University of Reading

Modelling of Hot Water Flooding
as an
Enhanced Oil Recovery Method

by

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Declaration

I confirm that this is my own work and the use of all materials from other sources has been properly and fully acknowledged.

Signed ....

Abstract

This dissertation describes and compares two numerical techniques that simulate one dimensional hot water injection. In total four equations are introduced in order to model hot water injection; the Buckley-Leverett equation, two mass balance equations for water and oil phases and an energy balance equation, all of which are highly non-linear. The objective of the mathematical model is to solve these equations under the appropriate initial and boundary conditions. This solution provides space and time distributions of water and oil pressures, saturations and temperature. One of the major difficulties with numerical modelling of this process is the dependence of the fluid properties on the pressure and temperature. In the first technique, the Buckley-Leverett equation is used to calculate oil and water saturation distributions which is a nonlinear hyperbolic equation. The second order Lax-Wendroff scheme is used to solve this equation. The results of the saturations are used in the mass balance equation, which is a nonlinear equation since its coefficients depend on temperature and pressure. A fully implicit central scheme is used in order to discretize the equation and then the Newton-Raphson method is used to solve this nonlinear system in order to find the pressure distribution. Finally, the pressure results are used in the nonlinear energy equation to obtain the temperature profile. In the second model, the implicit pressure/explicit saturation (IMPES) technique is used for the mass balance equations of water and oil phases in order to find the pressure and saturation distributions, then the results are used in the energy equation to get temperature profiles. Since all these equations are nonlinear and depend on each other, the energy equation needs to be coupled with material balance equations. Results show that saturation front in the first model lag behind that of the second model which can be a result of incompressibility assumption used in it. The second model has to be applied with some care as it can be easily become unstable, but if it is used in its stability domain the results are more reliable.

Contents

1 Introduction

A hydrocarbon reservoir is an underground volume comprised of porous rock containing a mixture of water and hydrocarbon fluids in the form of oil and gas, occupying the void space of the pores in the rock. Oils can be divided into two categories, light oils and heavy oils. Light oils have a low viscosity while heavy oils have a high viscosity. The viscosity of a fluid is a measure of how easily that fluid will flow, for instance, water has a very low viscosity while honey has a high viscosity.

When oil recovery is high due to high natural reservoir pressure. The rate of natural oil production will diminish with time, but there are some oil recovery methods to improve the production rate. Oil recovery processes involve the injection of fluid or a combination of fluid and chemicals into the oil reservoir via injection wells to force as much oil as possible towards and, hence, out of the production wells. Light oils are extracted under primary and secondary recovery methods which involve allowing the fluid to flow out under the natural pressure of its surrounding. These methods cannot be applied to the extraction of heavy oils, whose viscosity is far too high for such methods to be effective; their viscosity needs to be reduced. This is achieved by various thermal stimulation techniques like hot water flooding, steam injection, in-situ combustion and so far which raise the temperature of the oil, effectively reducing its viscosity. The approach which is under consideration here is hot water injection modeling. It is necessary to model and simulate this process in order to provide information about production and the future of the reservoir to get the best recovery.

All thermal recovery processes tend to raise the temperature of the crude in a reservoir to reduce the reservoir flow resistance by reducing the viscosity of the crude [12]. It is desirable to heat the reservoir efficiently, but inevitably some of the heat in the reservoir is lost through produced fluids, and some is lost to the adjacent overburden and underburden formations. The heat loss to the adjacent formations is controlled by conduction (heat transfer) which it can be readily estimated.

In hot water flooding, as can be seen in figure 1.1 many reservoir equivalent volumes of hot water are injected into a number of wells in order to reduce the viscosity and subsequently displace the oil in place more easily towards oil production wells. Hot water injection may be preferred in shallow reservoirs containing oils in the viscosity range of 100-1000 cp [4].

