sigma + it)| \geq \exp\left\{A_2 \frac{(\log Y)^{1-\sigma}}{\log\log Y}\right\}, \quad \frac{1}{2} < \sigma < 1,$$

$$\max_{T \leq t \leq T+Y} |\zeta(1 + it)| \geq A_3 \log\log Y,$$

where A_1, A_2, A_3 are positive, absolute constants.

Proof. See Theorem 9.4. [19] page 241. □

In his paper (1972), Levinson showed

$$\max_{1 \leq t \leq T} |\zeta(1 + it)| \geq e^\gamma \log\log T + O(1),$$

where γ is the Euler's constant (see Theorem 1 in [23]).

Now, we need a lower bound for $|\zeta(\sigma + it)|$ with σ close to 1 which is stronger than Theorem 2.3. The following result is essentially mentioned in [14] page 345.

This will be used to facilitate the proof of a result in Chapter 6 as part of our purpose in that chapter.

Proposition 2.4. *For $\frac{3}{4} \leq \sigma \leq 1 - \frac{\log\log\log N}{2\log\log N}$, we have*

$$\max_{1<t<N} |\zeta(\sigma+it)| \geq \exp\left\{(1+o(1))\frac{(\log N)^{1-\sigma}}{16(1-\sigma)\log\log N}\right\},$$

for $N \geq N_0$ independent of σ.

Proof. Take $\frac{3}{4} \leq \sigma \leq 1$ in [14, Corollary 3.4] we get

$$\max_{1<t<N} |\zeta(\sigma+it)| \geq \beta_\sigma(N^{\frac{1}{8}}) - 1.$$

Here

$$\beta_\sigma(N) = \sup_{\|a\|_2=1} \sqrt{\sum_{n=1}^{N} |b_n|^2},$$

where $a = (a_n)_{n\in\mathbb{N}} \in l^2$ and $b_n = \frac{1}{n^\sigma}\sum_{d|n} d^\sigma a_d$. Here $\|a\|_2 = \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}}$. We have a lower bound for $\beta_\sigma(N)$ (see [14, Page 354]) which tells us that

$$\beta_\sigma(N^{\frac{1}{8}}) \geq \max_{n\leq N^{\frac{1}{8}}} \sqrt{\eta_\sigma(n)},$$

where η_σ is the multiplicative function given by

$$\eta_\sigma(n) = \frac{1}{d(n)}\sum_{d|n} \sigma_{-\sigma}(d)^2.$$

Now, for n is squarefree we have

$$\eta_\sigma(n) = \prod_{p|n} \eta_\sigma(p) = \prod_{p|n} \frac{1}{2}\left(1 + (1+\frac{1}{p^\sigma})^2\right).$$

To set a large value of $\eta_\sigma(n)$, we take n to be a product of the primes up to some P. That is, $n = 2\cdot 3\cdots P$, where P is chosen so that

$$2\cdot 3\cdots P \leq N^{\frac{1}{8}} < 2\cdot 3\cdots P\cdot P'.$$

Here P' is the next prime after P. So, $\log n \sim P \sim \frac{1}{8}\log N$ by Prime Number Theorem. Therefore, we have

$$\eta_{\sigma,2}(2\cdot 3\cdots P) = \prod_{p\leq P} \frac{1}{2}\left(1 + (1+\frac{1}{p^\sigma})^2\right) \geq \prod_{p\leq P}(1+\frac{1}{p^\sigma}).$$

Thus,

$$\log \eta_{\sigma,2}(2\cdot 3\cdots P) \geq \sum_{p\leq P} \log(1+\frac{1}{p^\sigma}) \geq \sum_{p\leq P}\frac{1}{p^\sigma} - \frac{1}{2}\sum_{p\leq P}\frac{1}{p^{2\sigma}} \geq \sum_{p\leq P}\frac{1}{p^\sigma} - \nu,$$

for some absolute constant $\nu > 0$, since $x \geq \log(1+x) \geq x - \frac{x^2}{2}$, for $0 \leq x \leq 1$.

We end the proof of the Proposition by showing that for every $\epsilon > 0$, and $\frac{3}{4} \leq \sigma \leq 1 - \frac{\log\log\log N}{2\log\log N}$, we have

$$\sum_{p\leq P}\frac{1}{p^\sigma} \geq (1-\epsilon)\frac{(\log N)^{1-\sigma}}{8(1-\sigma)\log\log N}, \quad \text{for} \quad N \geq N_0(\epsilon).$$

Now, we have

$$\sum_{p\leq P}\frac{1}{p^\sigma} = \int_{2-}^{P} t^{-\sigma} d\pi(t) = \frac{\pi(P)}{P^\sigma} + \sigma \int_{2-}^{P} \frac{\pi(t)}{t^{\sigma+1}}dt.$$

By the Prime Number Theorem $\pi(x) \geq (1-\epsilon)\frac{x}{\log x}$ for $x \geq x_0(\epsilon)$. This tells us that,

$$\sum_{p\leq P}\frac{1}{p^\sigma} \geq (1-\epsilon)\frac{P^{1-\sigma}}{\log P} + \sigma(1-\epsilon)\int_2^P \frac{t^{-\sigma}}{\log t}dt - \gamma.$$

for some absolute constant $\gamma > 0$. Here

$$\sigma(1-\epsilon)\int_2^P \frac{t^{-\sigma}}{\log t}dt \geq \frac{\sigma(1-\epsilon)}{\log P}\int_2^P t^{-\sigma}dt = \frac{\sigma(1-\epsilon)}{(1-\sigma)\log P}\left(P^{1-\sigma} - 2^{1-\sigma}\right)$$

Thus, for any $\epsilon > 0$, and $P \geq P_0(\epsilon)$, we have

$$\sum_{p\leq P}\frac{1}{p^\sigma} \geq (1-\epsilon)\frac{P^{1-\sigma}-2}{\log P} + \sigma(1-\epsilon)\frac{P^{1-\sigma}-2}{(1-\sigma)\log P} - \gamma = (1-\epsilon)\frac{P^{1-\sigma}-2}{(1-\sigma)\log P} - \gamma.$$

Now, we have $P \sim \frac{1}{8}\log N$. So,

$$\frac{P^{1-\sigma}-2}{(1-\sigma)\log P} \sim \frac{(\log N)^{1-\sigma}}{8(1-\sigma)\log\log N},$$

when $1-\sigma \geq \frac{\log\log\log N}{2\log\log N}$ (actually, for $(1-\sigma)\log\log N \geq 1$). Therefore, from the above for $1-\sigma \geq \frac{\log\log\log N}{2\log\log N}$, we have

$$\begin{aligned}\max_{1<t<N}|\zeta(\sigma+it)| &\geq \beta_\sigma(N^{\frac{1}{8}}) - 1 \geq \max_{n\leq N^{\frac{1}{8}}}\sqrt{\eta_\sigma(n)} - 1 \\ &\geq \exp\left\{\frac{1}{2}\sum_{p\leq P}\frac{1}{p^\sigma} - \nu\right\} - 1 \geq \exp\left\{\frac{(1+o(1))(\log N)^{1-\sigma}}{16(1-\sigma)\log\log N}\right\},\end{aligned}$$

for $N \geq N_0$ independent of σ. The proof of Proposition 2.4 is completed. □

$O-$results for $\zeta(s)$ in the critical strip

We will give now some $O-$results for the Riemann Zeta function in the critical strip $\frac{1}{2} \le \sigma \le 1$. It is known that $\zeta(s)$ has finite order (see [29]). That is, there exists a positive constant A such that

$$\zeta(s) \ll t^A, \text{ for any } \ \sigma \text{ as } t \to \infty.$$

The Lindelöf function is defined by

$$\mu(\sigma) = \inf\{c > 0 : \zeta(\sigma + it) \ll t^c\}.$$

In 1908, Lindelöf proved $\mu(\sigma)$ is continuous, non-increasing, and convex. Since $\zeta(s)$ is bounded for $\sigma \ge 1 + \epsilon$ (each $\epsilon > 0$), it follows that $\mu(\sigma) = 0$ for $\sigma > 1$ and then from the functional equation that $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$. This equation also holds by continuity for $\sigma = 1$ and $\sigma = 0$. Therefore, if we define $y = \frac{1}{2} - \frac{\sigma}{2}$ to be the straight line joining the points $(0, \frac{1}{2})$ and $(1, 0)$ on the curve $\mu(\sigma)$, then by convexity property we see $\mu(\sigma) \le \frac{1}{2} - \frac{\sigma}{2}$ for $0 < \sigma < 1$. In particular, $\mu(\frac{1}{2}) \le \frac{1}{4}$. That is,

$$\zeta(\frac{1}{2} + it) \ll t^{\frac{1}{4}+\epsilon},$$

for every $\epsilon > 0$.

The exact value of $\mu(\sigma)$ is not known for $0 < \sigma < 1$. Lindelöf conjectured that $\mu(\frac{1}{2}) = 0$. It is equivalent to $\zeta(\frac{1}{2} + it) \ll t^\epsilon$ for any $\epsilon > 0$.

Note that the Riemann Hypothesis, which asserts that all of the non-trivial zeros of $\zeta(s)$ lie on the vertical line $\Re(s) = \frac{1}{2}$, implies the Lindelöf Hypothesis.

Much effort has gone into finding $\mu(\sigma)$ in the critical strip. We have $\zeta(\frac{1}{2}+it) \ll t^{\frac{1}{6}} \log t$ (see Theorem 5.12. in [29]). Small improvements on this result have been obtained by various different methods. The most recent improvement on this result is due to Huxley (2005), in which he showed $\zeta(1/2 + it) \ll t^{32/205} \log^c t$ for some c (see [18]). Furthermore, for $\sigma = 1$, Richert in his paper (1963) proved that $\zeta(1 + it) \ll (\log t)^{\frac{2}{3}}$. Moreover, he proved for $\frac{1}{2} \le \sigma \le 1$ that

$$|\zeta(\sigma + it)| \le A(1 + t^{B(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}}, \tag{2.1}$$

with $B = 100$ (see [27] page 98). More research on this subject has been done to improve (2.1). In 1975, Elson proved (2.1) with $B = 86$ and $A = 2100$, see [12]. Ching in his paper (1999) improved this obtaining (2.1) with $B = 46$ and $A = 175$. Moreover, Heath Brown (in unpublished work (see page 135 in [29])) proved (2.1) with $B = 18.8$ and some $A > 0$.

$O-$results for $\zeta(s)$ on the Riemann Hypothesis

If we assume the truth of the unproved Riemann Hypothesis the bounds can be improved significantly. This will give us the strongest conditional upper bound for the Riemann Zeta function available at present in the critical strip $\frac{1}{2} \le \sigma \le 1$. For the cases in which $\sigma = \frac{1}{2}$, and $\sigma = 1$, we have

$$|\zeta(1 + it)| \le (2e^{\gamma} + o(1)) \log\log t, \quad (\gamma \quad \text{is Euler's constant}),$$
$$\zeta(\frac{1}{2} + it) \ll \exp\{A\frac{\log t}{\log\log t}\}, \quad \text{for some} \quad A > 0.$$

See Theorem 14.9. and Theorem(A) 14.14. in [29]. For $\frac{1}{2} < \sigma < 1$, we have

$$\log \zeta(s) \ll \frac{(\log t)^{2-2\sigma} - 1}{(1-\sigma)\log\log t} + \log\log\log t.$$

See Section 14.33 in [29].

Chapter 3

Beurling prime systems

In this chapter we give the necessary background to Beurling (or generalised) prime systems and the associated Beurling zeta function. It is beneficial to give historical context to this subject.

In the late nineteenth century, Weber (see [30]) defined $N(x)$ to be the number of the integral ideals in a fixed algebraic number field $\mathcal{F}$ with the norm not exceeding x and proved that $N(x) = ax + O(x^{\theta})$, as $x \to \infty$ for some $a > 0$ and $\theta < 1$. Early in the twentieth century, Landau (see [22]) used Weber's result and the multiplicative structure to prove the Prime Ideal Theorem, which asserts that the number of the distinct prime ideals of the ring of integers in an algebraic number field $\mathcal{F}$ with the norm not exceeding x is a asymptotic to $\frac{x}{\log x}$, as x tends to infinity. His result showed that the only 'additive' result needed was Weber's.

3.1 Discrete g-prime systems

In 1937, Beurling (see [6]) considered number systems with only multiplicative structure, and was interested in finding conditions over the counting function of integers $N(x)$ which ensure the validity of the Prime Number Theorem. Beurling introduced generalised prime systems as follows:

Definition 9. *A generalised prime system $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying $1 < p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ and for which $p_n \longrightarrow \infty$*

as $n \longrightarrow \infty$.

The numbers $\{p_n\}_{n\geq 1}$ are called *generalised primes* (or *Beurling primes*). The associated system of *generalised integers* (or *Beurling integers*) $\mathcal{N} = \{n_i\}_{i\geq 1}$ can be formed from these. That is, the numbers of the form

$$p_1^{a_1}.p_2^{a_2}....p_k^{a_k} \qquad (*)$$

where $k \in \mathbb{N}$ and $a_1, a_1, ..., a_k \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$. We shall often refer to g-primes and g-integers for short. We remark here that $\mathcal{P}$ and $\mathcal{N} =< \mathcal{P} >$ are not sets, but multisets where elements can occur with a certain multiplicity. Beurling prime systems generalise the notion of prime numbers and the natural numbers obtained from them.

The generalised counting functions of primes and of integers are defined in the natural way as follows

$$\pi_{\mathcal{P}}(x) = \sum_{p\leq x, p\in\mathcal{P}} 1 \text{ and } \mathcal{N}_{\mathcal{P}}(x) = \sum_{n\leq x, n\in\mathcal{N}} 1. \tag{3.1}$$

The generalisation of the classical Prime Number Theorem proved by Beurling is as follows:

Theorem 3.1. *[Beurling's PNT] If* $\mathcal{N}_{\mathcal{P}}(x) = Ax + O(\frac{x}{\log^{\gamma} x})$ *for some* $A > 0$ *and* $\gamma > \frac{3}{2}$, *then* $\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$.

This is an analogue of the Prime Number Theorem. Beurling also showed that the condition $\gamma > \frac{3}{2}$ is necessary in the sense that there is a 'continuous analogue' of a g-prime system with $\gamma = \frac{3}{2}$ for which the Prime Number Theorem does not hold. In his paper 1970, Diamond (see [9]) showed (for discrete systems) Beurling's condition is sharp, namely, the Prime Number Theorem does not necessarily hold if $\gamma = \frac{3}{2}$.

Many of the known results involve the associated zeta function often referred to as a *Beurling zeta function* in the literature, which we define formally by the Euler product

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}.$$

This infinite product may be formally multiplied out to give the Dirichlet series $\zeta_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{1}{n^s}$. This is also the *Mellin transform* of $\mathcal{N}_{\mathcal{P}}$.

The important question in this work is: how do the distributions of $\mathcal{P}$ and $\mathcal{N}$ relate to each other?

Much of the research on this subject has been about connecting the asymptotic behaviour of the g-prime and g-integer counting functions defined in (3.1) as $x \longrightarrow \infty$. Specifically, given the asymptotic behaviour of $\pi_{\mathcal{P}}(x)$, what can be said about the behaviour of $\mathcal{N}_{\mathcal{P}}(x)$? On the other hand, given the asymptotic behaviour of $\mathcal{N}_{\mathcal{P}}(x)$, what can be said about the behaviour of $\pi_{\mathcal{P}}(x)$? Therefore, this research concentrates on finding conditions for which results of the type

$$\mathcal{N}_{\mathcal{P}}(x) = ax + E_1(x) \quad \Longrightarrow \quad \pi_{\mathcal{P}}(x) = \mathrm{li}(x) + E_2(x),$$

or

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + E_2(x) \quad \Longrightarrow \quad \mathcal{N}_{\mathcal{P}}(x) = ax + E_1(x),$$

hold. Here $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$, and the error terms $E_1(x)$,$E_2(x)$ (of $\mathcal{N}_{\mathcal{P}}(x)$ and $\pi_{\mathcal{P}}(x)$ respectively) are of orders less than x and $\mathrm{li}(x)$.

This research has also concentrated on determining the connections between the asymptotic behaviour (as $x \to \infty$) of g-integer counting function $\mathcal{N}_{\mathcal{P}}(x)$ and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$). Also, between $\pi_{\mathcal{P}}(x)$ and the zeros of $\zeta_{\mathcal{P}}(s)$.

Examples

1. Let $\mathcal{P}$ be the sequence of odd primes (i.e. $\mathcal{P} = \{3, 5, 7, ...\} = \mathbb{P}\backslash\{2\}$). Then the numbers $(*)$ forming $\mathcal{N}$ are all of the odd integers. That is, $\mathcal{N} = 2\mathbb{N}-1$. This shows that $\pi_{\mathcal{P}}(x) = \pi(x) - 1$ and

$$\mathcal{N}_{\mathcal{P}}(x) = \sum_{\substack{n\leq x,\\ n\ odd}} 1 = \sum_{k\leq \frac{x+1}{2}} 1 = \left[\frac{x+1}{2}\right].$$

The behaviour of these counting functions for large x is

$$\mathcal{N}_{\mathcal{P}}(x) = \frac{1}{2}x + O(1),$$

while $\pi(x) \sim \frac{x}{\log x}$, by the Prime Number Theorem.

2. For $\mathcal{P} = \{2, 2, 3, 3, 5, 5, 7, 7, ...\}$ (each prime occurs twice), with $(*)$ forming $\mathcal{N}$ to be the set of integers such that each integer occurs $d(n)$ times, where $d(n)$ is the number of divisors of n. That is,

$$\mathcal{N} = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, 6, 7, 7, ...\},$$

therefore $\pi_{\mathcal{P}}(x) = 2\pi(x) = 2\sum_{p\leq x} 1$ and

$$\mathcal{N}_{\mathcal{P}}(x) = \sum_{\substack{n\leq x,\\ n\in\mathbb{N}}} d(n).$$

Then the behaviour of these counting functions for large x is $\mathcal{N}_{\mathcal{P}}(x) \sim x\log x$ (see [2]) and $\pi_{\mathcal{P}}(x) \sim \frac{2x}{\log x}$,(by the Prime Number Theorem).

3.2 Continuous g-prime systems

The notion of g-primes as defined earlier can be generalised in such a way that we consider $\pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$ as general increasing functions not necessarily step functions. Such an extension is often referred to loosely as a 'continuous' g-prime system. Indeed Beurling's Prime Number Theorem is actually proven in this general setting. In the most general form, the 'continuous' g-prime systems are based on the analogue of $\Pi_{\mathcal{P}}(x)$ $(= \sum_{k=1}^{\infty} \frac{1}{k}\pi_{\mathcal{P}}(x^{1/k}))$ and are defined as follows:

Definition 10. *Let $\Pi_{\mathcal{P}}, \mathcal{N}_{\mathcal{P}}$ be functions such that $\Pi_{\mathcal{P}} \in S_0^+$ and $\mathcal{N}_{\mathcal{P}} \in S_1^+$ with $\mathcal{N}_{\mathcal{P}} = \exp_* \Pi_{\mathcal{P}}$. Then $(\Pi_{\mathcal{P}}, \mathcal{N}_{\mathcal{P}})$ is called an* outer g-prime system.

Note that, if $\Pi_{\mathcal{P}} \in S_0^+$, then automatically $\exp_* \Pi_{\mathcal{P}} \in S_1^+$. Hence any $\Pi_{\mathcal{P}} \in S_0^+$ defines an outer g-prime system. On the other hand, if $\mathcal{N}_{\mathcal{P}} \in S_1^+$, then $\mathcal{N}_{\mathcal{P}} = \exp_* \Pi_{\mathcal{P}}$ for some $\Pi_{\mathcal{P}} \in S_0$, but $\Pi_{\mathcal{P}}$ need not be increasing (see section 1.3 in [15]). Here we do not (yet) have the analogue of g-primes (i.e. $\pi_{\mathcal{P}}(x)$). We introduce $\pi_{\mathcal{P}}(x)$ as follows:

Definition 11. *A g-prime system is an outer g-prime system for which there exist $\pi_{\mathcal{P}} \in S_0^+$ such that*

$$\Pi_{\mathcal{P}}(x) = \sum_{k=1}^{\infty} \frac{1}{k}\pi_{\mathcal{P}}(x^{1/k}).$$

We say $\mathcal{N}_{\mathcal{P}}$ determines a g-prime system if there exists such an increasing $\pi_{\mathcal{P}} \in S_0$. As such by *Möbius inversion*, $\pi_{\mathcal{P}}(x)$ is given by

$$\pi_{\mathcal{P}}(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi_{\mathcal{P}}(x^{1/k}), \tag{3.2}$$

provided this series converges absolutely. To show that this sum always converges for $\Pi_{\mathcal{P}} \in S^+$, we let $a_k = \frac{\mu(k)}{k}$ and let $b_k = \Pi_{\mathcal{P}}(x^{1/k})$. The partial sums of the a_k are bounded in magnitude by q (some $q > 0$) since $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$. The sum $\sum_{k=1}^{\infty} |b_k - b_{k+1}|$ converges since b_k decreases to zero. By Abel's summation we have

$$\sum_{k=1}^{N} a_k b_k = \sum_{k=1}^{N-1} A_k(b_k - b_{k+1}) + A_N b_N,$$

where $A_n = \sum_{k=1}^{n} a_k$. Therefore,

$$\left|\sum_{k=M}^{N} a_k b_k\right| = \left|\sum_{k=M}^{N-1} A_k(b_k - b_{k+1}) + A_N b_N\right| \leq q \sum_{k=M}^{N-1} |b_k - b_{k+1}| + q\,|b_N|\,.$$

This shows that (3.2) always converges whenever $\Pi_{\mathcal{P}}$ is increasing.