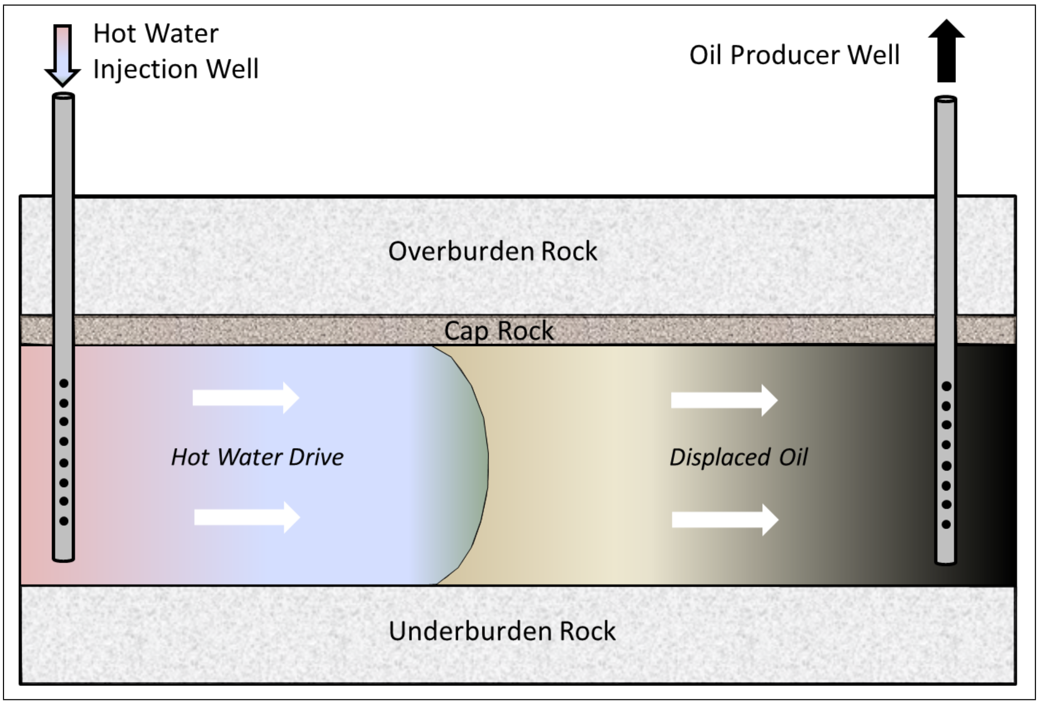

Figure 1.1: Schematic diagram of hot water injection process

The mathematical model representing the physical process of hot water injection requires rock and fluid properties in order to describe the fluid flow and heat transfer with a set of partial differential equations and algebraic equations, which are derived from physical principals. This set of equations is derived from four main principles: Conservation of mass of phases (water and oil); Darcy's Law for volumetric flow rates which describes how the fluid phases flow through the reservoir; volume balance equation, a condition which states that the fluid fills the rock pore volume; conservation of energy of phases. Since the resulting equations are too complex for more realistic models to be solved using analytic techniques, here is focused on numerical techniques.

In this dissertation, two different models are applied and analyzed for the hot water injection process. Chapter 2 contains the problem definition and characteristics of the model. Introducing some necessary concepts about rock and fluid properties, and required equations. Initial and boundary conditions and heat loss in our model are also included in this part. The first model is introduced in chapter 3, where in order to find the saturation distribution the Buckley-Leverett equation is used. The nonlinear mass balance equation is solved by a fully implicit central technique by using the results of oil and water saturations from the Buckley-Levertt equation. Subsequently, the saturation and pressure results are applied to a nonlinear energy equation discretized by a fully implicit method. Finally, the mass balance equation (pressure equation) and energy balance equation (temperature equation) are coupled to find the best result for pressure and temperature distribution, since these equations are highly nonlinear. In chapter 4 the second model is presented. In this model, implicit pressure explicit saturation (IMPES) technique is applied to our hot water model. During one time step, the results of IMPES are used in the temperature equation which is solved fully implicitly, and finally there is a coupling between IMPES technique and fully implicit temperature equation in order to find the final pressure, saturation and temperature distribution results. In both approaches, bottom hole pressures at the boundaries for the two model are also calculated using a well coupling method. Because of the complexity of the models we have tried to give easier understanding of the models by summarizing the models in flowchart diagrams at the end of each chapter. Chapter 5 shows and compares the results of the two models and some sensitivity analysis are presented as well. Finally chapter 6 outlines the conclusions which are drawn from the results.

2 Characteristics of the Model

In this project, we have tried to model the hot water flooding process in a reservoir which is initially saturated with oil and water. The reservoir is considered to be one-dimensional between an injection and production wells. A schematic diagram of the model is given in figure 2.1. Hot water is injected with a constant rate and temperature into the porous media which is filled with cold and heavy oil. In such a system fluid flow, heat transfer and heat losses are modeled in order to give a better understanding of the process and its effect on oil recovery.

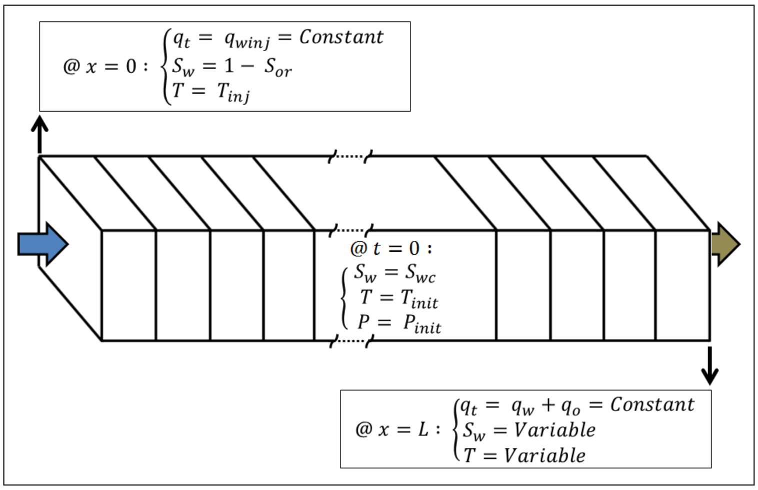

Figure 2.1: Schematic diagram of the problem

2.1 Assumptions

The following assumptions are made to model the process;

1. In all reservoir processes, every point within the reservoir is in thermodynamics equilibrium.
2. The injected fluid reaches thermal equilibrium instantaneously with the reservoir fluids and sand, meaning that all phases and rock in the same location have the same temperature.

3. The model simulates one-dimensional fluid flow and heat convection but two-dimensional heat conduction throughout the underburden – reservoir –overburden system.
4. There is a two-phase (water and oil) system which is immiscible.
5. There in no capillary pressure ($P_O = P_w = P$).
6. Gravity effects are neglected.

2.2 Rock and Fluid Properties Description

The data of rock and fluid properties are required to understand the concept of the model. Among these, Darcy's Law, porosity, saturation, permeabilities and phase viscosities and densities are introduced briefly below.

2.2.1 Darcy's Law

Darcy's Law describes the flow of a fluid through a porous medium. It determines how fast the phases flow through the reservoir and gives the phase velocities [1]. For one dimensional flow, the Darcy's phase velocities can be written

$$V_\alpha = -C_\alpha K_{abs}\left(\frac{\partial P_\alpha}{\partial x} - \rho_\alpha g\frac{\partial d}{\partial x}\right) \qquad \alpha = Oil,\ Water \tag{2.1}$$

where

$$C_\alpha = \frac{K_{r\alpha}}{\mu_\alpha} \tag{2.2}$$

denotes phase mobilities (which are phase relative permeabilities divided by phase viscosities) ρ denotes phase mass densities; $\frac{\partial d}{\partial x}$ represents the depth gradient; and K_{abs} is the absolute permeability of the reservoir; P_α is the pressure of each phase. The fluid flow is therefore due to a pressure gradient and a gravitational potential, g. In this project, by assumptions 5 and 6 Darcy' law is simplified to

$$V_\alpha = -C_\alpha K_{abs}\left(\frac{\partial P}{\partial x}\right) \qquad \alpha = Oil,\ Water \tag{2.3}$$

A fuller description of some of the terms in Darcy' law is now given in more detail.