In general though, $\pi_{\mathcal{P}}(x)$ (as given by (3.2)) need not be increasing (see example 2 in this section). We make the following definitions (see [4] and [15]):

Definition 12. *For an outer g-prime system* $(\Pi_{\mathcal{P}}, \mathcal{N}_{\mathcal{P}})$*, let* $\psi_{\mathcal{P}} = \Pi_{\mathcal{P}L}$*. That is,*

$$\psi_{\mathcal{P}}(x) = \int_{1-}^{x} \log t \;\, d\Pi_{\mathcal{P}}(t),$$

denote the generalized Chebyshev function*.*

We note that $\psi_{\mathcal{P}} \in S_0^+$, and that

$$\mathcal{N}_{\mathcal{P}} = \exp_* \Pi_{\mathcal{P}} \text{ is equivalent to } \psi_{\mathcal{P}} * \mathcal{N}_{\mathcal{P}} = \mathcal{N}_{\mathcal{P}L}. \tag{3.3}$$

Definition 13. *A g-prime system is* discrete *if* $\pi_{\mathcal{P}}$ *is a step function with integer jumps. In this case the g-primes are the discontinuities of* $\pi_{\mathcal{P}}$ *and the steps are the multiplicities. Note that, in this case*

$$\psi_{\mathcal{P}}(x) = \sum_{\substack{p^k \leq x, k \in \mathbb{N} \\ p \in \mathcal{P}}} \log p.$$

We can write this as

$$\psi_{\mathcal{P}}(x) = \sum_{n=1}^{\infty} \sum_{\substack{p \leq x^{\frac{1}{n}} \\ p \in \mathcal{P}}} \log p = \sum_{n=1}^{\infty} \vartheta_{\mathcal{P}}(x^{\frac{1}{n}}),$$

where

$$\vartheta_{\mathcal{P}}(x) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \log p.$$

Thus, the Beurling prime systems discussed in Section 3.1 are discrete systems. In the general case, the Beurling zeta function is now defined to be the *Mellin transform* of $\mathcal{N}_{\mathcal{P}}$ as follows:

Definition 14.

$$\zeta_{\mathcal{P}}(s) = \int_{1-}^{\infty} x^{-s} d\mathcal{N}_{\mathcal{P}}(x) = \exp\left\{\int_{1-}^{\infty} x^{-s} d\Pi_{\mathcal{P}}(x)\right\}.$$

The equality follows formally from $\mathcal{N}_{\mathcal{P}} = \exp_* \Pi_{\mathcal{P}}$, (see definition 12).

Throughout this thesis we shall assume these integrals converge for $\Re s > 1$. From definitions 11 and 14 we can relate the Beurling zeta function to $\pi_{\mathcal{P}}(x)$ as follows:

For $s = \sigma + it$ with $\sigma > 1$,

$$\begin{aligned} \zeta_{\mathcal{P}}(s) = \exp\left\{\int_{1-}^{\infty} x^{-s} d\Pi_{\mathcal{P}}(x)\right\} &= \exp\left\{\int_{1-}^{\infty} \sum_{k=1}^{\infty} \frac{x^{-s}}{k} \ d\pi_{\mathcal{P}}(x^{1/k})\right\} \\ = \exp\left\{\int_{1-}^{\infty} \sum_{k=1}^{\infty} \frac{\nu^{-sk}}{k} \ d\pi_{\mathcal{P}}(\nu)\right\} &= \exp\left\{-\int_{1-}^{\infty} \log(1-\nu^{-s}) \ d\pi_{\mathcal{P}}(\nu)\right\}. \end{aligned}$$

For a discrete systems this reduces to $\prod_{p \in \mathcal{P}} \frac{1}{1-p^{-s}}$.

Examples

1. Let $\Pi_{\mathcal{P}}(x) = \int_1^x \frac{t-t^{-1}}{\log t} \ dt$, $x \geq 1$. This means that clearly $\Pi_{\mathcal{P}} \in S_0^+$ and

$$\psi_{\mathcal{P}}(x) = \int_1^x \log t \ \Pi_{\mathcal{P}}^{'}(t) \ dt = \int_1^x t - \frac{1}{t} \ dt = \frac{1}{2}(x^2-1) - \log x, \quad x \geq 1.$$

By (3.2), we find

$$\begin{aligned}
\pi_{\mathcal{P}}(x) &= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi_{\mathcal{P}}(x^{1/k}) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_1^{x^{\frac{1}{k}}} \frac{t - \frac{1}{t}}{\log t}\ dt \\
&= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_1^{x} \frac{u^{\frac{1}{k}} - u^{-\frac{1}{k}}}{\log u} \left(u^{\frac{1}{k}-1}\right)\ du = \int_1^x \frac{1}{u \log u} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} (u^{\frac{2}{k}} - 1)\ du \\
&= \int_1^x \frac{1}{u \log u} \sum_{m=1}^{\infty} \frac{2^m (\log u)^m}{m!} \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^{1+m}} \right)\ du \\
&= \int_1^x \frac{1}{u} \sum_{m=1}^{\infty} \frac{2^m (\log u)^{m-1}}{m!\ \zeta(1+m)} du, \quad \text{for } x \geq 1.
\end{aligned}$$

This shows that $\pi_{\mathcal{P}} \in S_0^+$ and therefore we have a g-prime system. Moreover, in this case we have $\mathcal{N}_{\mathcal{P}}(x) = x^2$, since by (3.3) we have

$$\begin{aligned}
\int_1^x \log t\ d\mathcal{N}_{\mathcal{P}}(t) &= \int_1^x \mathcal{N}_{\mathcal{P}}\left(\frac{x}{t}\right)\ d\psi_{\mathcal{P}}(t) = \int_1^x \mathcal{N}_{\mathcal{P}}\left(\frac{x}{t}\right)\left(t - \frac{1}{t}\right)\ dt \\
&= x \int_1^x \mathcal{N}_{\mathcal{P}}(u) \left(\frac{x}{u} - \frac{u}{x}\right)\ \frac{du}{u^2}.
\end{aligned}$$

That is,

$$\mathcal{N}_{\mathcal{P}}(x) \log x - \int_1^x \frac{\mathcal{N}_{\mathcal{P}}(t)}{t} dt = x^2 \int_1^x \frac{\mathcal{N}_{\mathcal{P}}(u)}{u^3} du - \int_1^x \frac{\mathcal{N}_{\mathcal{P}}(u)}{u} du.$$

By differentiating and simplifying, we get $\frac{d}{dx}\left(\frac{\mathcal{N}_{\mathcal{P}}(x) \log x}{x^2}\right) = \frac{\mathcal{N}_{\mathcal{P}}(x)}{x^3}$. Therefore, $\log \mathcal{N}_{\mathcal{P}}(x) = 2 \log x + c$, but $\mathcal{N}_{\mathcal{P}}(1) = 1$, which means $c = 0$.

2. Let $\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1-t^{-c}}{\log t}\ dt$, $x \geq 1$, and $c > 0$. This means that $\Pi_{\mathcal{P}}$ and $\psi_{\mathcal{P}}$ are increasing, where

$$\psi_{\mathcal{P}}(x) = \int_1^x \log t\ \Pi'_{\mathcal{P}}(t)\ dt = \int_1^x 1 - \frac{1}{t^c}\ dt \quad x \geq 1.$$

For $c \leq 2$ we have a g-prime system (i.e. $\pi_{\mathcal{P}}(x)$ is increasing). For instance, take $c = 1$. Then $\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1-t^{-1}}{\log t}\ dt$, and $\psi_{\mathcal{P}}(x) = x - 1 - \log x, \quad x \geq 1$, and following the same arguments as in the first example, we have

$$\pi_{\mathcal{P}}(x) = \int_1^x \frac{1}{u} \sum_{m=1}^{\infty} \frac{(\log u)^{m-1}}{m!\ \zeta(1+m)} du \quad x \geq 1,$$

while $\mathcal{N}_{\mathcal{P}}(x) = x$, $x \geq 1$. For c sufficiently large, $\pi_{\mathcal{P}}(x)$ need not be increasing (see Theorem A1. in [15]).

Working with $\psi_{\mathcal{P}}(x)$ is often more convenient than working with $\Pi_{\mathcal{P}}(x)$. One reason is due to the following direct link between $\zeta_{\mathcal{P}}$ and $\psi_{\mathcal{P}}$

$$-\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) = \int_{1-}^{\infty} x^{-s} d\psi_{\mathcal{P}}(x).$$

From Definition 12 above, the following statements

$$\Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\alpha+\epsilon}), \ \forall \epsilon > 0 \tag{3.4}$$

and

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\epsilon}), \ \forall \epsilon > 0, \tag{3.5}$$

are equivalent for $\alpha \in [0, 1)$. Furthermore, we see that $\pi_{\mathcal{P}}(x) \leq \Pi_{\mathcal{P}}(x)$ and

$$\begin{aligned}
\Pi_{\mathcal{P}}(x) - \Pi_{\mathcal{P}}(\sqrt{x}) &= \sum_{k=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{k}})}{k} - 2\sum_{k=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{2k}})}{2k} \\
&= \sum_{k\geq 1} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{k}})}{k} - 2\sum_{k \ even} \frac{\pi_{\mathcal{P}}(x^{\frac{1}{k}})}{k} \\
&= \sum_{k\geq 1} \frac{(-1)^{k-1}\pi_{\mathcal{P}}(x^{\frac{1}{k}})}{k} \leq \pi_{\mathcal{P}}(x),
\end{aligned}$$

since $\pi_{\mathcal{P}}$ is increasing. This tells us that

$$0 \leq \Pi_{\mathcal{P}}(x) - \pi_{\mathcal{P}}(x) \leq \Pi_{\mathcal{P}}(\sqrt{x}).$$

Thus, $\pi_{\mathcal{P}}(x) = \Pi_{\mathcal{P}}(x) + O(\Pi_{\mathcal{P}}(\sqrt{x}))$. Then the following statements

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\alpha+\epsilon}), \ \forall \epsilon > 0 \ \text{and} \ \ \psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\epsilon}), \ \forall \epsilon > 0,$$

are equivalent for $\alpha \in [\frac{1}{2}, 1)$.

3.3 Some known results and comments

We now list some relevant known results from the literature which are necessary for this work. Initially the following known results were proved for discrete systems, but actually the proofs are only based on $\psi_{\mathcal{P}}(x)$ being increasing. So, they are valid for outer g-prime systems. We begin with Beurling's Prime Number Theorem as mentioned in section 3.1.

1. In 1937, Beurling (see [6]) proved that

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O\left(\frac{x}{(\log x)^{\gamma}}\right) \text{ for some } \gamma > \frac{3}{2} \Rightarrow \quad \pi_{\mathcal{P}}(x) \sim \frac{x}{\log x},$$

(generalises Prime Number Theorem), and he showed by example that the result can fail for $\gamma = \frac{3}{2}$.

2. In 1977, Diamond (see [10], Theorem 2) as a type of converse of Beurling's PNT, showed the following: *suppose that* $\int_2^\infty t^{-2}\left|\Pi_{\mathcal{P}}(t) - \frac{t}{\log t}\right| \; dt < \infty$. *Then there exists a positive constant c such that* $\mathcal{N}_{\mathcal{P}}(x) \sim cx$ *as* $x \to \infty$. Diamond in his work was seeking weakest possible conditions on $\pi_{\mathcal{P}}(x)$ which are sufficient to deduce that $\mathcal{N}_{\mathcal{P}}(x) \sim cx$ as $x \to \infty$. So, for example it follows from Diamond's work that

$$\Pi_{\mathcal{P}}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right) \text{ for some } \quad \delta > 0 \;\Rightarrow\; \mathcal{N}_{\mathcal{P}}(x) \sim cx.$$

3. In 1903, Landau (see [22]) proved that

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O(x^{\theta}), (\theta < 1), \tag{3.6}$$

implies $\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$. Furthermore, he proved that (3.6) implies

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-k\sqrt{\log x}})$$

for some $k > 0$.

4. In 2006, Diamond, Montgomery and Vorhauer (see [11]) showed Landau's result is best possible.That is, they proved that there is a discrete g-prime system for which (3.6) holds but

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \Omega(xe^{-q\sqrt{\log x}}) \text{ for some } \quad q > 0.$$

5. In 1969, Malliavin (see [24]) showed that for $\alpha \in (0, 1)$ and $a, c > 0$

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O(xe^{-c(\log x)^{\beta}}) \implies \Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-k(\log x)^{\alpha}}),$$

for some $k > 0$, where $\beta = 10\alpha$.

6. In his paper 1970, Diamond (see [7]) improved Malliavin's result and conversely he showed that if

$$\Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-c(\log x)^\alpha}),$$

holds for $\alpha \in (0,1)$ and some $c > 0$, then

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-b(\log x \log\log x)^\beta}), \text{ for some } \quad b > 0,$$

where $\beta = \frac{\alpha}{1+\alpha}$.

7. Balanzario (1998, [3]) showed (by giving a concrete continuous example) that there exists a continuous g-prime system for which

$$\Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^\alpha}), \tag{3.7}$$

and

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + \Omega_{\pm}(xe^{-c(\log x)^\beta}), \tag{3.8}$$

holds for some positive constants ρ and c with $\alpha = \beta = \frac{1}{2}$.

8. In 2006, Hilberdink (see Theorem 2.2 in [17]) extended Diamond's result in 6 (to $\alpha = 1$ case) as follows: *suppose* $\psi_{\mathcal{P}}(x) = x + O(x^\alpha)$ *for some* $\alpha \in (0,1)$. *Then there exist positive constants* ρ *and* c *such that*

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-c\sqrt{\log x \log\log x}}).$$

As we see from above, many authors have studied the error terms of the asymptotic behaviour of g-prime and g-integer counting functions and it seems three types occur commonly; namely those of the form

$$(i) \quad O\left(\frac{x}{(\log x)^\gamma}\right), \quad (ii) \quad O(xe^{-c(\log x)^\alpha}) \text{ and } (iii) \quad O(x^\theta),$$

where $\gamma > 1, c > 0$ and $\alpha, \theta \in (0,1)$. In our work, we study the asymptotic behaviours of Beurling counting functions of primes and integers with error terms (ii) and (iii).

From the previous two known results (6 and 7), if we assume (3.7) and let $\beta(\alpha)$ be the supremum of such β over all systems satisfying (3.7) for given $\alpha \in (0,1)$.

then by Diamond's result we see $\beta(\alpha) \geq \frac{\alpha}{1+\alpha}$. Further, from Balanzario's result we see that $\beta(\frac{1}{2}) \leq \frac{1}{2}$. Diamond and Bateman [5] raised the interesting problem to determine $\beta(\alpha)$ for $0 < \alpha < 1$.

In our work we study Balanzario's method in his paper and modify it to show (by adapting the method) that there is a (continuous) g-prime system for which (3.7) and (3.8) hold with $\beta = \alpha$ (any $0 < \alpha < 1$), this showing $\beta(\alpha) \leq \alpha$. Furthermore, we prove that there is a *discrete g-prime system* with the same property $\beta(\alpha) \leq \alpha$. This is more challenging since we need $\pi_{\mathcal{P}}(x)$ defined as a step function. For this we use the method developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a discrete system of Beurling primes satisfying this same property. We illustrate this in Chapter 4.

From the known results (listed above), we see that for $0 \leq \alpha, \beta < 1$ the statement

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha}), \tag{3.9}$$

does not necessarily imply

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(x^{\beta}), \;\; \rho > 0. \tag{3.10}$$

Actually, the example given in chapter 6 shows that (3.9)$\Longrightarrow$(3.10) is false for g-prime systems. For general g-prime systems that (3.10) does not imply (3.9) for discrete g-prime systems follows from a result of Diamond, Montgomery, Vorhauer paper [11] shows by using the probabilistic construction that there is a discrete system for which (3.10) does not imply (3.9).

Discrete g-prime systems where the functions $\mathcal{N}_{\mathcal{P}}(x)$ and $\psi_{\mathcal{P}}(x)$ are simultaneously 'well-behaved', that is (3.9) and (3.10) hold have been investigated by Hilberdink (see [17]). In particular, if (3.9) and (3.10) hold then one of α or β is at least $\frac{1}{2}$ (see Theorem 1. in [16]). We shall require the following two results in our subsequent work.

Lemma 3.2. *Suppose that for some $\alpha \in [0,1)$, we have (3.9) holds. Then $\zeta_{\mathcal{P}}(s)$ has analytic continuation to the half-plane $H_\alpha = \{s \in \mathbb{C} : \Re s > \alpha\}$ except for a simple (non removable) pole at $s = 1$ and $\zeta_{\mathcal{P}}(s) \neq 0$ in this region.*

Proof. See first part of Theorem 2.1 in [17]. □

Lemma 3.3. *Suppose for $0 \leq \alpha, \beta < 1$ both (3.9) and (3.10) hold. Then for $\sigma > \Theta = \max\{\alpha, \beta\}$, and uniformly for $\sigma \geq \Theta + \gamma$ (any $\gamma > 0$), $\zeta_{\mathcal{P}}(s)$ is of zero order for $\sigma > \Theta$. Furthermore,*

$$-\frac{\zeta_{\mathcal{P}}'}{\zeta_{\mathcal{P}}}(s) = O\big((\log t)^{\frac{1-\sigma}{1-\Theta}+\epsilon}\big),$$

and

$$\zeta_{\mathcal{P}}(s) = O\big(\exp\{(\log t)^{\frac{1-\sigma}{1-\Theta}+\epsilon}\}\big),$$

for all $\epsilon > 0$.

Proof. The proof of this lemma is given for discrete g-prime systems [17, Theorem 2.3], but holds more generally for outer g-prime systems as well (since no use is made of $\pi_{\mathcal{P}}(x)$). □

Assume that we have a discrete g-prime system such that (3.10) holds with $\beta < \frac{1}{2}$. It was shown in [16] that this implies $\zeta_{\mathcal{P}}(s)$ has non-zero order for $\beta < \sigma < \frac{1}{2}$. This shows that there is a link between the asymptotic behaviour (as $x \to \infty$) of the g-integer counting function $\mathcal{N}_{\mathcal{P}}(x)$ and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ (as $t \to \infty$). We illustrate the connection between $\mathcal{N}_{\mathcal{P}}(x)$ and $\zeta_{\mathcal{P}}(s)$ in Chapters 5 and 6.

Chapter 4

Examples of continuous and discrete g-prime systems

In this chapter we introduce a problem discussed by Diamond [7](as mentioned briefly in section 3.3), which is the following:

Assume $\Pi_{\mathcal{P}}(x) - \mathrm{li}(x) \ll xe^{-(\log x)^{\alpha}}$, for some $\alpha \in (0,1)$, so that

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-c(\log x)^{\beta}}), \tag{4.1}$$

for some $\rho, c > 0$ and $\beta > 0$. The problem is to determine the best possible β, given α. So, let $\beta(\alpha)$ be the supremum of such β over all systems satisfying (3.7) for given $\alpha \in (0,1)$. It follows from Malliavin's result that $\beta(\alpha) \leq 10\alpha$. Diamond in 1970 (see [8]) proved that $\beta(\alpha) \geq \frac{\alpha}{1+\alpha}$. In 1998, Balanzario [3] proved (by giving a concrete continuous example) that there exists a continuous g-prime system for which $\beta = \alpha = \frac{1}{2}$ in (3.7) and (3.8). Thus, $\beta(\frac{1}{2}) \leq \frac{1}{2}$.

In the first section of this chapter, we generalise Balanzario's result by adapting his method to show that for any $0 < \alpha < 1$ there is a continuous g-prime system for which (3.7) and (3.8) hold with $\beta = \alpha$. Thus, $\beta(\alpha) \leq \alpha$.

In the second section we do more challenging work using the theory developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a discrete g-prime system for which (3.7) and (3.8) hold with $\beta = \alpha$. Thus, $\beta(\alpha) \leq \alpha$ for discrete g-prime systems.

4.1 Continuous g-prime System

Theorem 4.1. *Let $0 < \alpha < 1$. Then there exists an outer g-prime system $\mathcal{P}$ for which*

$$\Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^{\alpha}}), \tag{4.2}$$

and

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + \Omega_{\pm}(xe^{-c(\log x)^{\alpha}}), \tag{4.3}$$

for some positive constants ρ and c. Thus, $\beta(\alpha) \leq \alpha$.

We define $\Pi_{\mathcal{P}}(x)$ (of g-primes) as in Balanzario's paper by

$$\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1 - t^{-k}}{\log t} \gamma(t) dt, \tag{4.4}$$

where

$$\gamma(t) = 1 - \sum_{n > n_0} \mu_n \frac{\cos(b_n \log t)}{t^{a_n}}, \quad t \geq 1.$$

Here k and n_0 are positive constants and μ_n, a_n and b_n are sequences to be chosen. In fact, we shall take $k = 4, n_0 = 3$, $\mu_n = \frac{2}{n^2}$, but it is notationally more convenient to use k, n_0 and μ_n. The sequences b_n and a_n are defined (in terms of another sequence x_n) as follows:

$$b_n = \exp\{(\log x_n)^{\alpha}\} \quad \text{ and } \quad a_n = \frac{1}{(\log x_n)^{1-\alpha}} \Big(= \frac{1}{(\log b_n)^{\theta}} \Big),$$

where $\theta = \frac{1}{\alpha} - 1$. Here $x_n = \exp\{e^{a\omega^n}\}$, for some $a > 0$ and $\omega > 1$ which we shall choose later. Note that, $a_n \to 0$ while $b_n \to \infty$ as $n \to \infty$. So, $x_{n+1} = \exp\{(\log x_n)^{\omega}\}$, with $x_1 = \exp\{e^{a\omega}\}$. We choose ω so that $\alpha\omega \geq 1$.

The function $\Pi_{\mathcal{P}}(x)$ is increasing since for $t \geq 1$,

$$\left| \sum_{n > n_0} \mu_n \frac{\cos(b_n \log t)}{t^{a_n}} \right| \leq \sum_{n > n_0} \frac{2}{n^2} \leq 1.$$

First, we show that (4.2) holds.