2.2.2 Porosity

Oil is contained in rocks which are a type of porous media. Porosity is the ratio of void space over the bulk volume of the rock [1],

$$\varphi = \frac{Pore\ Volume\ \ (V_p)}{Bulk\ Volume\ \ (V_b)} \tag{2.4}$$

2.2.3 Saturation

The pore volume space is not always filled with a single fluid. Saturation of each fluid (phase) is defined as the ratio of its volume over the total pore volume occupied by all phases [1],

$$S_i = \frac{Phase\ Volume\ (V_i)}{Pore\ Volume\ (\ V_p)} \tag{2.5}$$

By definition, the saturations are all non-negative, and sum to one.

2.2.4 Permeabilities

One of the main properties of porous rock is its capability to allow fluid flow through its connected pores which is known as permeability. There are two definitions of permeability in the oil industry; absolute and relative permeabilities. Under the condition of single phase flow, this capability is named absolute permeability. But when the porous media is filled by more than one phase, due to various ways the phases can occupy the pore volume, the phases adversely affect the flow of each other in a complicated manner [1]. This effect is described using phase relative permeabilities, K_{ro} and K_{rw}. The dependence of the relative permeabilities on the rock and fluid properties is very complicated [2]; the K_{ro} and K_{rw} considered here are non-negative functions of the saturation, S. In this case, it is necessary that relative permeabilities must tend to zero as its saturations approaches zero. There are different methods used to find relative permeabilities. In this project the Corey-type, which is a power law in the water saturation, $\mathrm{S_w}$, is chosen [3].

$$\begin{cases} K_{row}\,(S_w) = K_{ro}^{\max}\,(1 - S_{wn})^{no} & n_o = 3 \\ \\ K_{rw}\ (S_w) = K_{rw}^{\max}.S_{wn}^{nw} & n_w = 3 \end{cases} \tag{2.6}$$

where

$$S_{wn}\ (S_w) = \frac{S_w - S_{wi}}{1 - S_{wi} - S_{orw}} \tag{2.7}$$

$$\begin{cases} K_{row}\ (S_{wi}) = K_{ro}^{\max}\ ,\ K_{row}\ (1 - S_{or}) = 0 \\ \\ K_{rw}\ (S_w) = 0 \qquad ,\ K_{rw}\ (1 - S_{or}) = K_{rw}^{\max} \end{cases} \tag{2.8}$$

S_{wi}: Irreducible water saturation
S_{wc}: Connate water saturation
S_{orw}: Residual water saturation (water-oil system)
S_w: Water saturation
S_{wn}: Normalized water saturation

Figure 2.2 shows the results of water and oil relative permeabilities versus water saturation in the system by applying the Corey correlation.In describing two-phase flow mathematically, it is always the relative permeability ratio, $\frac{\mathrm{K_{ro}}}{\mathrm{K_{rw}}}$, versus water saturations (for oil and water system) that enters the equations.

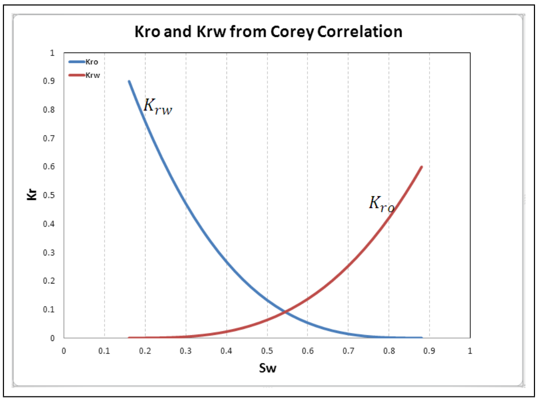

Figure 2.2: Oil and water relative permeabilities using Corey correlation

2.2.5 Hydrocarbon Viscosity

Phase viscosity represents the resistance of a phase to flow under the influence of a pressure gradient. The most obvious effect of thermal recovery on a reservoir fluids is the reduction of oil viscosity. In figure 2.3 two points are evident. First, the rate of viscosity improvement is greatest as the initial temperature increases. Little viscosity benefit is gained after reaching a certain temperature. Second, greater viscosity reductions are experienced in the more viscous low API gravity crudes (API is a degree of measurement for oil density) than in higher API gravity crudes. Heating from 100^{o}F to 200^{o}F reduces the viscosity, 98% for 10^{o}API crudes but only 73% for 30^{o}API oils. These observations show that the greatest viscosity reduction occurs with the more viscous oils at the initial temperature increases [4].

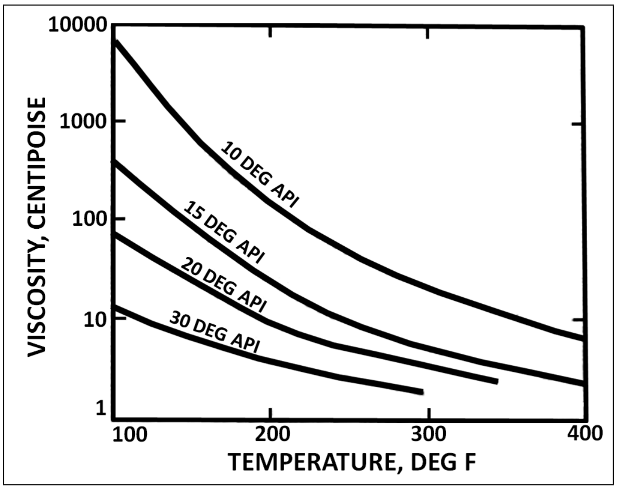

Figure 2.3: Effect of temperature on viscosity

Viscosity is a function of temperature and pressure, but water and oil viscosities are stronger functions of temperature in a thermal process rather than pressure. Since a thermal oil recovery method is modeled in this project, the effect of pressure is neglected [5].

$$\mu_o = 2.626 \times 10^8 \ (T - 459.59)^{-2.91} \tag{2.9}$$

$$\mu_w = \frac{2.185}{0.04012\,(T - 459.59) + 5.154 \times 10^{-6} \ (T - 459.59)^2 - 1} \tag{2.10}$$

2.2.6 Phase Mass Density

Phase density is defined as mass per unit volume for each phase. Water and oil densities in this context are considered to be a function of temperature and pressure. In the absence of experimental data, empirical relations are used to express densities of oil and water as functions of both temperature and pressure [5]:

$$\rho_w = 63 \ \exp\left(17.253 \times 10^{-5} \ (T - 459.59)\right) \ \exp\left(4 \times 10^{-6} \ (P - 1000)\right) \tag{2.11}$$

$$\rho_o = 59 \ \exp\left(-7.5885 \times 10^{-5}\,(T - 459.59)\right) \ \exp\left(1 \times 1^{-5} \ (P - 1000)\right) \tag{2.12}$$

Figure 2.4 shows the effect of temperature and pressure increase on the densities of both phases.