Proposition 4.2. *If $\Pi_{\mathcal{P}}(x)$ is given by (4.4), then*

$$\Pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^{\alpha}}).$$

Proof. We have

$$\Pi_{\mathcal{P}}(x) = \int_1^e \frac{1-t^{-k}}{\log t}\gamma(t)dt + \int_e^x \frac{1-t^{-k}}{\log t}\gamma(t)dt,$$

the first integral is just $O(1)$, therefore we get

$$\begin{aligned}\Pi_{\mathcal{P}}(x) &= \int_e^x \frac{1-t^{-k}}{\log t}dt - \sum_{n>n_0}\mu_n \int_e^x \frac{1-t^{-k}}{\log t}\cdot\frac{\cos(b_n\log t)}{t^{a_n}}dt + O(1)\\ &= \int_e^x \frac{dt}{\log t} - \sum_{n>n_0}\mu_n \int_e^x \frac{\cos(b_n\log t)}{t^{a_n}\log t}dt + O(1),\end{aligned}$$

because $k>1$. Now we show that the second term is $O(xe^{-(\log x)^\alpha})$. Notice that

$$\left|\int_e^x \frac{\cos(b_n\log t)}{t^{a_n}\log t}dt\right| = \left|\int_1^{\log x}\frac{\cos(b_n t)}{t}e^{t(1-a_n)}dt\right|$$

$$\begin{aligned}&= \left|\left[\frac{\sin(b_n t)}{tb_n}e^{t(1-a_n)}\right]_1^{\log x} - \frac{1}{b_n}\int_1^{\log x}\frac{\sin(b_n t)}{t}e^{t(1-a_n)}(1-a_n-\frac{1}{t})dt\right|\\ &\le 2\frac{x^{1-a_n}}{b_n\log x} + \frac{1}{b_n}\int_1^{\log x}\frac{e^{t(1-a_n)}}{t}dt \le 2\frac{x^{1-a_n}}{b_n\log x} + \frac{x^{1-a_n}}{b_n(1-a_n)} \le 3\frac{x^{1-a_n}}{b_n}.\end{aligned}$$

By the definition of a_n and b_n we have

$$\frac{x^{1-a_n}}{b_n} = x\exp\left\{-\left(\frac{\log x}{(\log x_n)^{\alpha\theta}} + (\log x_n)^\alpha\right)\right\}.$$

The minimum value of $u + \frac{\log x}{u^\theta}$ occurs when $u^{\theta+1} = \theta\log x$, (since $\frac{d}{du}(u+\frac{\log x}{u^\theta}) = 1 - \frac{\theta\log x}{u^{\theta+1}}$). Therefore, $u + \frac{\log x}{u^\theta} = u(1+\frac{\log x}{u^{\theta+1}}) \ge (\theta\log x)^{\frac{1}{1+\theta}}(1+\frac{1}{\theta}) = \frac{(\log x)^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$, and so

$$\frac{x^{1-a_n}}{b_n} \le x\exp\{-(\log x)^\alpha\}.$$

Hence,

$$\sum_{n>n_0}\mu_n\left|\int_1^x \frac{\cos(b_n\log t)}{t^{a_n}\log t}dt\right| \le \sum_{n>n_0}3\mu_n x e^{-(\log x)^\alpha} = O(xe^{-(\log x)^\alpha}).$$

This proves equation (4.2). □

We estimate $\mathcal{N}_{\mathcal{P}}(x)$ through the associated zeta function $\zeta_{\mathcal{P}}$ defined in Chapter 3 for $s=\sigma+it$ with $\sigma>1$. We have for $x>1$

$$\mathcal{N}_{\mathcal{P}}(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\zeta_{\mathcal{P}}(s)\frac{x^s}{s}ds, \quad b>1,$$

at all points of continuity of $\mathcal{N}_{\mathcal{P}}(x)$. The main difficulty will be to show (4.3), that is to find the $\Omega-$result for $\mathcal{N}_{\mathcal{P}}$. The proof forms the rest of this section.

Now, let $\mathcal{M}_{\mathcal{P}}(x) = \int_1^x \mathcal{N}_{\mathcal{P}}(t)dt$. Then for $x > 1$

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds, \quad b > 1.$$

We already know that (4.1) holds for some $\beta \geq \frac{\alpha}{1+\alpha}$ (see result 5 in 3.3). So, to prove that equation (4.3) is true it suffices to show that for some positive constants c, ρ

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho}{2}x^2 + \Omega_{\pm}\left(x^2 e^{-c(\log x)^{\alpha}}\right). \tag{4.5}$$

Actually, if (4.3) does not hold then

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + o(xe^{-c(\log x)^{\alpha}}),$$

so that,

$$\mathcal{M}_{\mathcal{P}}(x) = \int_1^x \{\rho t + o(te^{-c(\log t)^{\alpha}})\}dt = \frac{\rho}{2}x^2 + o\left(x^2 e^{-c(\log x)^{\alpha}}\right),$$

which contradicts (4.5). So, (4.3) must hold if (4.5) holds. Our aim is therefore to prove that (4.5) is true for some $c, \rho > 0$. For this purpose we estimate the integral of $\mathcal{M}_{\mathcal{P}}(x)$ and the simplest way to do so is by calculating the contribution of the singularities of the integrand $g(s) = \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}$. We rewrite $\zeta_{\mathcal{P}}(s)$ as an infinite product to enable us to read off the singularities of $g(s)$. The sequences $\{a_n\}$ and $\{b_n\}$ are defined earlier will give us the position of the singularities of $\zeta_{\mathcal{P}}(s)$ in the complex plane, and from this we can deduce the statement (4.5). Extend the sequences a_n, b_n and μ_n by defining for $n > n_0$, $a_{-n} = a_n$, $b_{-n} = -b_n$ and $\mu_{-n} = \mu_n$. Then we use the following proposition to rewrite the zeta function as required.

Proposition 4.3. *For* $\Re(s) > 1$,

$$\zeta_{\mathcal{P}}(s) = \frac{s+k-1}{s-1}\prod_{|n|>n_0}\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{\frac{\mu_n}{2}}. \tag{4.6}$$

Remark: Recall the definition of $\gamma(t)$ and let

$$\gamma_N(t) = 1 - \sum_{n_0 < n \leq N} \mu_n \frac{\cos(b_n \log t)}{t^{a_n}}, \quad t \geq 1.$$

Then $\gamma_N(t)$ converges uniformly to $\gamma(t)$ for $t \geq 1$ since $|\gamma(t) - \gamma_N(t)| \leq \sum_{n>N} \frac{2}{n^2} \leq \frac{2}{N}$.

Proof of Proposition 4.3. We have

$$\frac{\cos(b \log t)}{t^a} \cdot \frac{1 - t^{-k}}{\log t} = \frac{1}{2}(t^{-a+ib} + t^{-a-ib}) \frac{1 - t^{-k}}{\log t}.$$

So, for $\Re(s) > 1$, we have

$$-\frac{d}{ds} \int_1^\infty t^{-s} \frac{\cos(b \log t)}{t^a} \cdot \frac{1 - t^{-k}}{\log t} dt$$

$$= \frac{1}{2} \int_1^\infty (t^{-s-a-ib} + t^{-s-a+ib} - t^{-s-a-ib-k} - t^{-s-a+ib-k}) dt$$

$$= \frac{1}{2} \left\{ \frac{1}{s-1+a+ib} - \frac{1}{s-1+a+ib+k} + \frac{1}{s-1+a-ib} - \frac{1}{s-1+a-ib+k} \right\}$$

$$= \frac{1}{2} \left\{ \frac{d}{ds} \log \left(\frac{s-1+a+ib}{s-1+a+ib+k} \right) + \frac{d}{ds} \log \left(\frac{s-1+a-ib}{s-1+a-ib+k} \right) \right\}$$

$$= \frac{d}{ds} \log \left\{ \left(1 - \frac{k}{s-1+a+ib+k}\right)^{\frac{1}{2}} \left(1 - \frac{k}{s-1+a-ib+k}\right)^{\frac{1}{2}} \right\}.$$

Hence, we have

$$-\int_1^\infty t^{-s} \frac{\cos(b \log t)}{t^a} \cdot \frac{1 - t^{-k}}{\log t} dt$$

$$= \log \left\{ \left(1 - \frac{k}{s-1+a+ib+k}\right)^{\frac{1}{2}} \left(1 - \frac{k}{s-1+a-ib+k}\right)^{\frac{1}{2}} \right\} + \textit{constant}.$$

By taking the limit as $\Re(s)$ tends to infinity we see that the constant of integration is zero. Taking $a = b = 0$ gives

$$\int_1^\infty t^{-s} \frac{1 - t^{-k}}{\log t} dt = \log \frac{s+k-1}{s-1}.$$

Thus from the definition of $\gamma_N(t)$, we get

$$\int_1^\infty t^{-s} \frac{1 - t^{-k}}{\log t} \gamma_N(t) dt = \log \left(\frac{s+k-1}{s-1} \right) + \sum_{n_0 < |n| \leq N} \mu_n \log \left(1 - \frac{k}{s-1+a_n-ib_n+k} \right)^{\frac{1}{2}}$$

$$= \log \left\{ \frac{s+k-1}{s-1} \prod_{n_0<|n|<N} \left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{\frac{\mu_n}{2}} \right\}.$$

By taking the limit as $N \to \infty$, we conclude the proof since $\gamma_N(t) \to \gamma(t)$ as $N \to \infty$ and $\log \zeta_{\mathcal{P}}(s) = \int_1^\infty t^{-s} \; d\Pi_{\mathcal{P}}(t)$. $\square$

The representation of $\zeta_{\mathcal{P}}(s)$ given by (4.6) holds not only in the half plane $\Re(s) > 1$, but also in a larger region. Let $\mathcal{D}_\zeta$ be the region defined by

$$\mathcal{D}_\zeta = \{s = \sigma + it \in \mathbb{C} : \sigma > -k+2, s \neq \xi(1-a_n+ib_n) + (1-\xi)(1-a_n+ib_n-k),$$
$$\text{for any } 0 \leq \xi \leq 1, |n| \geq n_0\}.$$

By a theorem of Weierstrass on the uniform convergence of analytic functions, the function

$$\varphi(s) = \prod_{|n|>n_0} \left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{\frac{\mu_n}{2}},$$

is analytic in $\mathcal{D}_\zeta$. The equation

$$\zeta_{\mathcal{P}}(s) = \frac{s+k-1}{s-1}\varphi(s), \;\; \sigma > 1,$$

gives us an analytic continuation of $\zeta_{\mathcal{P}}(s)$ to $\mathcal{D}_\zeta$ with $s = 1$ removed, where $\zeta_{\mathcal{P}}(s)$ has a simple pole. Notice that, since the zeros of $\varphi(s)$ are of fractional order, we avoid problems of multiple-valuedness by restricting the domain of definition of $\zeta_{\mathcal{P}}(s)$ to $\mathcal{D}_\zeta$. We try to give a suitable upper bound for $|\zeta_{\mathcal{P}}(s)|$ in the extended domain of definition. For this purpose we need the following

Proposition 4.4. *If $s = \sigma + it$ is such that $\sigma > -k+2, \mu = \sum_{n>n_0} \mu_n$, and $s \in \mathcal{D}_\zeta$, then*

$$|\varphi(s)| \leq (k+1)e^{\mu}.$$

Proof. For $s = \sigma + it$, we find an upper bound for $\varphi(s)$ which holds for arbitrary positive sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ is decreasing to zero and $\{b_n\}$ is increasing to ∞. We have

$$\begin{aligned} b_{n+1} - b_n &= \exp\{(\log x_{n+1})^\alpha\} - \exp\{(\log x_n)^\alpha\} \\ &= \exp\{(\log x_n)^{\omega\alpha}\} - \exp\{(\log x_n)^\alpha\} \geq \delta, \;\; (\text{some } \delta > 0), \end{aligned}$$

where δ depends on α and ω. So, we choose ω sufficiently large such that $\delta \geq 2k$. Therefore the interval $(t-2k, t+2k)$ contains at most one element of $\{b_n\}$. We call this element (if exists) by $b_n(t)$, so we can write

$$|\varphi(s)| = \left|1 - \frac{k}{s-1+a_{n(t)}-ib_{n(t)}+k}\right|^{\frac{\mu_{n(t)}}{2}} \prod_{|n|>n_0, n\neq n(t)} \left|1 - \frac{k}{s-1+a_n-ib_n+k}\right|^{\frac{\mu_n}{2}}.$$

Since $a_n > 0$, we have $\sigma - 1 + k > 1$ and hence

$$\left|1 - \frac{k}{s-1+a_{n(t)}-ib_{n(t)}+k}\right|^{\frac{\mu_{n(t)}}{2}} \leq \left|1 + \frac{k}{\sigma-1+k}\right|^{\frac{\mu_{n(t)}}{2}} \leq 1+k.$$

Now, when $n \neq n(t)$,

$$\begin{aligned}
\left|1 - \frac{k}{s-1+a_n-ib_n+k}\right|^{\frac{\mu_n}{2}} &= \exp\left\{\frac{\mu_n}{2}\log\left|1 - \frac{k}{s-1+a_n-ib_n+k}\right|\right\} \\
&= \exp\left\{\frac{\mu_n}{2}\Re\log\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)\right\} \\
&= \exp\left\{\frac{\mu_n}{2}\Re\Big(-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots\Big)\right\},
\end{aligned}$$

where

$$|z| = \left|\frac{k}{s-1+a_n-ib_n+k}\right| \leq \frac{k}{|\Im(s)-b_n|} = \frac{k}{|t-b_n|} \leq \frac{k}{\delta} \leq \frac{k}{2k} = \frac{1}{2}.$$

Therefore

$$\begin{aligned}
|\varphi(s)| &\leq (k+1)\prod_{|n|>n_0, n\neq n(t)} \exp\left\{\frac{\mu_n}{2}\Big(|z| + \left|\frac{z^2}{2}\right| + \left|\frac{z^3}{3}\right| + \cdots\Big)\right\} \\
&\leq (k+1)\exp\left\{\frac{1}{4}\sum_{|n|>n_0}\mu_n\Big(1+\frac{1}{2}+\frac{1}{4}+\cdots\Big)\right\} \leq (k+1)e^{\mu},
\end{aligned}$$

as required. □

For $k=4, n_0=3$ and $\mu_n = 2n^{-2}$ we have

$$|\varphi(s)| \leq 5\exp\left\{\sum_{n>3}\frac{2}{n^2}\right\} < 9, \quad \text{if } \sigma > -2.$$

Corollary 4.5. *For $s \in \mathcal{D}_\zeta$ such that $|s-1|>1$ we have $|\zeta_{\mathcal{P}}(s)| \leq 45$.*

Proof.

$$|\zeta_{\mathcal{P}}(s)| = \left|\frac{s+k-1}{s-1}\varphi(s)\right| \le 9\left|1+\frac{k}{s-1}\right| \le 9\left(1+\frac{4}{|s-1|}\right)$$

. ☐

We need to find an $\Omega-$result for $\mathcal{M}_{\mathcal{P}}(x)$. In order to do this we estimate $\mathcal{M}_{\mathcal{P}}(x)$ at some particular sequence of x. We shall take x to be

$$x = x_n\left(1+\frac{r}{\log x_n}\right), \quad \text{where} \quad -1 \le r \le 1. \tag{4.7}$$

Note. Our choice of r (and hence x) is such that $\mathcal{M}_{\mathcal{P}}(x)$ equals the main term $\frac{\varrho}{2}x^2$ plus a large positive error term for some $r > 0$ and large negative error term for some $r < 0$.

We deform the vertical path of integration in the inversion formula

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds, \quad b > 1,$$

from the path $\Re(s) = b > 1$ to the left (see Figure 4.1).

Let $T_n = \exp\{(\log x_n)^\tau\}$, where $0 < \alpha < \tau < 1$. We remark here that the method works with any $\alpha < \tau < 1$, but for more convenience, we put $\tau = \frac{2\alpha}{\alpha+1}$ to fit in with the discrete case which comes later. Here Γ_1 joins $b - i\infty$ to $b - iT_n$. The points $b - iT_n$ to $-\frac{3}{2} - iT_n$ are joined by Γ_2. The segments Γ_5 and Γ_4 are symmetric to Γ_1 and Γ_2 with respect to the horizontal axis. We denote by Γ_3^* a comb formed by horizontal loops C_m, $n_0 < |m| \le n$, each going around the singular point $1 - a_m + b_m$. The collection of vertical line segments joining one loop to the next one is denoted by Γ_3. The points on Γ_3 have real part equal to $-\frac{3}{2}$. Furthermore, each C_m is made up of two horizontal line segments joined at the right hand side by small circle with centre at $1-a_m+ib_m$. The two horizontal line segments of C_m are extended to the left until they meet Γ_3. It is worthwhile pointing out that T_n lies between b_n and b_{n+1} (that is, between T_n and T_{n+1} there is one singular point of our zeta function), since $\log T_n > (\log x_n)^\alpha = \log b_n$, while,

$$\log b_{n+1} = (\log x_{n+1})^\alpha = (\log x_n)^{\alpha\omega} > (\log x_n)^\tau = \log T_n.$$

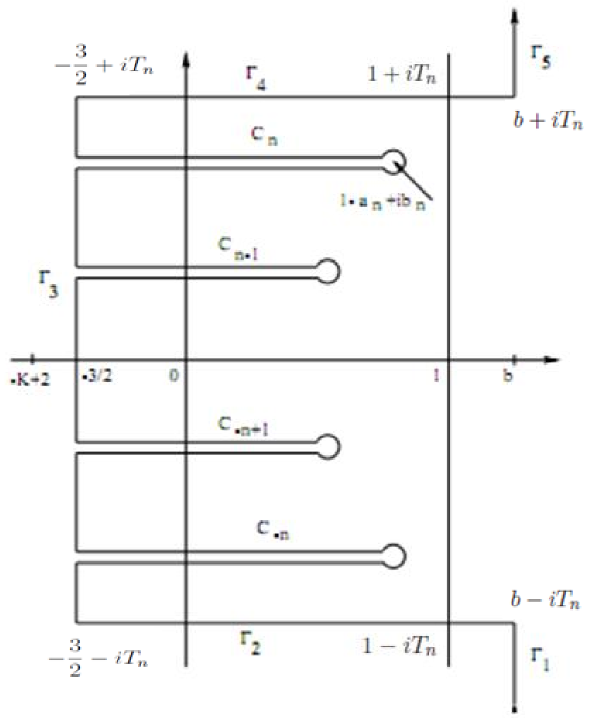

Figure 4.1:

Now we write

$$\mathcal{M}_{\mathcal{P}}(x) = I_1 + \cdots + I_5 + \sum_{n_0<|m|\leq n} J_m + \{k\varphi(1)\frac{x^2}{2} + x(1-k)\varphi(0)\},$$

where

$$I_m = \frac{1}{2\pi i}\int_{\Gamma_m} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds, \quad m = 1, 2, ..., 5,$$

$$J_m = \frac{1}{2\pi i}\int_{C_m} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds, \quad n_0 < |m| \leq n.$$

Here, as above, C_m is the mth horizontal loop with imaginary part equal to b_m. Consider first the integral I_3. In fact, we do not have one integral but many of them. This is because the vertical segment Γ_3 is broken at each horizontal loop C_m. However, on each vertical component of Γ_3 the integrand is bounded by the same constant which is 45. Thus, since $\mathcal{R}(s) = -\frac{3}{2}$ on Γ_3, we have

$$|I_3| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} 45\frac{x^{-\frac{3}{2}+1}}{\left|(\frac{3}{2}+it)(-\frac{1}{2}+it)\right|}dt = O(\frac{1}{\sqrt{x}}). \tag{4.8}$$

Let $b = 1 + \frac{1}{\log x_n}$. Then $|I_2|$ and $|I_4|$ are both $O\big((\frac{x}{T_n})^2\big)$, since

$$|I_2|\,, |I_4| \leq \frac{1}{2\pi}\int_{-\frac{3}{2}}^{1+(\log x_n)^{-1}} 45\frac{x^{2+(\log x_n)^{-1}}}{T_n^2}d\sigma \leq \frac{8}{T_n^2}x^{2+(\log x_n)^{-1}} = O(\frac{x^2}{T_n^2}). \quad (4.9)$$

Now we consider the integrals I_1 and I_5: Each of $|I_1|$ and $|I_5|$ is at most

$$\frac{1}{2\pi}\int_{T_n}^{\infty} 45\frac{x^{2+(\log x_n)^{-1}}}{t^2}dt \leq 8x^2 \exp\left\{1 + \left(\frac{1}{\log x_n}\right)^2\right\}\frac{1}{T_n} = O(\frac{x^2}{T_n}). \quad (4.10)$$

Therefore, we get

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + \sum_{n_0<|m|\leq n-1} J_m + \{J_{-n} + J_n\} + O\left(\frac{x^2}{T_n}\right). \quad (4.11)$$

We estimate each term in the right hand side of (4.11) separately. Since $\log x = \log x_n + o(1)$ we get

$$\frac{x^2}{T_n} = x^2\exp\{(\log x_n)^{\tau}\} = x^2\exp\{(\log x)^{\tau} + o(1)\}.$$

From this and from equation (4.11) we get

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + \sum_{n_0<|m|\leq n-1} J_m + \{J_{-n} + J_n\} + O\left(x^2 e^{-(\log x)^{\tau}}\right), \quad (4.12)$$

Proposition 4.6.

$$\sum_{n_0<|m|\leq n-1} J_m = O\left(x^2 e^{-(\log x)^{1-\frac{1-\alpha}{\omega}}}\right).$$

Proof. Let us consider the integral J_m and let γ_m be the circle centred at $1 - a_m + ib_m$ with the radius δ_1 parameterised by $\gamma_m(\vartheta) = 1 - a_m + ib_m + \delta_1 e^{i\vartheta}$, where $0 \leq \vartheta \leq 2\pi$. Therefore we have

$$\left|\int_{\gamma_m} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds\right| \leq 2\pi\delta_1 \sup_{s\in\gamma_m}\left|\zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}\right| \to 0,$$

as $\delta_1 \to 0$. Let $\delta_1 \to 0$, so we can write

$$|J_m| = \left|\frac{1}{2\pi i}\int_{C_m} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds\right| \leq \frac{1}{\pi}\int_{-\frac{3}{2}}^{1-a_m} 45\frac{x^{2-a_m}}{b_m^2}d\sigma \leq \frac{15x^2}{b_m^2}e^{-a_m\log x}.$$

But if $|m| \leq n-1$ then

$$e^{-a_m\log x} \leq e^{-a_{n-1}\log x} = \exp\left\{-\frac{\log x}{(\log x_{n-1})^{1-\alpha}}\right\} = \exp\left\{-\frac{\log x}{(\log x_n)^{\frac{1-\alpha}{\omega}}}\right\}.$$

$$\leq 2 \exp \left\{ -(\log x)^{1-\frac{1-\alpha}{\omega}} \right\}.$$

Hence,

$$\sum_{n_0 < |m| \leq n-1} J_m \leq 30\, x^2 e^{-(\log x)^{1-\frac{1-\alpha}{\omega}}} \sum_{|m|>n_0} \frac{1}{b_m^2} = O\left(x^2 e^{-(\log x)^{1-\frac{1-\alpha}{\omega}}} \right).$$