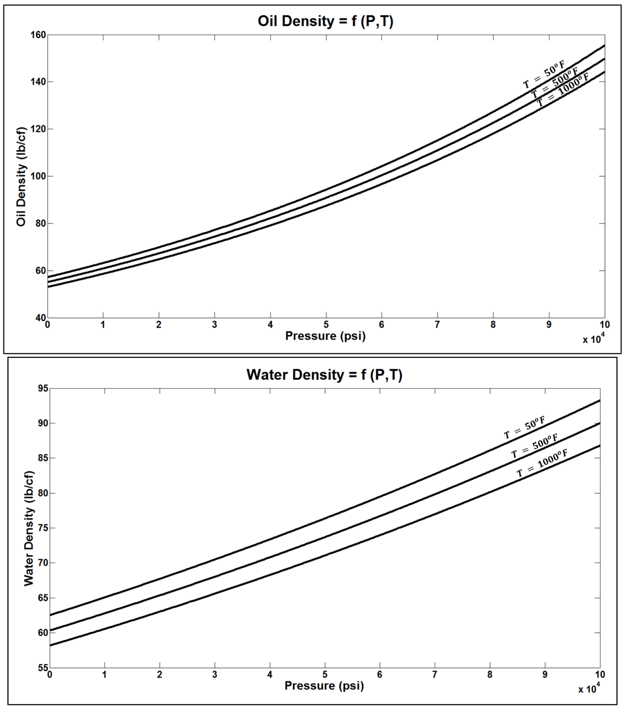

Figure 2.4: Effect of temperature and pressure on oil and water densities

2.3 Introducing Model Equations

In the two hot water models presented in this dissertation, four equations are required; the Buckley-Leverett equation, mass balance equations for water and oil, and an energy Balance equation.

2.3.1 Buckley-Leverett

The Buckley-Leverett (BL)equation is used in oil recovery in order to find the saturation distribution in 1D reservoir. In the BL mechanism oil is displaced by water from a rock in a similar as fluid is displaced from a cylinder by a leaky piston. In order to have better understanding of the Buckley-Leverett equation, it is first necessary to introduce the fractional flow equation.

2.3.1.1 Derivation of Fractional Flow for the Model

When oil is displaced by water in the system, from Darcy's equation we have

$$q_w = -1.127\ K_{abs}\frac{K_{rw}}{\mu_w}A_x\left(\frac{\partial P}{\partial x}\right) \tag{2.13}$$

$$q_o = -1.127\ K_{abs} \frac{K_{ro}}{\mu_o} A_x \left(\frac{\partial P}{\partial x}\right) \tag{2.14}$$

By adding the two equations

$$q_w + q_o = -1.127\ K_{abs} A_x \left(\frac{K_{rw}}{\mu_w} + \frac{K_{ro}}{\mu_o}\right) \frac{\partial P}{\partial x} \tag{2.15}$$

Substituting for

$$q = q_w + q_o \tag{2.16}$$

and

$$f_w = \frac{q_w}{q} \tag{2.17}$$

and solving for the fraction of water flowing, we obtain

$$f_w = \frac{1}{1 + \frac{K_{ro}}{\mu_o} \cdot \frac{\mu_w}{K_{rw}}} \tag{2.18}$$

since

$$K_r\left(S_w\right) \Rightarrow f_w(S_w) \tag{2.19}$$

Now, the Buckley-Leveret equation is derived for a 1D sample based on mass conservation and some assumptions [6], namely flow is linear and steady state, the fluid is incompressible, capillary pressure (P_c) is just a function of the saturation and pressure gradient for two phases is equal ($\frac{dP_c}{dS} = 0$), where $P_c = P_o - P_w$.