$\square$

We see that $1 - \frac{1-\alpha}{\omega} \geq \frac{2\alpha}{\alpha+1} = \tau$ since ω is taken sufficiently large, so equation (4.12) becomes

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + \{J_{-n} + J_n\} + O\left(x^2 e^{-(\log x)^{\tau}}\right), \tag{4.13}$$

It remains to study the expression $J_{-n} + J_n$. Denote by $J_n^{'}$ and $J_n^{''}$ the integrals along the line segments $C_n^{'}$ $C_n^{''}$ lying respectively above and below the branch cut C_n so that $J_n = J_n^{'} + J_n^{''}$. Now, if we write

$$s = 1 - a_n + ib_n + te^{i\theta},\ \ -\pi \leq \theta < \pi,$$

then the line segment $C_n^{''}$ is obtained by letting $\theta = -\pi$ and t run from 0 to $1 - a_n + \frac{3}{2}$. In this way we obtain $C_n^{''}$ with its direction reversed:

$$-C_n^{''} : \begin{cases} \theta = -\pi, \\ s = 1 - a_n + ib_n - t, \\ ds = -dt, \\ 0 \leq t \leq 1 - a_n + \frac{3}{2}. \end{cases}$$

To estimate J_n, split up as

$$J_n^{''} = \frac{1}{2\pi i} \left\{ \int_0^{(\log x)^{-\epsilon}} + \int_{(\log x)^{-\epsilon}}^{1-a_n+\frac{3}{2}} \right\} \frac{\zeta_{\mathcal{P}}(1-a_n+ib_n-t) x^{2-a_n+ib_n-t}}{(1-a_n+ib_n-t)(2-a_n+ib_n-t)}(dt), \tag{4.14}$$

where ϵ is arbitrary positive number. The second integral over $((\log x)^{-\epsilon}, 1-a_n+\frac{3}{2})$ is bounded in modulus by

$$\frac{45x^{2-a_n}}{2\pi b_n^2} \int_{(\log x)^{-\epsilon}}^{1-a_n+\frac{3}{2}} x^{-t} dt \leq \frac{45x^{2-a_n} e^{-(\log x)^{1-\epsilon}}}{2\pi b_n^2 \log x} = O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right).$$

To deal with the integral over $(0, (\log x)^{-\epsilon})$ rewrite the integrand as follows:

$$\frac{\zeta_{\mathcal{P}}(s)}{s(s+1)} = (s-1+a_n-ib_n)^{\frac{\mu_n}{2}} f_n(s), \quad \text{say},$$

where

$$f_n(s) = \frac{(s+3)}{s(s+1)(s-1)(s+a_n-ib_n+3)^{\frac{\mu_n}{2}}} \prod_{|m|>n_0, m\neq n} \left(1 - \frac{4}{s+a_m-ib_m+3}\right)^{\frac{\mu_m}{2}}. \tag{4.15}$$

Here $f_n(s)$ is analytic in a disc around the point $z_n = 1-a_n+ib_n$. Therefore we can write

$$f_n(s) = \sum_{j=0}^{\infty} \frac{f_n^{(j)}(z_n)}{j!}(s-z_n)^j.$$

The series is convergent if $|s-z_n| < \delta_0$ for some $\delta_0 > 1$. Let $a_{n,j} = \frac{f_n^{(j)}(z_n)}{j!}$. So, the integrand in (4.14) becomes

$$x^{2-a_n+ib_n-t}(te^{-i\pi})^{\frac{\mu_n}{2}} f_n(1-a_n+ib_n-t).$$

Thus, we have

$$\begin{aligned} J_n'' &= \frac{1}{2\pi i} x^{2-a_n+ib_n} e^{\frac{-i\pi\mu_n}{2}} \int_0^{(\log x)^{-\epsilon}} x^{-t} t^{\frac{\mu_n}{2}} f_n\left(1-a_n+ib_n-t\right) dt + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right) \\ &= \frac{1}{2\pi i} x^{2-a_n+ib_n} \frac{e^{\frac{-i\pi\mu_n}{2}}}{(\log x)^{\frac{\mu_n}{2}+1}} \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} f_n\left(1-a_n+ib_n-\frac{t}{\log x}\right) dt \\ &\qquad + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right) \end{aligned}$$

$$= \frac{1}{2\pi i} x^{2-a_n+ib_n} \frac{e^{\frac{-i\pi\mu_n}{2}}}{(\log x)^{\frac{\mu_n}{2}+1}} S_n + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right), \tag{4.16}$$

where

$$S_n = \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} \sum_{j=0}^{\infty} a_{n,j} \left(\frac{-1}{\log x}\right)^j dt.$$

Similarly, we can obtain

$$J_n' = -\frac{1}{2\pi i} x^{2-a_n+ib_n} \frac{e^{\frac{i\pi\mu_n}{2}}}{(\log x)^{\frac{\mu_n}{2}+1}} S_n + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right). \tag{4.17}$$

Since $J_n = J_n' + J_n''$, becomes

$$J_n = -\frac{\sin\frac{\pi\mu_n}{2} \quad x^{2-a_n+ib_n}}{\pi(\log x)^{\frac{\mu_n}{2}+1}} S_n + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right). \tag{4.18}$$

Since $J_{-n} = \bar{J}_n$, we have

$$J_n + J_{-n} = (J_n + \bar{J}_n =)2\Re(J_n).$$

Our next step is to estimate the integral S_n appearing in (4.18). For this we obtain lower and upper bounds for $f_n(s)$ in $D(z_n, 1)$ (that is $|s - z_n| \leq 1$, $s = 1 - a_n + ib_n - y$). For the upper bound we notice that $|s| \geq b_n$. Thus

$$\left|\frac{(s+3)}{s(s+1)(s-1)}\right| \leq \frac{b_n+6}{(b_n-2)^3} \leq \frac{2b_n}{(b_n/2)^3} = \frac{16}{b_n^2}.$$

Also

$$|s + a_n - ib_n + 3|^{\frac{\mu_n}{2}} > (4 - |s - 1 + a_n - ib_n|)^{\frac{\mu_n}{2}} \geq 3^{\frac{\mu_n}{2}} \geq 1.$$

Now we want to estimate from above the product appearing in the definition of f_n in (4.15). As in the proof of Proposition 4.4 we have

$$\left|1 - \frac{4}{s + a_m - ib_m + 3}\right| \leq 1 + \frac{4}{|\Im(s) - b_m|} \leq 1 + \frac{k}{\delta} \leq 1 + \frac{k}{2k} = \frac{3}{2}, \quad \text{for} \quad m \neq n.$$

Thus the product in (4.15) is in modulus less than

$$\prod_{|m|>n_0, m\neq n} \left(\frac{3}{2}\right)^{\frac{\mu_m}{2}} \leq \prod_{|m|>n_0} \left(\frac{3}{2}\right)^{\frac{1}{m^2}} \leq \left(\frac{3}{2}\right)^{2\sum_{j=1}^{\infty}\frac{1}{j^2}} < 4.$$

Thus we have proved

Proposition 4.7. *For $|s - (1 - a_n + ib_n)| \leq 1$, then $|f_n(s)| \leq \frac{64}{b_n^2}$.*

This and Cauchy's inequalities give the following

Corollary 4.8. *For all $j = 1, 2, 3, ...$ $|a_{n,j}| \leq \frac{64}{b_n^2}$.*

Now we estimate the lower bound for $f_n(s)$ in $D(z_n, 1)$:

$$|s| \leq |s - 1 + a_n - ib_n| + |1 - a_n + ib_n| \leq 1 + 1 + |a_n| + |b_n| \leq 3 + b_n.$$

Thus

$$\left|\frac{(s+3)}{s(s+1)(s-1)}\right| \geq \frac{|s+3|}{(b_n+4)^3} \geq \frac{|s|-3}{(b_n+4)^3} \geq \frac{b_n-1-3}{(b_n+4)^3} \geq \frac{\frac{1}{2}b_n}{(2b_n)^3} = \frac{1}{16b_n^2}.$$

Each term in the infinite product in (4.15) is

$$\left|1-\frac{4}{s+a_m-ib_m+3}\right| \geq 1-\frac{4}{|s+a_m-ib_m+3|} \geq 1-\frac{4}{|\Im(s)-b_m|} \geq 1-\frac{k}{\delta} \geq 1-\frac{k}{2k} = \frac{1}{2}.$$

Therefore

$$\prod_{|m|>n_0, m\neq n} \left|1-\frac{4}{s+a_m-ib_m+3}\right|^{\frac{\mu_m}{2}} \geq \prod_{|m|>n_0} \left(\frac{1}{2}\right)^{\frac{1}{m^2}} > \frac{1}{10}.$$

Thus we have

Proposition 4.9. *For $|s-(1-a_n+ib_n)| \leq 1$, we have*

$$|f_n(s)| \geq \frac{1}{160b_n^2}.$$

With all these inequalities we can estimate the integral S_n, the function occurring in (4.18), as follows:

$$\begin{aligned} S_n &= \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} f_n\left(1-an+ib_n-\frac{t}{\log x}\right) dt \\ &= \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} \sum_{j=0}^{\infty} a_{n,j} \left(\frac{-t}{\log x}\right)^j dt \\ &= a_{n,0} \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} dt + \sum_{j=1}^{\infty} a_{n,j} \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} \left(\frac{-t}{\log x}\right)^j dt. \end{aligned}$$

For the second term we get, by Corollary 4.8,

$$\begin{aligned} \left|\sum_{j=1}^{\infty} a_{n,j} \int_0^{(\log x)^{1-\epsilon}} e^{-t} t^{\frac{\mu_n}{2}} \left(\frac{-t}{\log x}\right)^j dt\right| &\leq \sum_{j=1}^{\infty} \frac{64}{b_n^2} \left(\frac{1}{\log x}\right)^{j\epsilon} \int_0^{\infty} e^{-t} t^{\frac{\mu_n}{2}} dt \\ &\leq \left(\frac{1}{\log x}\right)^{\epsilon} \frac{64}{b_n^2} \sum_{j=0}^{\infty} \left(\frac{10}{98}\right)^{j\epsilon} \leq 148 \left(\frac{1}{\log x}\right)^{\epsilon} \frac{1}{b_n^2}. \end{aligned}$$

Now since

$$\int_{(\log x)^{1-\epsilon}}^{\infty} e^{-t} t^{\frac{\mu_n}{2}} dt \leq \int_{(\log x)^{1-\epsilon}}^{\infty} e^{-t} t dt \leq 2\log x e^{-(\log x)^{1-\epsilon}},$$

The integral S_n in (4.18) is

$$S_n = a_{n,0}\left(\Gamma(\frac{1}{2}\mu_n + 1) + O\left(\log x e^{-(\log x)^{1-\epsilon}}\right)\right) + O\left(\frac{1}{b_n^2(\log x)^{\epsilon}}\right). \tag{4.19}$$

Since $a_{n,0} = f_n(1 - a_n + ib_n)$ and $\Gamma(\frac{1}{2}\mu_n + 1) \to 1$ as $n \to \infty$, from Proposition 4.9 we find

$$|S_n| \geq \frac{d_0}{b_n^2}\left(1 - 2\log x e^{-(\log x)^{1-\epsilon}} - \frac{d_1}{(\log x)^{\epsilon}}\right) \geq d e^{-2(\log x)^{\alpha}}, \quad d > 0$$

for some $d_0, d_1 > 0$ and for x sufficiently large, that is for n is sufficiently large (since x is a sequence depending on n). We use this lower bound of the integral S_n appearing in equation (4.18). Now consider the other factor in that equation,

$$\frac{\sin\frac{\pi\mu_n}{2}}{\pi} x^{2-a_n}\left(\frac{1}{\log x}\right)^{\frac{\mu_n}{2}+1} \geq \frac{\mu_n}{\pi} x^2 e^{-a_n \log x} \cdot \frac{1}{2(\log x)^2}$$

$$\geq a x^2 e^{-\frac{\log x}{(\log x_n)^{1-\alpha}}} \cdot \frac{1}{(\log x)^2(\log\log x_n)^2}, \qquad \text{for some} \quad a > 0,$$

using $\mu_n = \frac{1}{n^2}$ and $n \leq \frac{\log\log x_n}{\omega - 1}$. From the above bound on S_n and (4.18), we get

$$|J_n| \geq a x^2 \exp\left\{-\left(\frac{c_0 \log x}{(\log x_n)^{1-\alpha}} + 2(\log x_n)^{\alpha}\right)\right\} \geq a x^2 e^{-c(\log x)^{\alpha}}, \;\; 0 < \alpha < 1, \tag{4.20}$$

for some constants $a, c_0, c > 0$ and for sufficiently large n.

Our aim is to obtain large values for $2\Re(J_n)$ compared with the other error term of (4.13). For this purpose we recall equation (4.18)

$$J_n = J_n^{'} + J_n^{''} = -\frac{\sin\frac{\pi\mu_n}{2} \quad x^{2-a_n+ib_n}}{\pi(\log x)^{\frac{\mu_n}{2}+1}} S_n + O\left(x^2 e^{-(\log x)^{1-\epsilon}}\right),$$

We can rewrite the above equation as follows,

$$A = \frac{J_n}{x^{2-a_n}} = B x^{ib_n} + C,$$

where $B = -\frac{\sin(\frac{\pi\mu_n}{2})}{\pi}\left(\frac{1}{\log x}\right)^{\frac{\mu_n}{2}+1} S_n$, and $C = O\left(x^{a_n} e^{-(\log x)^{1-\epsilon}}\right)$. We get

$$\Re\left\{\frac{A - C}{|B|}\right\} = \Re\left(\exp\{ib_n \log x + i \arg B\}\right) = \cos(b_n \log x + \arg B)$$

That is,

$$\Re\left\{\frac{A - C}{|B|}\right\} = \cos\left(\left(b_n \log x_n + \arg B\right) + b_n \log\left(1 + \frac{r}{\log x_n}\right)\right). \tag{4.21}$$

From definition of B we have $\arg B = \arg S_n + \pi$. The main term (involving the Γ function) on the right hand side of equation (4.19) is independent of r. Now, as r runs from -1 to $+1$, the argument of S_n (and therefore $\arg B$) does not exceed 2π, since the last two terms are much smaller than the first one. This tells us that, as r runs from -1 to $+1$,

$$\Big(b_n \log x_n + \arg B\Big) + b_n \log\left(1 + \frac{r}{\log x_n}\right),$$

runs through an interval centred somewhere in

$$\Big(b_n \log x_n - 2\pi, b_n \log x_n + 2\pi\Big).$$

The highest point is at least

$$b_n \log x_n - 2\pi + b_n \log\left(1 + \frac{1}{\log x_n}\right),$$

whereas the lowest point is at most

$$b_n \log x_n + 2\pi + b_n \log\left(1 + \frac{-1}{\log x_n}\right).$$

Therefore the length of (4.21) is

$$\geq b_n \log\left(1 + \frac{1}{\log x_n}\right) - 4\pi \geq \frac{b_n}{1 + \log x_n} \quad - 4\pi \longrightarrow \infty, \;\; as \;\; n \longrightarrow \infty.$$

For large n we choose values (r^+ and r^-) of r appropriately such that

$$\Re\left\{\frac{A - C}{|B|}\right\} = +1 \;\; \text{and} \;\; \Re\left\{\frac{A - C}{|B|}\right\} = -1.$$

For the first case we have $\Re(\frac{J_n}{x^{2-a_n}}) = \Re(A) = |B| + \Re(C)$, that is,

$$\Re(J_n) = |B|\, x^{2-a_n} + \Re(C) x^{2-a_n}$$

$$\geq |J_n| - |C|\, x^{2-a_n} + \Re(C) x^{2-a_n}$$

$$= |J_n| + O(x^2 e^{-(\log x)^{1-\epsilon}})$$

$$\geq A_0 x^2 e^{-c(\log x)^{\alpha}}.$$

Therefore, for sufficiently large n we have

$$\Re(J_n) \geq A_0 x^2 e^{-c(\log x)^{\alpha}}, \;\; \text{for } r = r^+ \text{ and } \; A_0, c > 0. \tag{4.22}$$

Similarly we can get

$$\Re(J_n) \leq -A_0 x^2 e^{-c(\log x)^\alpha}, \text{ for } r = r^- \text{ and } A_0, c > 0. \tag{4.23}$$

From the above inequalities and the following equation

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + 2\Re(J_n) + O\left(x^2 e^{-(\log x)^{\frac{2\alpha}{\alpha+1}}}\right), \tag{4.24}$$

we have

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + \Omega_{\pm}(x^2 e^{-c(\log x)^\alpha}), \tag{4.25}$$

for some positive constant c. This proves (4.5) and hence (4.3). The proof of Theorem 4.1 is completed.

4.2 Discrete g-prime System

In the above section, we found a continuous g-prime system for which $\beta = \alpha$. Now we show that it may be adapted to give a discrete version. Finding discrete system satisfying this same property is generally more challenging. The reason for this is that if we have $\Pi_{\mathcal{P}}(x)$ defined as a step function, then seeing the singularities of the Beurling zeta function is difficult.

We shall use the method developed by Diamond, Montgomery, Vorhauer [11] and later Zhang [31] which uses (the theory of) probability measures to find discrete systems of Beurling primes.

Theorem 4.10. *Let $0 < \alpha < 1$. Then there is a discrete g-prime system $\mathcal{P}$ for which*

$$\pi^d_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^{\alpha}}), \tag{4.26}$$

and

$$\mathcal{N}^d_{\mathcal{P}}(x) = \rho x + \Omega_{\pm}(xe^{-c(\log x)^{\alpha}}), \tag{4.27}$$

for some positive constants ρ and c. Thus $\beta(\alpha) \leq \alpha$ for discrete systems.

To find the g-prime satisfying (4.26) we use the following lemmas from Zhang's paper [31].

Lemma 1. *Let $f(\nu)$ be a nonnegative-valued Lebesgue measurable function on $(-\infty, \infty)$ with support $[1, \infty)$. Assume that there is increasing function $F(x)$ on $(-\infty, \infty)$ with support $[1, \infty)$ satisfying*

$$\int_1^x f(\nu)d\nu \ll F(x),$$

$$\int_x^{x+1} f(\nu)d\nu \ll \sqrt{F(x)(1+\log x)},$$

$$\log x = o(F(x)),$$

$$\int_1^x \nu^{-1}\sqrt{F(\nu)}d\nu \ll \sqrt{F(x)},$$

and

$$F(x+1) \ll F(x).$$

Let

$$1 \le \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_k < \nu_{k+1} < \cdots$$

be a sequence such that $\nu_k \to \infty$ *as* $k \to \infty$ *and such that*

$$p_k = \int_{\nu_{k-1}}^{\nu_k} f(\nu) d\nu,$$

satisfies $0 < p_k < 1$ *for* $k > k_0$. *Then there is a subsequence* $\nu_{k_j}, j = 1, 2, ...$ *such that*

$$\sum_{\nu_{k_j} \le x} \nu_{k_j}^{it} - \sum_{\nu_k \le x} \nu_k^{it} p_k \ll \sqrt{F(x)} \left(\sqrt{1 + \log x} + \sqrt{\log(t+1)} \right), \tag{4.28}$$

for $1 \le x < \infty$ *and* $t \ge 0$.

Lemma 3. *If the sequence* ν_k *in* ***Lemma 1*** *satisfies also*

$$\sum_{\nu_k \le x} \nu_k^{it} p_k - \int_1^x \nu^{it} f(\nu) d\nu \ll \sqrt{F(x)} \left(\sqrt{1 + \log x} + \sqrt{\log(t+1)} \right), \tag{4.29}$$

for $F(x) \ge c \log(t+1)$ *with a constant* $c > 0$ *then there is a subsequence* $\nu_{k_j}, j = 1, 2, ...$ *such that*

$$\sum_{\nu_{k_j} \le x} \nu_{k_j}^{it} - \int_1^x \nu^{it} f(\nu) d\nu \ll \sqrt{F(x)} \left(\sqrt{1 + \log x} + \sqrt{\log(t+1)} \right), \tag{4.30}$$

for $1 \le x < \infty$ *and* $t \ge 0$.

Lemma 4. *Let* $f(x)$ *be a Lebesgue measurable function on* $(-\infty, \infty)$ *with support* $[1, \infty)$ *satisfying*

$$0 \le f(x) \ll \frac{1 - x^{-1}}{\log x}.$$

Then the function

$$F(x) = \frac{x}{1 + \log x},$$

satisfies the conditions of ***Lemma 1*** *and both the sequences*

$$(1) \quad \nu_k = \sqrt{\log(k + k_0)},\ k = 0, 1, 2, ...$$

and

$$(2) \quad \nu_k = \log(k + k_0) \log\log(k + k_0),\ k = 0, 1, 2, ...$$

*satisfy the conditions of **Lemma 1** and **Lemma 3**. Therefore both* (1) *and* (2) *have a subsequence* ν_{k_j}, $j = 1, 2, ...$ *satisfying*

$$\sum_{\nu_{k_j} \le x} \nu_{k_j}^{it} - \int_1^x \nu^{it} f(\nu) d\nu \ll \sqrt{x} \left(1 + \sqrt{\frac{\log(t+1)}{1 + \log x}} \right), \tag{4.31}$$

for $1 \le x < \infty$ *and* $t \ge 0$.

Now, consider the continuous function

$$h(\nu) = \frac{1 - \nu^{-k}}{\log \nu} \gamma(\nu), \quad \text{with } \gamma(\nu) = 1 - \sum_{n > n_0} \mu_n \frac{\cos(b_n \log \nu)}{\nu^{a_n}}, \quad \nu \ge 1.$$

That is, the function $h = \Pi_{\mathcal{P}}'$ where $\Pi_{\mathcal{P}}$ from Theorem 4.1. Here k, n_0, μ_n, b_n and a_n as in Theorem 4.1. The function $h(\nu) \ll \frac{1-\nu^{-1}}{\log \nu}$. So, by **Lemma 4** there is a sequence $1 \le \alpha_0 \le \alpha_1 \le \alpha_2 \cdot\cdot \le \alpha_j \le \alpha_{j+1} \le \cdots$ such that $\alpha_j \to \infty$ as $j \to \infty$ for which

$$\sum_{\alpha_j \le x} \alpha_j^{-it} - \int_1^x \nu^{-it} h(\nu) d\nu \ll \sqrt{x} \left(1 + \sqrt{\frac{\log(t+1)}{1 + \log x}} \right),$$

for $1 \le x < \infty$ and $t \ge 0$. In particular, when $t = 0$ we have

$$\sum_{\alpha_j \le x} 1 - \int_1^x h(\nu) d\nu = O(\sqrt{x}). \tag{4.32}$$

We shall take $\{\alpha_j\}_{j \ge 0}$ as our g-primes. By Proposition 4.2 we get

$$\pi_{\mathcal{P}}^d(x) := \sum_{\alpha_j \le x} 1 = \mathrm{li}(x) + O(xe^{-(\log x)^\alpha}) + O(\sqrt{x}) = \mathrm{li}(x) + O(xe^{-(\log x)^\alpha}),$$

with α as in section 4.1. We let

$$\Pi_{\mathcal{P}}^d(x) = \sum_{n \ge 1} \frac{\pi_{\mathcal{P}}^d(x^{\frac{1}{n}})}{n}.$$

Then

$$\Pi_{\mathcal{P}}^d(x) = \mathrm{li}(x) + O(xe^{-(\log x)^\alpha}), \tag{4.33}$$

since $\Pi_{\mathcal{P}}^d(x) = \pi_{\mathcal{P}}^d(x) + O(\sqrt{x})$. This proves (4.26). We estimate $\mathcal{N}_{\mathcal{P}}^d(x)$ through its associated zeta function $\zeta_{\mathcal{P}}^d$ given by

$$\begin{aligned} \zeta_{\mathcal{P}}^d(s) &= \int_{1-}^\infty x^{-s} d\mathcal{N}_{\mathcal{P}}^d(x) = \exp\left\{ \int_{1-}^\infty x^{-s} d\Pi_{\mathcal{P}}^d(x) \right\} \\ &= \exp\left\{ \int_{1-}^\infty - \log(1 - x^{-s}) d\pi_{\mathcal{P}}^d(x) \right\}. \end{aligned} \tag{4.34}$$

This can be written as

$$\zeta_{\mathcal{P}}^d(s) = \zeta_{\mathcal{P}}(s) \exp\{F_2(s) - F_1(s)\}, \tag{4.35}$$

where $\zeta_{\mathcal{P}}(s)$ as in Section 4.1 is analytic in $\mathcal{D}_\zeta$ and

$$F_1(s) = \int_1^\infty \left\{\nu^{-s} d\pi_{\mathcal{P}}^d(\nu) + \log(1-\nu^{-s}) d\pi_{\mathcal{P}}^d(\nu)\right\},$$

and

$$F_2(s) = \int_1^\infty \nu^{-s} \left\{d\pi_{\mathcal{P}}^d(\nu) - h(\nu)d\nu\right\}.$$

We see $\log(1-\nu^{-s}) = \nu^{-s} + O(\nu^{-2\sigma})$ for $\nu > 1$, which tells us that integral function $F_1(s)$ converges uniformly for $\sigma \geq \frac{1}{2} + \delta$ each $\delta > 0$. Therefore, $F_1(s)$ is analytic for $\sigma > \frac{1}{2}$. Similarly, so is $F_2(s)$ since $\Pi_{\mathcal{P}}^d(x) - \Pi_{\mathcal{P}}(x) = O(\sqrt{x})$, and hence $-F_1(s) + F_2(s)$ is holomorphic in the half-plane $H_{\frac{1}{2}} = \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$. Thus, $\zeta_{\mathcal{P}}^d(s)$ is analytic in $\mathcal{D}_\zeta \cap H_{\frac{1}{2}}$. Let

$$\mathcal{M}_{\mathcal{P}}^d(x) = \int_1^x \mathcal{N}_{\mathcal{P}}^d(t)dt.$$

Then

$$\mathcal{M}_{\mathcal{P}}^d(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\mathcal{P}}^d(s)\frac{x^{s+1}}{s(s+1)}ds, \quad b > 1.$$

To prove that equation (4.27) is true it suffices to show that for some positive constant c,

$$\mathcal{M}_{\mathcal{P}}^d(x) = \frac{\rho}{2}x^2 + \Omega_\pm\left(x^2 e^{-c(\log x)^\alpha}\right), \tag{4.36}$$

for some $\rho > 0$ and α as in Section 4.1. Our aim is to prove that (4.36) is true for some $\rho, c > 0$. For this purpose we estimate the integral of $\mathcal{M}_{\mathcal{P}}^d(x)$ and the simplest way to do so by calculating the singularities of the integrand $f(s) = \zeta_{\mathcal{P}}^d(s)\frac{x^{s+1}}{s(s+1)} = \zeta_{\mathcal{P}}(s)\exp\{F_2(s) - F_1(s)\}\frac{x^{s+1}}{s(s+1)}$. Since $F_1(s)$ and $F_2(s)$ are holomorphic for $\sigma > \frac{1}{2}$, the singularities of $f(s)$ are the same singularities of $\zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}$.

By Proposition (4.4) and Corollary (4.5) we have $|\zeta_{\mathcal{P}}(s)| \leq 45$. It remains to estimate $\exp\{F_2(s) - F_1(s)\}$. To do this we need the following modification of **Lemma 5** from Zhang's paper.

Lemma 4.11. *Let a function $F(x,t)$ defined for $1 \le x < \infty$ and $t \ge 0$ be locally of bounded variation in x and satisfy $F(1,t) = 0$ and*

$$F(x,t) \ll \sqrt{x}\left(1 + \sqrt{\frac{\log(t+1)}{1+\log x}}\right).$$

Given $x \ge x_0 > 1$, let $\sigma \ge \frac{1}{2} + \delta,\ \ \delta > 0$. Then

$$\int_1^\infty \nu^{-\sigma}\ \ dF(\nu,t) \ll \sqrt{\log(t+1)}.$$

Proof. Using integration by parts, the integral on the left hand side is

$$\sigma \int_1^\infty \nu^{-\sigma-1} F(\nu,t)\ \ d\nu \ll \int_1^\infty \nu^{-\sigma-\frac{1}{2}}\left(1 + \sqrt{\frac{\log(t+1)}{1+\log\nu}}\right) d\nu \ll \sqrt{\log(t+1)},$$

since $\int_1^\infty \nu^{-\sigma-\frac{1}{2}} d\nu$ converges. □

Let

$$g(x,t) = \sum_{\alpha_j \le x} \alpha_j^{-it} - \int_1^x \nu^{-it} h(\nu) d\nu,\ \ x \ge 1.$$

So, $g(x,t)$ satisfies the conditions of Lemma 4.11. Thus, by Lemma 4.11, we have

$$F_2(\sigma+it) = \int_1^\infty \nu^{-\sigma}\ \ (\nu^{-it}\ \ dg(\nu,0)) \ll \sqrt{\log t},\ \ t \ge 2, \tag{4.37}$$

for $\sigma \ge \frac{1}{2} + \delta,\ \ \delta > 0$.

We shall need to use $T_n = \exp\{(\log x_n)^\tau\}$, such that $0 < \alpha < \tau < 1$. Therefore,

$$F_2(\sigma + iT_n) = O((\log x_n)^{\frac{\tau}{2}}). \tag{4.38}$$

Also,

$$\begin{aligned} F_1(s) &= \int_1^\infty \left\{\nu^{-s} d\pi_{\mathcal{P}}^d(\nu) + \log(1-\nu^{-s}) d\pi_{\mathcal{P}}^d(\nu)\right\} \\ &= \int_1^\infty \nu^{-s} d\pi_{\mathcal{P}}^d(\nu) - \sum_{m \ge 1} \frac{1}{m} \int_1^\infty \nu^{-ms} d\pi_{\mathcal{P}}^d(\nu) \\ &= -\sum_{m \ge 2} \frac{1}{m} \int_1^\infty \nu^{-ms} d\pi_{\mathcal{P}}^d(\nu). \end{aligned}$$

This shows that the integral for $F_1(s)$ converges unifomly for $\sigma \ge \frac{1}{2} + \delta$ with each $\delta > 0$. Therefore,

$$F_1(s) = O(1), \quad \text{for} \quad \sigma \ge \frac{1}{2} + \delta,\ \ \delta > 0. \tag{4.39}$$

Hence, we see from equations (4.38) and (4.39) that for $s = \sigma + iT_n$, we have

$$\Re\left\{F_2(s) - F_1(s)\right\} \le |F_2(s) - F_1(s)| \le b(\log x_n)^{\frac{\tau}{2}},\ b > 0. \tag{4.40}$$

This tells us that

$$\Re\left\{F_2(s) - F_1(s)\right\} \ge -b(\log x_n)^{\frac{\tau}{2}}, \ \text{ for some } b > 0. \tag{4.41}$$

From (4.37) and (4.39) we have proved the following

Corollary 4.12. *For $\sigma + it \in \mathcal{D}_\zeta \cap H_{\frac{1}{2}}$, we have*

$$\zeta_{\mathcal{P}}^d(\sigma + it) = O(e^{b\sqrt{\log t}}).$$

Our aim is to find an Ω-result for $M_{\mathcal{P}}^d(x)$. In order to do this we estimate $M_{\mathcal{P}}^d(x)$ at some particular sequence of x. We shall take x to be as in (4.7). Following the same method as in 4.1 we obtain

$$M_{\mathcal{P}}^d(x) = I_1^d + \cdots + I_5^d + \sum_{n_0 < |m| \le n} J_m^d + \{k\varphi(1)\frac{x^2}{2} + x(1-k)\varphi(0)\}, \tag{4.42}$$

where $\{k\varphi(1)\frac{x^2}{2} + x(1-k)\varphi(0)\}$ means the residues at $s = 0, 1$ as in 4.1, so that

$$I_m^d = \frac{1}{2\pi i}\int_{\Gamma_m} \zeta_{\mathcal{P}}^d(s)\frac{x^{s+1}}{s(s+1)}ds, \quad m = 1, 2, ..., 5,$$

and

$$J_m^d = \frac{1}{2\pi i}\int_{C_m} \zeta_{\mathcal{P}}^d(s)\frac{x^{s+1}}{s(s+1)}ds, \quad n_0 < |m| \le n.$$

Where C_m is the mth horizontal loop with imaginary part equal to b_m, (see Figure 4.1). By (4.8), (4.9) and (4.10) with $\zeta_{\mathcal{P}}^d$ instead of $\zeta_{\mathcal{P}}$ we have

$$\begin{aligned} I_1^d + \cdots + I_5^d &= O\left(\frac{x^2}{T_n}e^{(\log x_n)^{\frac{\tau}{2}}}\right) \\ &= O\left(x^2 \exp\left\{(\log x_n)^{\frac{\tau}{2}} - (\log x_n)^\tau\right\}\right) = O\left(x^2 e^{-c(\log x_n)^\tau}\right), \end{aligned}$$

for some $c > 0$, since $(\log x_n)^{\frac{\tau}{2}} = o\left((\log x_n)^\tau\right)$. We put $\tau = \frac{2\alpha}{\alpha+1}$ as in section 4.1 we get

$$I_1^d + \cdots + I_5^d = O\left(x^2 e^{-c(\log x)^{\frac{2\alpha}{\alpha+1}}}\right).$$

From this we see equation (4.42) becomes

$$\mathcal{M}^d_{\mathcal{P}}(x) = \frac{\rho x^2}{2} + \sum_{n_0<|m|\leq n-1} J^d_m + \{J^d_{-n} + J^d_n\} + O\left(x^2 e^{-c(\log x)^{\frac{2\alpha}{\alpha+1}}}\right), \ \rho > 0. \quad (4.43)$$

To estimate the second term in the right hand side of equation (4.43) we need to prove again Proposition 4.6 with $\zeta^d_{\mathcal{P}}(s)$ instead of $\zeta_{\mathcal{P}}(s)$ as follows

Proposition 4.13.

$$\sum_{n_0<|m|\leq n-1} J^d_m = O\left(x^2 e^{-q(\log x)^{1-\frac{1-\alpha}{\omega}}}\right), \quad \textit{for some } \ q > 0.$$

Proof. Let us consider the integral J_m and let γ_m be the circle centred at $1 - a_m + ib_m$ with the radius δ_1 parameterised by $\gamma_m(\vartheta) = 1 - a_m + ib_m + \delta_1 e^{i\vartheta}$, where $0 \leq \vartheta \leq 2\pi$. Therefore we have

$$\left| \int_{\gamma_m} \zeta^d_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} ds \right| \leq 2\pi\delta \sup_{s\in\gamma_m} \left| \zeta^d_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} \right| \to 0,$$

as $\delta \to 0$. Let $\delta \to 0$, so we can write

$$\left|J^d_m\right| = \left| \frac{1}{2\pi i} \int_{C_m} \zeta^d_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} ds \right| \leq \frac{b_0 e^{(\log x_n)^{\frac{\tau}{2}}}}{\pi} \int_{-\frac{3}{2}}^{1-a_m} \frac{x^{2-a_m}}{b_m^2} d\sigma$$

$$\leq \frac{b_1 x^2}{b_m^2} \exp\left\{(\log x_n)^{\frac{\tau}{2}} - a_m \log x\right\},$$

for some $b_0, b_1 > 0$. But if $|m| \leq n-1$ then

$$\exp\left\{(\log x_n)^{\frac{\tau}{2}} - a_m \log x\right\} \leq \exp\left\{(\log x_n)^{\frac{\tau}{2}} - a_{n-1} \log x\right\}$$

$$= \exp\left\{(\log x_n)^{\frac{\tau}{2}} - \frac{\log x}{(\log x_{n-1})^{1-\alpha}}\right\} = \exp\left\{(\log x_n)^{\frac{\tau}{2}} - \frac{\log x}{(\log x_n)^{\frac{1-\alpha}{\omega}}}\right\}.$$

Since $\log x = \log x_n + o(1)$, the last term is

$$\leq 2\exp\left\{(\log x)^{\frac{\tau}{2}} - (\log x)^{1-\frac{1-\alpha}{\omega}}\right\} \leq 2\exp\left\{-q(\log x)^{1-\frac{1-\alpha}{\omega}}\right\},$$

for some $q > 0$, since $(\log x)^{\frac{\tau}{2}} = o\left((\log x)^{1-\frac{1-\alpha}{\omega}}\right)$ for ω sufficiently large. Hence

$$\sum_{n_0<|m|\leq n-1} J^d_m = O\left(x^2 e^{-q(\log x)^{1-\frac{1-\alpha}{\omega}}} \sum_{|m|>n_0} \frac{1}{b_m^2}\right) = O\left(x^2 e^{-q(\log x)^{1-\frac{1-\alpha}{\omega}}}\right).$$

$\square$

We see that $1 - \frac{1-\alpha}{\omega} \geq \frac{2\alpha}{\alpha+1} = \tau$ since ω is taken sufficiently large, so equation (4.43) becomes

$$\mathcal{M}_{\mathcal{P}}^{d}(x) = \frac{\rho x^2}{2} + \{J_{-n}^{d} + J_{n}^{d}\} + O\left(x^2 e^{-c(\log x)^{\frac{2\alpha}{\alpha+1}}}\right), \tag{4.44}$$

for some constants ρ, q, $c > 0$. It remains to study the expression $J_{-n}^{d} + J_{n}^{d}$.

Denote by $J_{n}^{d'}$ and $J_{n}^{d''}$ the integrals along the line segments $C_n^{'}$ $C_n^{''}$ (see Figure 4.1) lying respectively above and below the branch cut C_n so that $J_n^d = J_n^{d'} + J_n^{d''}$. Now, if we write

$$s = 1 - a_n + ib_n + te^{i\theta}, \;\; -\pi \leq \theta < \pi,$$

then the line segment $C_n^{''}$ is obtained by letting $\theta = -\pi$ and t run from 0 to $1 - a_n + \frac{3}{2}$. In this way we obtain $C_n^{''}$ with its direction reversed:

$$-C^{''} : \begin{cases} \theta = -\pi, \\ s = 1 - a_n + ib_n - t, \\ ds = -dt, \\ 0 \leq t \leq 1 - a_n + \frac{3}{2}. \end{cases}$$

To estimate J_n^d, split it up into

$$J_n^{d''} = \frac{1}{2\pi i}\left\{\int_0^{(\log x)^{-\epsilon}} + \int_{(\log x)^{-\epsilon}}^{1-a_n+\frac{3}{2}}\right\} \frac{\zeta_{\mathcal{P}}^{d}(1 - a_n + ib_n - t)x^{2-a_n+ib_n-t}}{(1 - a_n + ib_n - t)(2 - a_n + ib_n - t)}(dt), \tag{4.45}$$

where ϵ is arbitrary positive number. The second integral over $((\log x)^{-\epsilon}, 1-a_n+\frac{3}{2})$ is bounded in modulus by

$$\frac{b_2 x^{2-a_n} e^{(\log x_n)^{\frac{\tau}{2}}}}{2\pi b_n^2}\int_{(\log x)^{-\epsilon}}^{1-a_n+\frac{3}{2}} x^{-t}dt \leq \frac{b_2 x^{2-a_n}\exp\left\{(\log x_n)^{\frac{\tau}{2}} - (\log x)^{1-\epsilon}\right\}}{2\pi b_n^2 \log x},$$

for some $b_2 > 0$. By taking ϵ to be as small as we please we find that the last term of the above inequality is $O\left(x^{2-a_n}e^{-c(\log x)^{1-\epsilon}}\right)$, for some $c > 0$.

To deal with the integral over $(0, (\log x)^{-\epsilon})$ rewrite the integrand as follows:

$$\frac{\zeta_{\mathcal{P}}^{d}(s)}{s(s+1)} = (s - 1 + a_n - ib_n)^{\frac{\mu_n}{2}} g_n(s) \;\; \text{say},$$

where

$$g_n(s) = e^{F_2(s)-F_1(s)} f_n(s).$$

Here $g_n(s)$ is analytic in a disc around the point $z_n = 1 - a_n + ib_n$. Therefore By Proposition 4.7 and equation (4.40) we obtain

$$|g_n(s)| \leq \frac{64}{b_n^2} \exp\left\{\Re\left(F_2(s) - F_1(s)\right)\right\} \leq \frac{q_1}{b_n^2} e^{(\log x_n)^{\frac{\tau}{2}}}, \tag{4.46}$$

for some $q_1 > 0$. While, by Proposition 4.9 and equation (4.41) we have

$$|g_n(s)| \geq \frac{1}{160 b_n^2} \exp\left\{\Re\left(F_2(s) - F_1(s)\right)\right\} \geq \frac{q_2}{b_n^2} e^{-(\log x_n)^{\frac{\tau}{2}}}, \tag{4.47}$$

for some $q_2 > 0$. Therefore, by (4.18) with $g_n(s)$ instead of $f_n(s)$ and by (4.47) we obtain

$$\left|S_n^d\right| \geq b \exp\left\{-\left((\log x_n)^{\frac{\tau}{2}} + 2(\log x_n)^{\alpha}\right)\right\}, \quad b > 0, \tag{4.48}$$

for x is sufficiently large, that is for n is sufficiently large. We use this lower bound of the integral S_n^d. Now considering the other factor in (4.17) (with S_n is replaced by S_n^d), we have

$$\begin{aligned}\frac{\sin\frac{\pi\mu_n}{2}}{\pi} x^{2-a_n+} \left(\frac{1}{\log x}\right)^{\frac{\mu_n}{2}+1} &\geq \frac{\mu_n}{\pi} x^2 e^{-a_n log x} \cdot \frac{1}{2(\log x)^2} \\ &\geq q_3 x^2 e^{-\frac{\log x}{(\log x_n)^{1-\alpha}}} \cdot \frac{1}{(\log x)^2\left(\log\log x_n\right)^2}, \quad q_3 > 0.\end{aligned}$$

where $\mu_n = \frac{1}{n^2}$ and $n \leq \frac{\log\log x_n}{\omega - 1}$. From the above and (4.48), we get

$$\left|J_n^d\right| \geq \frac{q_3 x^2}{(\log x)^2\left(\log\log x_n\right)^2} \exp\left\{-\left(\frac{\log x}{(\log x_n)^{1-\alpha}} + 2(\log x_n)^{\alpha} + (\log x_n)^{\frac{\tau}{2}}\right)\right\},$$

for some $q_3 > 0$ and large n. That is, for some $q_3, c_0 > 0$ and large n we have

$$\left|J_n^d\right| \geq q_3 x^2 \exp\left\{-\left(c_0(\log x)^{\alpha} + (\log x)^{\frac{\tau}{2}}\right)\right\}. \tag{4.49}$$

This gives

$$\left|J_n^d\right| \geq q_3 x^2 e^{-c(\log x)^{\alpha}}, \ \ 0 < \alpha < 1,$$

for some positive constants q_3, c and for sufficiently large n. Since ω is taken sufficiently large, we see (4.44) becomes

$$\mathcal{M}_{\mathcal{P}}^d(x) = \frac{\rho x^2}{2} + \left\{J_{-n}^d + J_n^d\right\} + O\left(x^2 e^{-(\log x)^{\frac{2\alpha}{1+\alpha}}}\right). \tag{4.50}$$

We next aim to obtain a large value for $J_n^d + J_{-n}^d = 2\Re(J_n^d)$ compared with the other error term of (4.50). To achieve this we can use similar arguments as those discussed in Section 4.1 to show that for sufficiently large n we have

$$\Re(J_n^d) \geq Ax^2 e^{-c(\log x)^\alpha}, \quad \text{for } r = r^+ \text{ and } A, c > 0, \tag{4.51}$$

and

$$\Re(J_n^d) \leq -Ax^2 e^{-c(\log x)^\alpha}, \quad \text{for } r = r^- \text{ and } A, c > 0. \tag{4.52}$$

Hence we get

$$\mathcal{M}_{\mathcal{P}}^d(x) = \frac{\rho x^2}{2} + \Omega_{\pm}(x^2 e^{-c(\log x)^\alpha}), \tag{4.53}$$

for some positive constants ρ and c. This proves equation (4.36) and hence (4.27). The proof of Theorem 4.10 is completed.