By applying mass balance of water around a control volume (see figure 2.5) of length $\triangle x$ we get the following system for a time period of $\triangle t$:

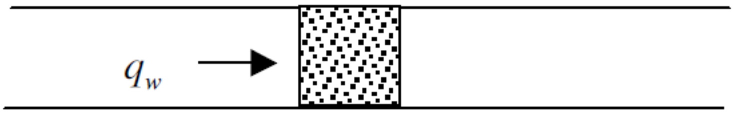

Figure 2.5: Mass Balance Element for Fractional Flow Equation

The material balance may be written:

$$\left[(\rho_w q_w)_x - (\rho_w q_w)_{x+\triangle x}\ \right] \triangle t = A \triangle x \varphi \left[(\rho_w S_w)^{t+\Delta t} - (\rho_w S_w)^t\right] \tag{2.20}$$

which, when $\Delta x \to 0$ and $\Delta t \to 0$, reduces to the continuity equation:

$$-\frac{\partial}{\partial x}\left(\rho_w q_w\right) = A\varphi \frac{\partial}{\partial t}(\rho_w S_w) \tag{2.21}$$

by assuming an incompressible fluid $\rho_w = constant$ and we have that $q_w = f_w q$, Therefore

$$-\frac{\partial f_w}{\partial x} = \frac{A\varphi}{q}\frac{\partial S_w}{\partial t} \tag{2.22}$$

Since $f_w(S_w)$, equation 2.22 may be rewritten as [6]

$$-\frac{\partial f_w}{\partial S_w}\frac{\partial S_w}{\partial x} = \frac{A\varphi}{q}\frac{\partial S_w}{\partial t} \tag{2.23}$$

where fractional water flow is defined as

$$\begin{cases} f_s\ (S_{cv}) = \frac{1}{1+\frac{K_{ro}}{K_w}\cdot\frac{\mu_w}{\mu_o}} = \frac{1}{1+\frac{(K_{ro}^{\max})}{K_{rw}^{\max}}\cdot\left(\frac{1-S_n}{S_n}\right)^3\cdot\frac{\mu_w}{\mu_o}} \\ S_n = \frac{S_w-S_{wc}}{1-S_{wc}-S_{or}} \qquad\qquad S_{wc} < S_w < (1-S_{or}) \end{cases} \tag{2.24}$$

$$\frac{\partial f_w}{\partial S_w} = 3\left(\frac{\mu_w}{\mu_o}\right)\left(\frac{K_{ro}^{\max}}{K_{rw}^{\max}}\right)\frac{(1-S_n)^2}{(1-S_{or}-S_{wc})\,.S_n^4.\left(1+\frac{K_{ro}}{K_{rw}}.\frac{\mu_w}{\mu_o}.\left(\frac{1-S_n}{S_n}\right)^3\right)^2} \tag{2.25}$$

$$f_w = f\ (K_{ro}^{\max},\ K_{rw}^{\max},\ \mu_o, \mu_w, S_{wc}, S_{or}, S_w)$$

Figure 2.6 shows the fractional water flow function and its derivatives as a function of water saturation

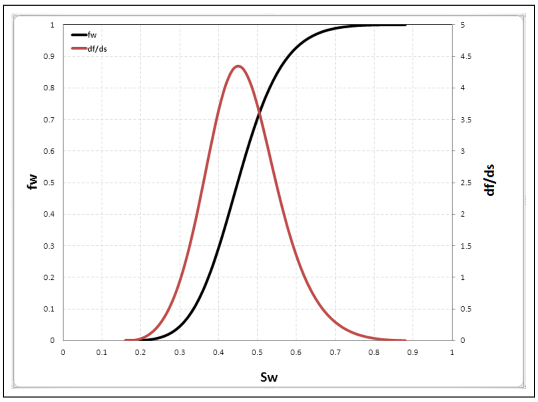

Figure 2.6: Fractional water function and its derivative versus water saturation

Equation 2.22 is known as the Buckley-Leverett equation which is a first order hyperbolic equation. The equation can be solved analytically by the method of characteristics and graphically [7]. In this project a second order numerical scheme, the Lax-Wendroff scheme, is used to solve it. The method is explained in more detail in the following chapters.