Chapter 5

Connecting the error term of $\mathcal{N}_{\mathcal{P}}(x)$ and the size of $\zeta_{\mathcal{P}}(s)$

When proving results linking the asymptotic behaviour of $\Pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$ one often uses as a go-between the Beurling zeta function $\zeta_{\mathcal{P}}(s)$. Thus an assumption made on $\Pi_{\mathcal{P}}(x)$ is translated into a property of $\zeta_{\mathcal{P}}(s)$ which is then shown to imply a property of $\mathcal{N}_{\mathcal{P}}(x)$ and similarly vice versa. The property on $\zeta_{\mathcal{P}}(s)$ is often related to its size along the vertical line (or holomorphicity). For example, if $\mathcal{N}_{\mathcal{P}}(x) = cx + O(x^{\alpha}), \;\; \alpha < 1$. Then $\zeta_{\mathcal{P}}(s)$ is holomorphic in $H_{\alpha}\backslash\{1\}$ and $\zeta_{\mathcal{P}}(\sigma + it) = O(t)$ for $\sigma > \alpha$. That is, $\zeta_{\mathcal{P}}$ has at most polynomial growth on vertical lines to the left of 1. Furthermore, bounds on the vertical growth can be shown via the inverse Mellin transform to imply $\mathcal{N}_{\mathcal{P}}(x) = cx + O(x^{\alpha})$. Here we investigate the connection when $\mathcal{N}_{\mathcal{P}}(x) = cx + \Omega(x^{1-\epsilon})$, and where $\zeta_{\mathcal{P}}(\sigma + it)$ may have infinite order. Therefore, if we assume that $\zeta_{\mathcal{P}}(s)$ has polynomial growth along some curve for $\sigma < 1$, what can be said about the behaviour of $\mathcal{N}_{\mathcal{P}}(x)$ (as $x \to \infty$) and vice versa?

We concentrate in this chapter on determining the connections between the asymptotic behaviour of the g-integer counting function $\mathcal{N}_{\mathcal{P}}(x)$ and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$). We aim to find this link and apply it in chapter 6.

5.1 From $\zeta_{\mathcal{P}}$ to $\mathcal{N}_{\mathcal{P}}$

We start with showing how assumptions on growth of $\zeta_{\mathcal{P}}(s)$ imply estimates on the error term of $\mathcal{N}_{\mathcal{P}}(x)$. Note that in fact, the following theorem is purely analytical as there is no use of g-prime systems (only the fact that $\mathcal{N}_{\mathcal{P}} \in S_1^+$).

Theorem 5.1. *Suppose that for some $\alpha \in [0, 1)$, $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue ρ.*

Further assume that for some $c < 1$,

$$\zeta_{\mathcal{P}}(\sigma + it) = O(t^c), \quad \textit{for} \;\; \sigma \geq 1 - \frac{1}{f(\log t)},$$

where f is a positive, strictly increasing continuous function, tending to infinity. Then for $\gamma = 1 - c$,

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-\frac{\gamma}{2}h^{-1}(\gamma^{-1}\log x)}),$$

where $h(u) = uf(u)$.

Proof. We use the bound $\zeta_{\mathcal{P}}(s) = O(t^c)$, for some $c < 1$ to find an approximate formula for

$$\mathcal{M}_{\mathcal{P}}(x) = \int_0^x \mathcal{N}_{\mathcal{P}}(y)dy = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds.$$

This holds for any $b > 1$. Pushing the contour to the left of the line $\Re s = b$ past the simple pole at 1, we get for any $T > 0$

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho}{2}x^2 + \frac{1}{2\pi i}\int_{\eta_T} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\int_{1-\frac{1}{f(\log T)}+iT}^{b+iT} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds$$
$$+ \frac{1}{2\pi i}\int_{b-iT}^{1-\frac{1}{f(\log T)}-iT} \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\left(\int_{b+iT}^{b+i\infty} + \int_{b-i\infty}^{b-iT}\right)\zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds.$$

Here η_T is the contour $s = 1 - \frac{1}{f(\log t)} + it$ for $a < |t| \leq T$ and $s = 1 - \frac{1}{f(\log a)} + it$ for $|t| \leq a$. The constant a is chosen such that $a > e$ and $1 - \frac{1}{f(\log a)} > \alpha$, (see Figure 5.1).

The modulus of the integral over the horizontal line $[1-\frac{1}{f(\log T)}+iT, b+iT]$ is

$$\left|\int_{1-\frac{1}{f(\log T)}+iT}^{b+iT}\zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds\right| = O\Big(\int_{1-\frac{1}{f(\log T)}}^{b}\frac{T^c x^{u+1}}{T^2}du\Big)$$
$$= O(\frac{x^{b+1}}{T^{2-c}\log x}) \to 0 \text{ as } T\to\infty.$$

Similarly for the integral over $[b-iT, 1-\frac{1}{f(\log T)}-iT]$.

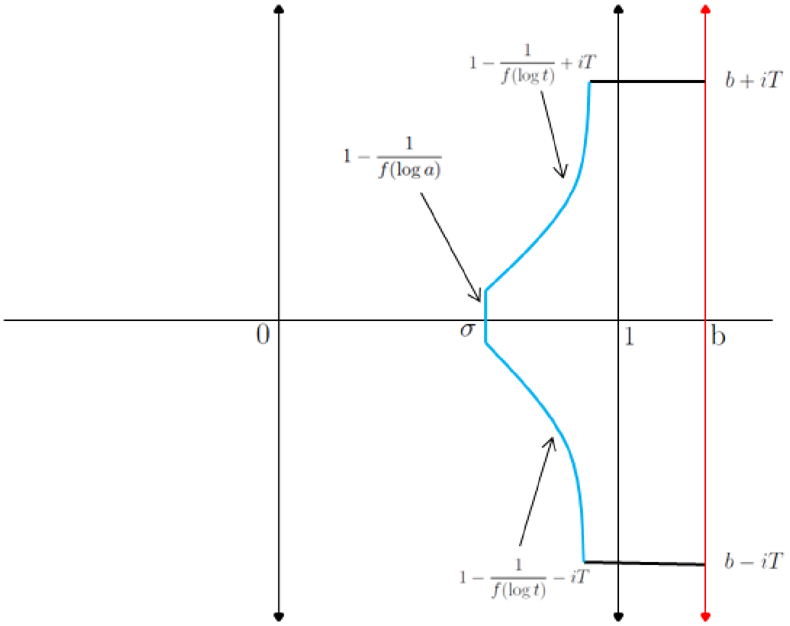

Figure 5.1: contour η_T

Now, letting $T\to\infty$ we get

$$\mathcal{M}_{\mathcal{P}}(x) = \frac{\rho}{2}x^2 + \frac{1}{2\pi i}\int_\eta \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds,$$

where η is the contour $s = 1-\frac{1}{f(\log t)}+it$ for $|t|>a>e$ and $s = 1-\frac{1}{f(\log a)}+it$ for $|t|\le a$. Therefore,

$$\left|\mathcal{M}_{\mathcal{P}}(x)-\frac{\rho}{2}x^2\right| = \left|\frac{1}{2\pi i}\int_\eta \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}ds\right|,$$

$$= O\left(\int_a^\infty \frac{\left|\zeta_{\mathcal{P}}(1-\frac{1}{f(\log t)}-it)\right|}{t^2}x^{2-\frac{1}{f(\log t)}}dt\right) + O(x^{2-\frac{1}{f(\log a)}}),$$

$$= O\left(x^2 \int_{\log a}^{\infty} \exp\left\{-(2u + \frac{\log x}{f(u)}) + (c+1)u\right\} du\right) + O(x^{2-\frac{1}{f(\log a)}}),$$

$$= O\left(x^2 \int_{\log a}^{\infty} \exp\left\{-((1-c)u + \frac{\log x}{f(u)})\right\} du\right) + O(x^{2-\frac{1}{f(\log a)}}).$$

To estimate the integral, we split it into

$$\int_{\log a}^{\infty} \exp\left\{-(\gamma u + \frac{\log x}{f(u)})\right\} du = \left(\int_{\log a}^{A} + \int_{A}^{\infty}\right) \exp\left\{-(\gamma u + \frac{\log x}{f(u)})\right\} du,$$

for some $A > \log a$ and $\gamma = 1 - c$.

The first integral over $(\log a, A)$ is $\leq e^{-\frac{\log x}{f(A)}} \int_{\log a}^{A} e^{-\gamma u} du = O(e^{-\frac{\log x}{f(A)}})$, whilst the second integral over (A, ∞) is $\leq \int_{A}^{\infty} e^{-\gamma u} du = O(e^{-\gamma A})$.

Now, choose A optimally such that these $O-$terms are of the same order (i.e. $Af(A) = \gamma^{-1} \log x$), and $h(A) = Af(A)$. Then $A = h^{-1}(\gamma^{-1} \log x)$. Hence

$$\int_{\log a}^{\infty} \exp\left\{-(\gamma u + \frac{\log x}{f(u)})\right\} du = O(\exp(-\gamma h^{-1}(\gamma^{-1} \log x)).$$

Therefore

$$\left|\mathcal{M}_{\mathcal{P}}(x) - \frac{\rho}{2}x^2\right| = O\left(x^2 \exp\{-\gamma h^{-1}(\gamma^{-1} \log x)\}\right), \ \ \gamma = 1 - c. \tag{5.1}$$

The function $\mathcal{N}_{\mathcal{P}}$ is increasing function, so for every $0 < y < x$, we have

$$\int_{0}^{x} \mathcal{N}_{\mathcal{P}}(u) du - \int_{0}^{x-y} \mathcal{N}_{\mathcal{P}}(u) du = \int_{x-y}^{x} \mathcal{N}_{\mathcal{P}}(u) du \leq y \mathcal{N}_{\mathcal{P}}(x).$$

On the other hand

$$\int_{0}^{x+y} \mathcal{N}_{\mathcal{P}}(u) du - \int_{0}^{x} \mathcal{N}_{\mathcal{P}}(u) du = \int_{x}^{x+y} \mathcal{N}_{\mathcal{P}}(u) du \geq y \mathcal{N}_{\mathcal{P}}(x).$$

Therefore

$$\frac{\mathcal{M}_{\mathcal{P}}(x) - \mathcal{M}_{\mathcal{P}}(x-y)}{y} \leq \mathcal{N}_{\mathcal{P}}(x) \leq \frac{\mathcal{M}_{\mathcal{P}}(x+y) - \mathcal{M}_{\mathcal{P}}(x)}{y}.$$

Using equation (5.1) the left hand side of the above inequality is

$$= \frac{1}{y}\left(\frac{\rho}{2}(x^2 - (x-y)^2) + O\left(x^2 \exp\{-\gamma h^{-1}(\gamma^{-1} \log(x-y))\}\right)\right).$$

That is, the left hand side is

$$= \frac{1}{y}\left(\rho x y - \frac{\rho y^2}{2} + O\left(x^2 \exp\{-\gamma h^{-1}(\gamma^{-1} \log(x-y))\}\right)\right). \tag{5.2}$$

Similarly, the right hand side of the same inequality is

$$= \frac{1}{y}\left(\rho xy + \frac{\rho y^2}{2} + O\left(x^2 \exp\{-\gamma h^{-1}(\gamma^{-1}\log x)\}\right)\right),$$

hold for any $0 < y < x$.

Now, for some $\epsilon > 0$ and some $d > 0$ we have $h(x) - h(x-d) = xf(x) - (x-d)f(x-d) = x\big(f(x) - f(x-d)\big) + df(x-d) \geq \epsilon > 0$. That is, $h(x) - \epsilon \geq h(x-d)$. Therefore, with $y = o(x)$, we see

$$h^{-1}(\gamma^{-1}\log(x-y)) \geq h^{-1}(\gamma^{-1}\log x - \epsilon) \geq h^{-1}(\gamma^{-1}\log x) - d,$$

for some $\epsilon > 0$ and some $d > 0$. So in the $O-$term in (5.2) we can replace $x - y$ by x.

Now, setting $y = x\exp\{-\frac{\gamma}{2}h^{-1}(\gamma^{-1}\log x)\}$ we get

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x\exp\{-\frac{\gamma}{2}h^{-1}(\gamma^{-1}\log x)\}\right).$$

$\square$

It is useful to observe how the size of the error term of $\mathcal{N}_{\mathcal{P}}(x)$ depends on where $\zeta_{\mathcal{P}}(\sigma + it) = O(t^c)$, for some $c < 1$ as $\sigma \to 1$. Here are some examples showing how different functions f lead to different error terms for $\mathcal{N}_{\mathcal{P}}(x)$.

Examples

1. For $f(x) = \frac{x}{\log x}$. We have $h(x) = \frac{x^2}{\log x}$ gives $h^{-1}(x) \sim \sqrt{\frac{1}{2}x\log x}$ and

$$h^{-1}(\log x) \geq (1-\epsilon)\sqrt{\frac{1}{2}\log x \log\log x}, \quad \forall \epsilon > 0,\ \ x \geq x_0(\epsilon).$$

Thus Theorem 5.1 says,

$$\begin{aligned}&\zeta_{\mathcal{P}}(\sigma + it) = O(t^c),\ \ (c < 1) \ \text{ for } \ \sigma \geq 1 - \tfrac{\log\log t}{\log t} \ \text{ implies}\\ &\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x\exp\{-a\sqrt{\log x \log\log x}\}\right),\end{aligned}$$

for some $\rho > 0$, and for every $a < \sqrt{\frac{1-c}{8}}$.

2. For $f(x) = x^{\theta}$ for some $\theta > 0$. We have $h(x) = x^{1+\theta}$ and

$$h^{-1}(\log x) = (\log x)^{\frac{1}{1+\theta}}.$$

That is,

$$\zeta_{\mathcal{P}}(1 - \frac{1}{(\log t)^{\theta}} + it) = O(t^{c}), \ \ (c < 1) \ \ \text{implies}$$
$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x \exp\{-b(\log x)^{\frac{1}{1+\theta}}\}\right),$$

for some $\rho > 0$, where $b = \frac{\gamma^{\frac{\theta}{1+\theta}}}{2}$ and $\gamma = 1 - c$.

5.2 From $\mathcal{N}_{\mathcal{P}}$ to polynomial growth of $\zeta_{\mathcal{P}}$

Our purpose in this section is to obtain a kind of converse of Theorem 5.1. That is, we find the region where $\zeta_{\mathcal{P}}(\sigma+it) = O(t^c)$, for some $c > 0$, if we assume that we have a bound for the error term of $\mathcal{N}_{\mathcal{P}}(x)$. In the other words, the reason of the following theorem is to obtain polynomial growth for $\zeta_{\mathcal{P}}(\sigma+it)$, $\sigma < 1$. This depends on σ and the bound of the error term of $\mathcal{N}_{\mathcal{P}}(x)$. We shall need to assume a priori that $\zeta_{\mathcal{P}}$ is has an analytic continuation to the left of $\sigma = 1$ and that $\zeta_{\mathcal{P}}(\sigma+it)$ is bounded above by $O(e^t)$ for $0 \leq \sigma < 1$.

Theorem 5.2. *Suppose that for some $\alpha \in [0,1)$, $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half plane H_α except for a simple pole at $s = 1$ with residue ρ and for $\sigma > \alpha$, $\zeta_{\mathcal{P}}(\sigma+it) = O(e^t)$, $(t > 0, t \to \infty)$.*

Further assume that

$$\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(xe^{-k(x)}),$$

for some positive, increasing function k tending to infinity such that $k'(x) = o(\frac{1}{x})$. Then for some $c > 0$,

$$\zeta_{\mathcal{P}}(\sigma+it) = O(t^c),$$

for $1 - \frac{k(\frac{e^t}{t})}{t} \leq \sigma < 1 - \frac{\log t}{t}$, where t is sufficiently large.

Proof. The usual *Mellin transform*

$$\zeta_{\mathcal{P}}(s) = \int_{1-}^{\infty} x^{-s} d\mathcal{N}_{\mathcal{P}}(x), \quad \sigma > 1$$

cannot be used directly for $\sigma < 1$, since the error term is not small enough to ensure analytic continuation to $\sigma < 1$. Instead we use a formula which is based on:

$$\sum_{n\geq 1} \frac{a_n}{n^s} e^{-(\lambda n)^\delta} = \frac{1}{2\pi i\delta} \int_{c-i\infty}^{c+i\infty} \Gamma(\frac{\omega}{\delta}) g(s+\omega) \lambda^{-\omega} d\omega, \tag{5.3}$$

where $g(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$, $\lambda > 0$, $\delta > 0$, and $c > 0$, $c > \sigma_1 - \sigma$, where σ_1 is the abscissa of absolute convergence of $g(s)$. (See [28] page 301.)

We generalize equation (5.3) (with $\delta = 1$) in terms of the Beurling zeta function. The reason for doing this is to find an estimate for $\zeta_{\mathcal{P}}(s)$ for $\sigma < 1$. That is, we show

$$\int_{1-}^{\infty} x^{-s} e^{-\lambda x} d\mathcal{N}_{\mathcal{P}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\omega) \zeta_{\mathcal{P}}(s+\omega) \lambda^{-\omega} d\omega, \tag{5.4}$$

holds for $\lambda > 0$ and $c > 0$, $c > 1 - \sigma$.

To see this, notice that the right hand side of equation (5.4) is equal

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\omega) \Big(\int_{1-}^{\infty} x^{-(s+\omega)} d\mathcal{N}_{\mathcal{P}}(x) \Big) \lambda^{-\omega} d\omega,$$

and observe that we can invert the order of integrations by 'absolute convergence' since gamma is exponentially small. It becomes

$$\int_{1-}^{\infty} x^{-s} \Big(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\omega) (\lambda x)^{-\omega} d\omega \Big) d\mathcal{N}_{\mathcal{P}}(x) = \int_{1-}^{\infty} x^{-s} e^{-\lambda x} d\mathcal{N}_{\mathcal{P}}(x).$$

Note that both sides of (5.4) are entire functions.

Now we integrate by parts the left hand side of equation (5.4) and on the right we push the contour to the left of the lines $\Re\omega = 0$ and $\Re\omega = 1 - \sigma$. We get

$$\lambda \int_{1}^{\infty} e^{-\lambda x} x^{-s} \mathcal{N}_{\mathcal{P}}(x) dx + s \int_{1}^{\infty} e^{-\lambda x} x^{-s-1} \mathcal{N}_{\mathcal{P}}(x) dx = \zeta_{\mathcal{P}}(s) + \rho \lambda^{s-1} \Gamma(1-s)$$
$$+ \frac{1}{2\pi i} \Big(\int_{c-i\infty}^{c'-i\infty} + \int_{c'-i\infty}^{c'+i\infty} + \int_{c'+i\infty}^{c+i\infty} \Big) \Gamma(\omega) \zeta_{\mathcal{P}}(s+\omega) \lambda^{-\omega} d\omega,$$

for some negative constant $c' > -1$ and $c' + \sigma > \alpha$, since the integrand of the right hand side of equation (5.4) has singularities at $\omega = 0$ and $\omega = 1 - s$ with residues $\zeta_{\mathcal{P}}(s)$ and $\rho\lambda^{s-1}\Gamma(1-s)$ respectively. $\big[$The contribution from the horizontal line $[c' + iy, c + iy]$ is

$$\Big| \int_{c'+iy}^{c+iy} \Gamma(\omega) \zeta_{\mathcal{P}}(s+\omega) \lambda^{-\omega} d\omega \Big| = \Big| \int_{c'}^{c} \Gamma(x+iy) \zeta_{\mathcal{P}}(\sigma + x + i(y+t)) \lambda^{-(x+iy)} dx \Big|$$
$$= O\Big(y^{-\frac{1}{2}} \exp\Big\{ |y+t| - \frac{\pi\,|y|}{2} \Big\} \int_{c'}^{c} y^{x} \lambda^{-x} dx \Big) \to 0 \text{ as } y \to \infty.$$

since $|\Gamma(x+iy)| \ll e^{-\frac{\pi|y|}{2}} |y|^{x-\frac{1}{2}} \sqrt{2\pi}$ (See [28] page 151), and $|\zeta_{\mathcal{P}}(\sigma+it)| \ll e^{t}$.

Similarly for the integral over $[c - iy, c' - iy]$.$\big]$ Therefore,

$$\begin{aligned} \zeta_{\mathcal{P}}(s) = {} & \lambda \int_{1}^{\infty} e^{-\lambda x} x^{-s} \mathcal{N}_{\mathcal{P}}(x) dx + s \int_{1}^{\infty} e^{-\lambda x} x^{-s-1} \mathcal{N}_{\mathcal{P}}(x) dx \\ & - \rho \lambda^{s-1} \Gamma(1-s) - \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \Gamma(\omega) \zeta_{\mathcal{P}}(s+\omega) \lambda^{-\omega} d\omega. \end{aligned} \tag{5.5}$$

As we mentioned earlier, we are interested in finding an estimate for $\zeta_{\mathcal{P}}(s)$ for $\sigma < 1$. So, we take $\alpha < \sigma < 1$, and try to estimate each term in the right hand side of equation (5.5) separately. For the first integral we have

$$\lambda \int_1^\infty e^{-\lambda x} x^{-s} \mathcal{N}_{\mathcal{P}}(x) dx = \lambda \int_1^\infty e^{-\lambda x} x^{-s} \big(\rho x + O(x e^{-k(x)})\big) dx$$

$$= \lambda\rho \int_1^\infty e^{-\lambda x} x^{-s+1} dx + O\bigg(\lambda \int_1^\infty x^{-\sigma+1} e^{-(\lambda x + k(x))} dx\bigg).$$

$$= \rho\lambda^{s-1}\Gamma(2-s) - \rho\lambda \int_0^1 e^{-\lambda x} x^{-s+1} dx + O\bigg(\lambda \int_1^\infty x^{-\sigma+1} e^{-(\lambda x + k(x))} dx\bigg),$$

since $\sigma < 1$. Hence

$$\lambda \int_1^\infty e^{-\lambda x} x^{-s} \mathcal{N}_{\mathcal{P}}(x) dx = \rho\lambda^{s-1}\Gamma(2-s) + O\bigg(\lambda \int_1^\infty x^{-\sigma+1} e^{-(\lambda x + k(x))} dx\bigg) + O(1), \tag{5.6}$$

since $\int_0^1 e^{-\lambda x} x^{-s+1} dx = O(1)$.

For the second integral of equation (5.5) we have

$$s \int_1^\infty e^{-\lambda x} x^{-s-1} \mathcal{N}_{\mathcal{P}}(x) d = s \int_1^\infty e^{-\lambda x} x^{-s-1} \big(\rho x + O(x e^{-k(x)})\big) dx$$

$$= s\rho \int_1^\infty e^{-\lambda x} x^{-s} dx + O\bigg(t \int_1^\infty x^{-\sigma} e^{-(\lambda x + k(x))} dx, \bigg)$$

$$= s\rho\lambda^{s-1}\Gamma(1-s) - s\rho \int_0^1 e^{-\lambda x} x^{-s} dx + O\bigg(t \int_1^\infty x^{-\sigma} e^{-(\lambda x + k(x))} dx, \bigg)$$

since $\sigma < 1$. Hence

$$s \int_1^\infty e^{-\lambda x} x^{-s-1} \mathcal{N}_{\mathcal{P}}(x) dx = s\rho\lambda^{s-1}\Gamma(1-s) + O(\frac{t}{1-\sigma}) + O\bigg(t \int_1^\infty x^{-\sigma} e^{-(\lambda x + k(x))} dx\bigg), \tag{5.7}$$

since $\left| s\rho \int_0^1 e^{-\lambda x} x^{-s} dx \right| = O(t \int_0^1 x^{-\sigma} dx) = O(\frac{t}{1-\sigma})$.

Finally, for the vertical line over $[c^{'} - i\infty, c^{'} + i\infty]$ we have

$$\left| \int_{c^{'}-i\infty}^{c^{'}+i\infty} \Gamma(\omega)\zeta_{\mathcal{P}}(s+\omega)\lambda^{-\omega} d\omega \right| = \left| \int_{-\infty}^{\infty} \Gamma(c^{'}+iy)\zeta_{\mathcal{P}}(\sigma + c^{'} + i(y+t))\lambda^{-(c^{'}+iy)} dy \right|$$

$$= O\bigg(\lambda^{-c^{'}} \int_{-\infty}^{\infty} (|y|+1)^{c^{'}-\frac{1}{2}} \exp\left\{ |y+t| - \frac{\pi\,|y|}{2} \right\} dy\bigg)$$

$$= O\big(\lambda^{-c^{'}} e^t\big).$$

From the above, equation (5.5) becomes

$$\begin{aligned}
\zeta_{\mathcal{P}}(s) &= \rho\lambda^{s-1}\Gamma(2-s) + s\rho\lambda^{s-1}\Gamma(1-s) - \rho\lambda^{s-1}\Gamma(1-s) + O(\frac{t}{1-\sigma}) + O\big(\lambda^{-c^{'}}e^{t}\big)\\
&+ O\bigg(\lambda\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx\bigg) + O\bigg(t\int_1^\infty x^{-\sigma}e^{-(\lambda x+k(x))}dx\bigg),\\
&= O\bigg(\lambda\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx\bigg) + O\bigg(t\int_1^\infty x^{-\sigma}e^{-(\lambda x+k(x))}dx\bigg)\\
&\quad + O(\frac{t}{1-\sigma}) + O\big(\lambda^{-c^{'}}e^{t}\big),
\end{aligned} \tag{5.8}$$

since for $\sigma < 1$, the gamma terms cancel each other.

Our aim is to find for which σ we have $\zeta_{\mathcal{P}}(s) = O(t^c)$ for some $c > 0$. So, putting $\lambda = e^{-t}$, we see that the last term of the right hand side of equation (5.8) is $O(1)$, since $c^{'} > -1$. We see also with $\lambda = e^{-t}$ that the term

$$\lambda\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx = O(t).$$

Indeed, we split the integral into the ranges $(1, B)$ and (B, ∞) for some $B > 1$. For the first integral we have

$$\begin{aligned}
\int_1^B x^{1-\sigma}e^{-(\lambda x+k(x))}dx &\le B^{1-\sigma}\int_1^B e^{-k(x)}dx\\
\le B^{1-\sigma}\bigg(Be^{-k(B)} + \int_1^B xk^{'}(x)e^{-k(x)}dx\bigg) &\ll B^{2-\sigma}e^{-k(B)},
\end{aligned}$$

since $k^{'}(x) = o(\frac{1}{x})$. For the range (B, ∞) the second integral is

$$\begin{aligned}
\int_B^\infty x^{1-\sigma}e^{-(\lambda x+k(x))}dx &\le e^{-k(B)}\int_B^\infty x^{1-\sigma}e^{-\lambda x}dx\\
&= \lambda^{\sigma-2}e^{-k(B)}\int_{\lambda B}^\infty y^{1-\sigma}e^{-y}dy\\
&\le \lambda^{\sigma-2}e^{-k(B)}\Gamma(2-\sigma).
\end{aligned}$$

Therefore,

$$\lambda\int_1^\infty x^{1-\sigma}e^{-(\lambda x+k(x))}dx \ll \lambda B^{2-\sigma}e^{-k(B)} + \lambda^{\sigma-1}e^{-k(B)}\Gamma(2-\sigma).$$

Now, setting $B = \sqrt{\frac{1}{\lambda}}$, we get

$$\lambda\int_1^\infty x^{1-\sigma}e^{-(\lambda x+k(x))}dx \ll \lambda^{\frac{\sigma}{2}}e^{-k(\sqrt{\frac{1}{\lambda}})} + \lambda^{\sigma-1}e^{-k(\sqrt{\frac{1}{\lambda}})}\Gamma(2-\sigma).$$

For $1-\sigma > \frac{\log t}{t}$ we see $\lambda^{\sigma-1} < \lambda^{-\frac{\log t}{t}} = t$. Therefore, the first term in the right hand side of the above inequality $\to 0$ as $t \to \infty$, whilst the second term is $O(t)$. Thus, equation (5.8) becomes

$$\zeta_{\mathcal{P}}(s) = O\left(t \int_1^\infty x^{-\sigma} e^{-(\lambda x + k(x))} dx\right) + O(\frac{t}{1-\sigma}). \tag{5.9}$$

To estimate the first integral in the right hand side of equation (5.9) we split it into the ranges $(1, A)$ and (A, ∞) for some $A > 1$.

The first integral is less than

$$\int_1^A \frac{e^{-k(x)}}{x^\sigma} dx \leq \frac{A^{1-\sigma} e^{-k(A)}}{1-\sigma} + \frac{1}{1-\sigma} \int_1^A \frac{xk^{'}(x)}{x^\sigma} e^{-k(x)} dx \ll \frac{A^{1-\sigma} e^{-k(A)}}{1-\sigma},$$

since $k^{'}(x) = o(\frac{1}{x})$, whilst the second integral over (A, ∞) is

$$\leq \frac{e^{-k(A)}}{A^\sigma} \int_A^\infty e^{-\lambda x} dx \leq \frac{e^{-k(A)}}{\lambda A^\sigma}.$$

So these tell us that

$$\int_1^\infty x^{-\sigma} e^{-(\lambda x + k(x))} dx \ll \frac{A^{1-\sigma} e^{-k(A)}}{1-\sigma} + \frac{e^{t-k(A)}}{A^\sigma} = \frac{e^{-k(A)}}{A^\sigma}\left(\frac{A}{1-\sigma} + e^t\right).$$

Choose $A = (1-\sigma)e^t$, equation (5.9) becomes

$$\begin{aligned}\zeta_{\mathcal{P}}(s) &= O\left(\frac{t}{1-\sigma}\big((1-\sigma)e^t\big)^{1-\sigma} e^{-k((1-\sigma)e^t)}\right) + O(\frac{t}{1-\sigma}) \\ &= O\big(t^2 \exp\big\{t(1-\sigma) - k((1-\sigma)e^t)\big\}\big) + O(t^2),\end{aligned} \tag{5.10}$$

since $\frac{1}{1-\sigma} < \frac{t}{\log t}$, and $(1-\sigma)^{1-\sigma} \to 1$ as $1-\sigma \to 0$. Therefore

$$\zeta_{\mathcal{P}}(\sigma + it) = O(t^c) \quad \text{for some } \; c > 0, \tag{5.11}$$

when

$$\exp\big\{t(1-\sigma) - k((1-\sigma)e^t)\big\} \leq t^c.$$

Certainly (5.11) holds when

$$k((1-\sigma)e^t) \geq (1-\sigma)t. \tag{5.12}$$

Now, we have

$$k((1-\sigma)e^t) > k(\frac{e^t \log t}{t}) > k(\frac{e^t}{t}),$$

since $1-\sigma > \frac{\log t}{t} > \frac{1}{t}$. This shows that (5.11) holds when

$$1-\sigma \leq \frac{k\left(\frac{e^t}{t}\right)}{t}.$$

Therefore, for

$$1-\frac{k\left(\frac{e^t}{t}\right)}{t} \leq \sigma < 1-\frac{\log t}{t},$$

we have $\zeta_{\mathcal{P}}(s) = O(t^c)$ for some $c > 0$. $\square$

We now illustrate Theorem 5.2 with some examples (of course, in each case we assume that $\zeta_{\mathcal{P}}$ has an analytic continuation to H_α).

Examples

1. For $k(x) = (\log x)^\alpha$, for some $\alpha \in (0,1)$. This means $k\left(\frac{e^x}{x}\right) = (x-\log x)^\alpha \sim x^\alpha$, and

$$k\left(\frac{e^x}{x}\right) \leq (1+\epsilon)x^\alpha, \ \ \forall \epsilon > 0, \ \ x \geq x_0(\epsilon).$$

   That is,

$$\begin{aligned} \mathcal{N}_{\mathcal{P}}(x) &= \rho x + O\left(x\exp\{-(\log x)^\alpha\}\right), \ \ \rho > 0 \text{ implies} \\ \zeta_{\mathcal{P}}(\sigma+it) &= O(t^c), \quad \text{for} \quad \sigma \geq 1-\frac{(1+\epsilon)}{t^{1-\alpha}} \quad \text{and} \ \ c > 0. \end{aligned}$$

2. For $k(x) = \frac{\log x \log\log\log x}{\log\log x}$. This means $k\left(\frac{e^x}{x}\right) \sim \frac{x\log\log x}{\log x}$, and

$$k\left(\frac{e^x}{x}\right) \leq (1+\epsilon)\frac{x\log\log x}{\log x}, \ \ \forall \epsilon > 0, \ \ x \geq x_0(\epsilon).$$

   That is,

$$\begin{aligned} \mathcal{N}_{\mathcal{P}}(x) &= \rho x + O\left(x\exp\left\{-\frac{\log x\log\log\log x}{\log\log x}\right\}\right), \ \ \rho > 0 \text{ implies} \\ \zeta_{\mathcal{P}}(\sigma+it) &= O(t^c), \quad \text{for} \quad \sigma \geq 1-\frac{(1+\epsilon)\log\log t}{\log t} \quad \text{and} \ \ c > 0. \end{aligned}$$

Chapter 6

Application to a particular example.

In this chapter we investigate a particular example of a g-prime system $\mathcal{P}_0$. In this example, $\psi_{\mathcal{P}}$ is given explicitly (see Definition 15) and hence gives very precise knowledge about the asymptotic behaviour of $\psi_{\mathcal{P}}(x)$. We write $\mathcal{N}_0, \Pi_0$ and π_0 for the associated Beurling counting functions and ζ_0 for the associated Beurling zeta function.

Definition 15. *Let*

$$\psi_0(x) = [x] - 1, \;\; x \geq 1. \tag{6.1}$$

Notice that (6.1) does not in itself give an outer g-prime system. For this we need Π_0 increasing. But

$$\Pi_0(x) = \int_1^x \frac{d([t]-1)}{\log t} = \sum_{2 \leq n \leq x} \frac{1}{\log n}$$

is increasing. That is, Π_0 (and ψ_0) $\in S_0^+$ which tells us that we have an outer g-prime system. Furthermore, we show at the end of this chapter that $(\Pi_0, \mathcal{N}_0)$ is a g-prime system by showing $\pi_0 \in S_0^+$ (i.e. π_0 is increasing).

We want to investigate the behaviour of $\mathcal{N}_0(x)$ as $x \to \infty$. It is immediate from Diamond's work (see point 2 in Section 3.3) that $\mathcal{N}_0(x) \sim \tau x$, for some $\tau > 0$, so we can write

$$\mathcal{N}_0(x) = \tau x + E(x). \tag{6.2}$$

Here $E(x) = o(x)$. We find $O-$results and $\Omega-$results for $E(x)$ as an application of Theorem 5.1 and Theorem 5.2.

Now, equation (6.1) (which implies $\psi_0(x) = x + O(1)$) tells us that $\zeta_0(s)$ has an analytic continuation to the half plane $\{s \in \mathbb{C} : \Re s > 0\}$ except for a simple pole at $s = 1$ and $\zeta_0(s) \neq 0$ in this region (see Lemma 3.2). Moreover,

$$-\frac{\zeta_0'}{\zeta_0}(s) = \int_{1-}^{\infty} x^{-s} d\psi_0(x) = \zeta(s) - 1. \tag{6.3}$$

Here, the ζ appearing in the right hand side of the above equation is the Riemann-zeta function. This tells us that $\frac{\zeta_0'}{\zeta_0}(s)$ has an analytic continuation to $\mathbb{C}\backslash\{1\}$. Let $L(s) = \frac{\zeta_0'}{\zeta_0}(s) + \frac{1}{s-1} = \frac{(s-1)\zeta_0'(s)+\zeta_0(s)}{(s-1)\zeta_0(s)}$. We see from (6.3) that $L(s)$ is entire, which means it has an entire primitive $H(s)$. This implies $\zeta_0(s) = c\frac{e^{H(s)}}{s-1}$ (some c) and so ζ_0 has an analytic continuation to $\mathbb{C}\backslash\{1\}$ (note that $\zeta_0 \neq 0$). This also tells us that $\log \zeta_0(s)$ is exists and analytic on $\mathbb{C}\backslash(-\infty, 1]$.

The constant τ

It is worthwhile to point out that the constant τ appearing in the right hand side of equation (6.2) can be calculated numerically as follows:

We have

$$\tau = e^{-\gamma} \lim_{x\to\infty} \exp\left\{\int_1^x \frac{d\Pi_0(\nu)}{\nu} - \log\log x\right\},$$

where γ is the Euler's constant (see [26, Page 46]).

Now, $d\Pi_0(\nu) = \frac{d\psi_0(\nu)}{\log \nu}$, therefore

$$\tau = e^{-\gamma} \lim_{x\to\infty} \exp\left\{\sum_{2\leq n\leq x} \frac{1}{n \log n} - \log\log x\right\}. \tag{6.4}$$

Thus by calculating (6.4) numerically one can get $\tau \approx 1.24$.

6.1 O-Results for $\mathcal{N}_0(x) - \tau x$

We will now find some $O-$results for $E(x)$. First, using result 7 in section 3.3 (with $\psi_0(x) = x + O(1)$), we get

$$\mathcal{N}_0(x) = \tau x + O\left(x \exp\{-c\sqrt{\log x \log\log x}\}\right), \tag{6.5}$$

for some $c > 0$.

We can improve on this by using Theorem 5.1. The real reason which allows us to improve on (6.5) is that $\zeta_0(s)$ is connected to the Riemann zeta function and we can use all the available information on $\zeta(s)$.

Theorem 6.1. *We have*

$$\mathcal{N}_0(x) = \tau x + O\left(x \exp\{-b(\log x)^{\frac{3}{5}}\big(\log\log x\big)^{\frac{2}{5}}\}\right), \tag{6.6}$$

for some $b > 0$. Furthermore, on the Riemann Hypothesis this can be improved to

$$\mathcal{N}_0(x) = \tau x + O\left(x \exp\left\{-\frac{(1-\epsilon)\log x \log\log\log x}{4\log\log x}\right\}\right), \quad \textit{for every } \epsilon > 0. \tag{6.7}$$

Proof. Firstly, we show that (6.6) holds. We have

$$\zeta(s) \ll (1 + t^{100(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}},$$

uniformly for $0 \le \sigma \le 2,\ t \ge 2$, (see (2.1) in Chapter 2 with $B = 100$). By (6.3) we get

$$\left|-\frac{\zeta_0'}{\zeta_0}(s)\right| \ll (1 + t^{100(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}}. \tag{6.8}$$

Now, for $\sigma \in (0,1)$, we have

$$\begin{aligned}\log\zeta_0(\sigma+it) &= -\int_{[\sigma+it,2+it]} \frac{\zeta_0'}{\zeta_0}(s)ds + \log\zeta_0(2+it)\\ &= -\int_\sigma^2 \frac{\zeta_0'}{\zeta_0}(u+it)du + O(1).\end{aligned}$$

Therefore,

$$\Re\big\{\log\zeta_0(\sigma+it)\big\} = \log|\zeta_0(\sigma+it)| \le \int_\sigma^2 \left|\frac{\zeta_0'}{\zeta_0}(u+it)\right| du + O(1). \tag{6.9}$$

Using the bound (6.8) we obtain

$$|\zeta_0(\sigma+it)| \ll \exp\big\{(1 + t^{100(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}}\big\}, \tag{6.10}$$

uniformly for $0 < \sigma \le 2,\ t \ge 2$.

Our aim here is to apply Theorem 5.1. For this purpose we have to show for which region (σ near 1), $\zeta_0(\sigma+it) \ll t^c$ for some positive constant $c<1$. So, in order for $|\zeta_0(\sigma+it)| \ll t^c$, to hold for some $c<1$, we need

$$\exp\left\{(1+t^{100(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}}\right\} \leq t^c.$$

That is, we need

$$1+e^{100(1-\sigma)^{\frac{3}{2}}\log t} \leq c(\log t)^{\frac{1}{3}}.$$

This certainly holds for t sufficiently large if

$$100(1-\sigma)^{\frac{3}{2}}\log t \leq \frac{1}{4}\log\log t.$$

Therefore, for

$$\sigma \geq 1-\left(\frac{\log\log t}{400\log t}\right)^{\frac{2}{3}},$$

we have

$$\zeta_0(\sigma+it) = O(t^c), \quad \text{for some positive constant } c<1.$$

Thus, we can apply Theorem 5.1. We have $f(x) = (\frac{400x}{\log x})^{\frac{2}{3}}$, which tells us that $h(x) = x^{\frac{5}{3}}(\frac{400}{\log x})^{\frac{2}{3}}$ and

$$h^{-1}(\log x) \sim (\log x)^{\frac{3}{5}}\big(\frac{3\log\log x}{2000}\big)^{\frac{2}{5}}.$$

Thus, for this example we have

$$\mathcal{N}_0(x) = \tau x + O\left(x\exp\{-b(\log x)^{\frac{3}{5}}\big(\log\log x\big)^{\frac{2}{5}}\}\right),$$

for some $b>0$. This concludes the proof of (6.6).

Now we show (6.7). On the Riemann Hypothesis we have

$$\log\zeta(s) \ll \frac{(\log t)^{2-2\sigma}-1}{(1-\sigma)\log\log t} + \log\log\log t,$$

for $\sigma_0 \leq \sigma \leq 1$, (see Chapter 2 on $O-$results). Using these bounds in (6.9) we obtain

$$\log|\zeta_0(\sigma+it)| \leq A\exp\big\{\frac{a(\log t)^{2(1-\sigma)}-1}{(1-\sigma)\log\log t} + a\log\log\log t\big\},$$

for some $A, a > 0$.

However, $\frac{e^u-1}{u} \leq e^u$ for all $u \geq 0$. Therefore, in order for $\log|\zeta_0(\sigma+it)| \leq c\log t$, for some positive constant $c < 1$, it is sufficient to have

$$\exp\left\{a(\log t)^{2(1-\sigma)} + a\log\log\log t\right\} \leq A_1 \log t,$$

for some $A_1, a > 0$. That is,

$$a(\log t)^{2(1-\sigma)} \leq \log\log t + \log A_1 - a\log\log\log t.$$

So, for $\sigma \geq 1 - \frac{\log\log\log t - k_1}{2\log\log t}$, the above holds for some suitable $k_1 > 0$, if t is sufficiently large. That is, in this region we have

$$\zeta_0(\sigma+it) = O(t^c), \quad \text{for every} \quad c > 0.$$

Now, apply Theorem 5.1 with $f(x) = \frac{2\log x}{\log\log x - k_1}$, which tells us that $h(x) = \frac{2x\log x}{\log\log x - k_1}$ and

$$h^{-1}\left(\frac{\log x}{\gamma}\right) \sim \frac{\log x \log\log\log x}{2\gamma \log\log x},$$

where $\gamma = 1 - c$. Hence, on the Riemann Hypothesis we have

$$\mathcal{N}_0(x) = \tau x + O\left(x \exp\left\{-\frac{(1-\epsilon)\log x \log\log\log x}{4\log\log x}\right\}\right),$$

for every $\epsilon > 0$. This concludes the proof of (6.7).

□

6.2 $\Omega-$Results for $\mathcal{N}_0(x) - \tau x$

Now we turn our attention to find lower bounds for $E(x)$. First, we have a result that follows from existing theory.

Theorem 6.2. *We have*

$$\mathcal{N}_0(x) = \tau x + \Omega(x^{1-\delta}), \ \forall \delta > 0. \tag{6.11}$$

Proof. If (6.11) is not true then $\mathcal{N}_0(x) = \tau x + o(x^{1-\delta})$, which implies that $\mathcal{N}_0(x) = \tau x + O(x^{1-\delta})$, for some $\delta > 0$. Thus we have a 'well-behaved' system (see section 3).

$$\psi_0(x) = x + O(1)$$
$$\mathcal{N}_0(x) = \tau x + O(x^{1-\delta}), \quad \delta > 0,$$

By Lemma 3.3 for $1 - \delta < \sigma < 1$, we have

$$\zeta(s) - 1 = -\frac{\zeta_0'}{\zeta_0}(s) = O\big((\log t)^{\frac{1-\sigma}{\delta}+\epsilon}\big), \ \forall \epsilon > 0.$$

This is a contradiction, since we know by Theorem 2.3 that

$$|\zeta(\sigma + it)| = \Omega(\exp\left\{c(\log t)^{1-\sigma-\epsilon}\right\}), \quad \text{for some } c > 0 \quad \text{and any} \quad \epsilon > 0.$$

$\square$

In order to improve the above result, we will apply Theorem 5.2. Our strategy here is to show that the conditions of Theorem 5.2 are satisfied (i.e. $\zeta_0(\sigma + it)$ has at most polynomial growth in a region just to the left of $\sigma = 1$). This will force a contradiction with the knowledge of lower bounds of the Riemann zeta function (since $\zeta_0(s)$ is connected to the Riemann zeta function).

Theorem 6.3. *We have*

$$\mathcal{N}_0(x) = \tau x + \Omega(xe^{-ck(x)}), \quad \textit{for every } c > 1, \tag{6.12}$$

where $k(x) = \frac{\log x \log\log\log\log x}{\log\log\log x}$.

Before we prove Theorem 6.3, we recall the following Proposition from Chapter 2.

Proposition 6.4. *For* $\frac{3}{4} \leq \sigma \leq 1 - \frac{\log\log\log N}{2\log\log N}$, *we have*

$$\max_{1<t<N} |\zeta(\sigma + it)| \geq \exp\left\{(1 + o(1))\frac{(\log N)^{1-\sigma}}{16(1-\sigma)\log\log N}\right\},$$

for $N \geq N_0$ *independent of* σ.

Proof of Theorem 6.3. If (6.12) is not true then

$$\mathcal{N}_0(x) = \tau x + O(xe^{-ck(x)}) \text{ for some } c > 1.$$

We know that $\zeta_0(s)$ has an analytic continuation to $\mathbb{C}\backslash\{1\}$ with a simple pole at $s = 1$ and $\zeta_0(s) \neq 0$ in this region. Moreover, by (6.10) we have $\zeta_0(\sigma + it) \ll e^{\epsilon t}$, $\forall \epsilon > 0$ for $\lambda \leq \sigma < 1$, (some fixed λ). Therefore, we can apply Theorem 5.2 as the conditions are satisfied. We obtain

$$\zeta_0(\sigma + it) = O(t^b), \quad \text{for some} \quad b > 0,$$

for $1 - \frac{\log t}{t} \geq \sigma \geq 1 - \frac{(1-\epsilon)c \log\log\log t}{\log\log t}$, (any $\epsilon > 0$, and $t > t_0(\epsilon)$ since $\frac{ck(\frac{e^t}{t})}{t} \sim \frac{c \log\log\log t}{\log\log t}$). Furthermore, (6.9) tells us that

$$\log|\zeta_0(s)| \leq \int_\sigma^2 |\zeta(u + it) - 1|\, du + O(1) \ll \log t,$$

since $\zeta(s) = O(\log t)$ for $1 - \frac{a}{\log t} \leq \sigma \leq 2$, (any $a > 0$), see Theorem 3.5. in [29]. Let $B(t) = \frac{(1-\epsilon)c \log\log\log t}{\log\log t}$. Therefore, for $1 - B(t) \leq \sigma \leq 2$,

$$\log|\zeta_0(\sigma + it)| \leq A \log t \quad \text{for some} \quad A > 0.$$

Consider concentric circles with centre $\vartheta + it$ (for some $\vartheta > 1$) and radii $R_1 = \vartheta - 1 + B(t) - \lambda(t)$ and $R_2 = \vartheta - 1 + B(t) - 2\lambda(t)$, (with $\lambda(t) = \frac{1}{\log\log t}$). Apply the Borel-Carathéodory Theorem to $\log \zeta_0(z)$, (see 9.1 in [29]). Therefore, for $\sigma \geq 1 - B(t) + \lambda(t)$ and $t \geq t_0$, we obtain

$$\begin{aligned}
|\log \zeta_0(\sigma + it)| &\leq \frac{2R_2}{R_1 - R_2} A \log t + \frac{R_1 + R_2}{R_1 - R_2} |\log \zeta_0(\vartheta + it)| \\
&\leq \frac{A_1}{\lambda(t)} \log t + \frac{D}{\lambda(t)} \ll \log t \log\log t,
\end{aligned}$$

for some $A_1, D > 0$.

Now, let C be the circle (see Figure 6.1) with centre $1 - dB(t) + \lambda(t) + it$, for some $d \in (\frac{1}{2}, 1)$ and radius $R = rB(t)$ for some positive constant $r < 1 - d$. By Cauchy's integral formula

$$-\frac{\zeta_0'}{\zeta_0}(s) = \frac{1}{2\pi i} \int_C \frac{\log \zeta_0(z)}{(z - s)^2} dz \quad \text{for} \quad s \in C.$$

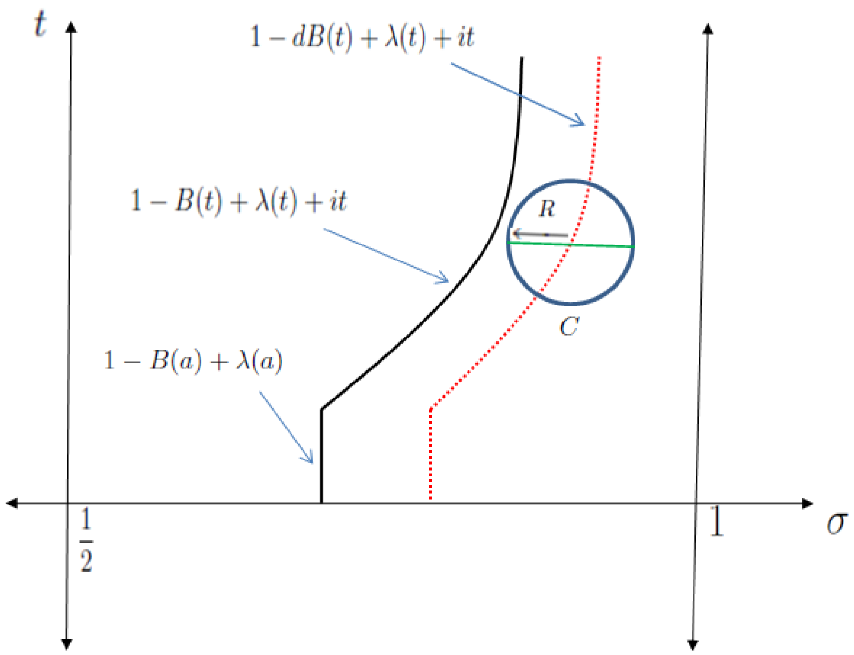

Figure 6.1: circle C

Therefore, for $s \in C$, we have

$$\left|\frac{\zeta_0'}{\zeta_0}(s)\right| \leq \frac{1}{R} \max_{z \text{ on } C} |\log \zeta_0(z)| \ll \frac{1}{B(t)} \log t \log\log t = o(\log^2 t).$$

So, this tells us that for $s \in C$

$$\zeta(s) = o(\log^2 t), \quad \text{for} \quad \sigma \geq 1 - \frac{(1-\epsilon)c \log\log\log t}{\log\log t} \tag{6.13}$$

However, by Proposition 6.4, for $1 - \sigma = \frac{(1-\epsilon)c \log\log\log T}{\log\log T}$, with $(1-\epsilon)c > 1$, we have

$$\begin{aligned}
\max_{1<t<T} |\zeta(\sigma + it)| &\geq \exp\left\{\frac{\big(1 + o(1)\big)(\log T)^{1-\sigma}}{16(1-\sigma)\log\log T}\right\} \\
&= \exp\left\{\big(1 + o(1)\big)\frac{(\log\log T)^{(1-\epsilon)c}}{16(1-\epsilon)c \log\log\log T}\right\} \\
&> e^{2\log\log T} = (\log T)^2.
\end{aligned}$$

This is a contradiction with (6.13). $\square$

In the previous sections we have been trying to obtain good lower and upper bounds for $\mathcal{N}_0(x) - \tau x$. That is, upper bounds for $\mathcal{N}_0(x) - \tau x$ which holds for all sufficiently large values of x and lower bound for $\mathcal{N}_0(x) - \tau x$ which holds for a sequence of x's tending to infinity [and not necessarly for all (sufficiently large) values of x]. For the upper bound of $\mathcal{N}_0(x) - \tau x$ we have shown some unconditional $O-$results and one result was conditional with the unproved Riemann Hypothesis.

Set $\Delta(x) = \mathcal{N}_0(x) - \tau x$. A comparison of the $O-$results and $\Omega-$results (based on Theorem 6.1 and Theorem 6.3) of this chapter, we have shown that

$$\Delta(x) = \Omega\left(x \exp\left\{ -\frac{c \log x \log\log\log\log x}{\log\log\log x} \right\}\right) \quad \text{for every } \ c > 1,$$

while on the Riemann Hypothesis,

$$\Delta(x) \ll x \exp\left\{ -\frac{(1-\epsilon) \log x \log\log\log x}{4 \log\log x} \right\}, \quad \text{for every } \ \epsilon > 0.$$

This shows that there is a small gap between these results which reflects the great difficulty in determining the behaviours of $\zeta_0(s)$ in the strip $\frac{1}{2} < \sigma < 1$. The interesting question is: What is the true order of this error term?

6.3 $\mathcal{P}_0$ is a $g-$prime system

We end this chapter by showing that the pair $(\Pi_0, \mathcal{N}_0)$ is a g-prime system. That is, we show $\pi_0 \in S_0^+$, (i.e. π_0 is increasing).

Theorem 6.5. $(\Pi_0, \mathcal{N}_0)$ *is a g-prime system.*

We prove Theorem 6.5 by showing $\pi_0 \in S_0^+$. Writing

$$\vartheta_0(x) = \int_1^x \log y \ d\pi_0(y), \tag{6.14}$$

which tells us that $\pi_0 \in S_0^+ \Leftrightarrow \vartheta_0 \in S_0^+$. Therefore, we will show that $\vartheta_0 \in S_0^+$ and this will complete the proof of Theorem 6.5.

Now, we have

$$\psi_0(x) = \sum_{n=1}^{\infty} \vartheta_0(x^{\frac{1}{n}}),$$

(see definition 13 in chapter 3) and by the Möbius Inversion Formula we get

$$\vartheta_0(x) = \sum_{n=1}^{\infty} \mu(n)\psi_0(x^{\frac{1}{n}}).$$

Therefore, with $\psi_0(x) = [x] - 1, \ x \geq 1$, we have

$$\vartheta_0(x) = \sum_{n=1}^{\infty} \mu(n)\big([x^{\frac{1}{n}}] - 1\big).$$

[**Note:** The above series is finite since the terms are zero for $n > \frac{\log x}{\log 2}$.] The following Proposition will complete the proof.

Proposition 6.6. *Let* $\vartheta_0(x) = \sum_{n=1}^{\infty} \mu(n)\big([x^{\frac{1}{n}}] - 1\big), \ x \geq 1$. *Then the following hold:*

(i) $\quad \vartheta_0(x) = \vartheta_0(k)$ *for* $k \leq x < k+1, \ k \in \mathbb{N}$, (*i.e.* $\vartheta_0(x) = \vartheta_0([x])$.)

(ii) *Define* $f(n,k) = [k^{\frac{1}{n}}] - [(k-1)^{\frac{1}{n}}], \ n, k \in \mathbb{N}$. *Then*

$$f(n,k) = \begin{cases} 1 & \textit{if } k = q^n, \quad \textit{for some } q \in \mathbb{N}, \\ 0 & \textit{if } k \neq q^n, \quad \textit{for any } q \in \mathbb{N}. \end{cases}$$

(iii) *We call* $k \in \mathbb{N}$ *a* perfect power *if there exist natural numbers* $q > 1$, *and* $n > 1$ *such that* $k = q^n$. *Then*

$$\vartheta_0(k) - \vartheta_0(k-1) = \begin{cases} 1 & \text{if } k \text{ is not a perfect power }, \; k \geq 2 \\ 0 & \text{if } k \text{ is a perfect power }. \end{cases}$$

Proof. (i) For $k \leq x < k+1, \; k \in \mathbb{N}$

$$\vartheta_0(x) - \vartheta_0(k) = \sum_{n=1}^{\infty} \mu(n)\big([x^{\frac{1}{n}}] - [k^{\frac{1}{n}}]\big).$$

Thus, by showing $[k^{\frac{1}{n}}] \leq x^{\frac{1}{n}} < [k^{\frac{1}{n}}] + 1$, we will have completed the proof of (i) since the above sum will equal zero.

We have $k^{\frac{1}{n}} \leq x^{\frac{1}{n}} < (k+1)^{\frac{1}{n}}$, so, it is clear that $x^{\frac{1}{n}} \geq [x^{\frac{1}{n}}] \geq [k^{\frac{1}{n}}]$. It remains to show that $x^{\frac{1}{n}} < [k^{\frac{1}{n}}] + 1$.

Assume that $x^{\frac{1}{n}} \geq [k^{\frac{1}{n}}] + 1$, for some $n, k \in \mathbb{N}$. We have $[k^{\frac{1}{n}}] \in \mathbb{N} \cup \{0\}$, so we let $q = [k^{\frac{1}{n}}]$. Clearly $q \leq k^{\frac{1}{n}} < q+1$. That is

$$q^n \leq k < (q+1)^n.$$

Thus,

$$k + 1 \leq (q+1)^n \leq x,$$

since $x^{\frac{1}{n}} \geq [k^{\frac{1}{n}}] + 1 = q + 1$. This is a contradiction (since $k \leq x < k+1$). Hence $\vartheta_0(x) = \vartheta_0(k)$ for $k \leq x < k+1, \; k \in \mathbb{N}$.

(ii) First we prove that $f(n,k)$ is either 1 or 0. We have $[k^{\frac{1}{n}}] \geq [(k-1)^{\frac{1}{n}}] \geq 0$, so we need to show that

$$[k^{\frac{1}{n}}] - [(k-1)^{\frac{1}{n}}] \leq 1.$$

Suppose for a contradiction that $[k^{\frac{1}{n}}] > 1 + [(k-1)^{\frac{1}{n}}]$, for some $n, k \in \mathbb{N}$. Let $q = [(k-1)^{\frac{1}{n}}]$. Then the assumption implies $k^{\frac{1}{n}} \geq [k^{\frac{1}{n}}] > q + 1$. That is,

$$k > (q+1)^n. \tag{6.15}$$

However, $q \leq (k-1)^{\frac{1}{n}} < q+1$. That is, $k < (q+1)^n + 1$. Therefore, as both sides are integers

$$k \leq (q+1)^n.$$

This is in contradiction with (6.15). Hence

$$0 \leq [k^{\frac{1}{n}}] - [(k-1)^{\frac{1}{n}}] \leq 1.$$

However $[k^{\frac{1}{n}}]$ and $[(k-1)^{\frac{1}{n}}]$ are integers, therefore $f(n,k)$ is either 1 or 0.

Now, for $k = q^n$ for some $q \in \mathbb{N}$, then $[k^{\frac{1}{n}}] = q$ and

$$[(k-1)^{\frac{1}{n}}] = [(q^n - 1)^{\frac{1}{n}}] < q.$$

Therefore, $f(n,k) = 1 \quad \text{if} \quad k = q^n \quad \text{for some} \quad q \in \mathbb{N}.$

We end the proof of (ii) by showing that $f(n,k) = 0$ whenever $k \neq q^n$, $q \in \mathbb{N}$.

Suppose that $[k^{\frac{1}{n}}] - [(k-1)^{\frac{1}{n}}] = 1$ for some k such that $k \neq q^n,\ q \in \mathbb{N}$ (i.e. $[k^{\frac{1}{n}}] - 1 = [(k-1)^{\frac{1}{n}}]$). We have $[(k-1)^{\frac{1}{n}}] \leq (k-1)^{\frac{1}{n}} < [(k-1)^{\frac{1}{n}}] + 1$. That is,

$$[k^{\frac{1}{n}}] - 1 \leq (k-1)^{\frac{1}{n}} < [k^{\frac{1}{n}}].$$

Let $q = [k^{\frac{1}{n}}]$ we get $k^{\frac{1}{n}} \geq q$. That is,

$$k \geq q^n. \tag{6.16}$$

However, $(k-1)^{\frac{1}{n}} < q$. That is, $k < q^n + 1$. Therefore, as both sides are integers

$$k \leq q^n. \tag{6.17}$$

From (6.16) and (6.17) we get $k = q^n,\ \ q \in \mathbb{N}$. This is a contradiction. Therefore,

$$f(n,k) = 0 \quad \text{if} \quad k \neq q^n, \quad \text{for some} \quad q \in \mathbb{N}.$$

Remark: For the particular case when k is a perfect power, we have that

$$f(n, q^r) = \begin{cases} 1 & \text{if } n \mid r, \\ 0 & \text{if } n \nmid r. \end{cases}$$

(iii) We have from (ii) that

$$f(n,k) = \begin{cases} 1 & \text{if } k = q^n, \quad \text{for some} \quad q \in \mathbb{N}, \\ 0 & \text{if } k \neq q^n, \quad q \in \mathbb{N}. \end{cases}$$

Therefore, for k is not a perfect power, we have

$$\vartheta_0(k) - \vartheta_0(k-1) = \sum_{n=1}^{\infty} \mu(n) f(n,k) = 1 + \sum_{n=2}^{\infty} \mu(n) f(n,k) = 1.$$

Now, for k is a perfect power, let r be a maximal natural number greater than or equal 2 such that $k = q^r$, $q > 1$, $q \in \mathbb{N}$. So, by the above Remark we have

$$\vartheta_0(k) - \vartheta_0(k-1) = \vartheta_0(q^r) - \vartheta_0(q^r - 1) = \sum_{n=1}^{\infty} \mu(n) f(n, q^r) = \sum_{n|r} \mu(n).$$

By Theorem 2.1 of [2] we have the last sum is zero (since $r > 1$). The proof of Proposition 6.6 is completed.

$\square$

Theorem 6.5 follows as $\vartheta_0(x)$ is increasing. Furthermore we have the following Corollary:

Corollary 6.7. $\vartheta_0(x)$ *is increasing step function with jump* 1*. In fact, the jumps appear only when* x *is not a perfect power.*

Bibliography

[1] T. M. Apostol, *Mathematical Analysis Second Edition*, Addison-Wesley, 1974.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[3] E. P. Balanzario, An example in Beurling's theory of primes, *Acta Arithmetica*, **87** (1998), 121–139.

[4] P. T. Bateman and H. G. Diamond, *Introduction to Analytic Number Theory-An Introduction Course*, World Scientific, 2004.

[5] P.T. Bateman and H.G. Diamond, Asymptotic distribution of Beurling's generalised prime numbers, *J. Studies in Number Theory. Ann*, **6** (1969), 152–212.

[6] A. Beurling, Analyse de la loi asymptotique de la distribution des nombres premiers generalises, *Acta Arithmetica*, **68** (1937), 255-291.

[7] H. Diamond, Asymptotic distribution of Beurling's generalized integers, *Illinois J. Math.*, **14** (1970), 12-28.

[8] H. Diamond, The prime number theorem for Beurling's generalized numbers, *J. Number Theory*, **1** (1969), 200–207.

[9] H. Diamond, A set of generalized numbers showing Beurling's Theorem to be sharp, *Illinois J. Math.*, **14** (1970), 29-34.

[10] H. Diamond, When do Beurling's generalized integers have density?, *J. Reine Angew. Math.*, **295** (1977), 22-39.

[11] H. Diamond, H. Mongomery, and U. Vorhauer, Beurling primes with large oscillation, *Math. Ann.*, **334** (2006), 1–36.

[12] W. Ellison and M. Mends, *The Riemann Zeta-function theory and application*, Actualits Scientifiques et Industrielles. Hermann, Paris, 1975.

[13] R. S. Hall. *Theorems about Beurling generalised primes and associated zeta function*. PhD thesis, University of Illinois,USA, 1967.

[14] T. Hilberdink, An arithmetical mapping and applications to Ω-results for the Riemann zeta function, *Acta Arithmetica*, **139** (2009), 341-367.

[15] T. Hilberdink, Generalised prime systems with periodic integer counting function, *Acta Arithmetica*, **152** (2012), 217–241.

[16] T. Hilberdink, Well-behaved Beurling primes and integers, *J. Number Theory*, **112** (2004), 332–344.

[17] T. Hilberdink, L. Lapidus, Beurling zeta functions, generalised primes, and fractal membranes, *Acta. Appl. Math.*, **94** (2006), 21–48.

[18] M. N. Huxley, Exponential sums and the Riemann zeta function V, *London Mathematical Society*, **90** (2005), 1–41.

[19] A. Ivić, *The Riemann Zeta-function Theory and Applications*, New York:Wiley, 1985.

[20] J. P. Kahane, Sur les nombres premiers généralisés de Beurling. Preuve d'une conjecture de Bateman et Diamond, *Journal de théorie des nombres de Bordeaux*, **9** (1997), 251–266.

[21] J. C. Lagarias, Beurling generalised integers with the Delone property, *Forum Math*, **11** (1999), 295–312.

[22] A. Landau, Neuer beweis des primzahlsatzes und beweis des primidealsatzes, *Math Ann*, **56** (1903), 645–670.

[23] N. Levinson, $\Omega-$results for the Riemann zeta-function, *Acta Arithmetica*, **20** (1972), 317330.

[24] P. Malliavin, Sur le reste de la loi asymptotique de repartition des nombres premiers generalises de Beurling, *Acta Math*, **106** (1961), 281–298.

[25] B. Nyman, A general prime number theorem, *Acta Mathematica*, **81** (1949), 299–307.

[26] R. Olofsson, Properties of the Beurling generalized prime, *J. Number Theory*, **131** (2011), 45–58.

[27] H. E. Richert, Zur abschtzung der Riemannschen zetafunktion in der nhe der vertikalen, *Mathematische Annalen*, **169** (1967), 97–101.

[28] E. C. Titchmarsh, *The Theory of Functions theory and application*, New York:Wiley, 1985.

[29] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford Clarendon Press, 1986.

[30] H. Weber, Über einen in der zahlentheorie angewandten satz der integralrechnung, *Nachr. Akad. Wiss. Göttingen*, **1896** (1896), 275–281.

[31] W. Zhang, Beurling primes with RH, Beurling primes with large oscillation, *Math. Ann.*, **337** (2007), 671–704.