2.3.2 Mass Balance Equation

The main flow equation in reservoir engineering can simply be derived by applying material balance to a control volume, as shown in figure 2.7. Mass accumulation inside a control volume is the difference between input and generated mass and output and consumed mass as below:

$$(\dot{m}_i - \dot{m}_o) - (\dot{m}_{cons} - \dot{m}_{gen}) = \frac{\partial M}{\partial t} \quad , \quad \dot{m} = Mass\ Flux = \rho\ .\ q \tag{2.26}$$

where$q$ is flow rate , ρ is density , M is mass and t is time. Time
Based on what we have in equation 2.26 for 1D flow, the input mass rate for x direction shown in figure 2.7 will be:

$$\dot{m}_{ix} = \ \rho\ .\ u_x\ .dA_x \tag{2.27}$$

dA_x, u_x and ρ are the normal cross sectional area in x direction, velocity and density respectively.

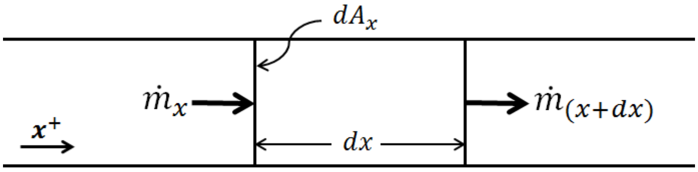

Figure 2.7: Material Balance Control Volume

Using the Euler approximation for the mass rate

$$\dot{m}_{i(x+dx)} = \dot{m}_o = \ \dot{m}_{i(x)} + \frac{\partial \dot{m}_{ix}}{\partial x}\ .\ dx \tag{2.28}$$

The generation and consumption terms in reservoir engineering are production and injection in wells and can be specified as:

$$\begin{aligned} \dot{m}_{cons} &= \ \rho\ .\ \ q_{prod} \quad , \qquad q_{prod} = Production\ Rate \\ \dot{m}_{gen} &= \ \rho\ .\ \ q_{inj} \quad , \qquad q_{inj} = Injection\ \ Rate \end{aligned} \tag{2.29}$$

By substituting equations 2.27 , 2.28 and 2.29 in equation 2.26:

$$-\frac{\partial}{\partial x}\left(\rho.u_x.dy.dz\right).dx - (\rho.q_p - \rho.q_i) = \ \frac{\partial}{\partial t}\ (\rho.\varphi.\ dx.dy.dz) \tag{2.30}$$

Using $V_b\ =\ dx\ .\ dy.\ dz$ and dividing both sides to V_b gives:

$$-\frac{\partial}{\partial x}\left(\rho.u_x\right) - \frac{1}{V_b}\left(\rho.q_{prod} - \rho.q_{inj}\right) = \ \frac{\partial}{\partial t}\ \ (\rho.\varphi) \tag{2.31}$$

Equation 2.31 is the most general type of the mass conservation law in its one dimensional form [8]. To make it more usable in reservoir engineering, Darcy law (equation 2.3) is used to substitute the velocities. Hence, the result for multi-phase flow in porous media will be

$$\frac{\partial}{\partial x}\left(\frac{\rho_\alpha}{\mu_\alpha}K_{abs}K_{r\alpha}\frac{\partial P}{\partial x}\right)-\frac{\rho_\alpha}{V_b}\left(q_{prod}-q_{inj}\right)=\frac{\partial}{\partial t}\ \left(\rho_\alpha.S_\alpha.\varphi\right) \tag{2.32}$$

In this dissertation, α denotes water and oil phases.

2.3.3 Energy Balance Equation

Thermal simulation is all about energy balance and temperature calculations. The energy conservation law is very similar to the mass conservation law and can be written as:

$$(\dot{e}_i-\ \dot{e}_o)-(\dot{e}_{cons}-\dot{e}_{gen})=\frac{\partial E}{\partial t}\quad,\ \ \dot{e}=Energy\ Flux \tag{2.33}$$

In hot water modeling, the energy consumption and generation are related to injection or production streams. There are two main heat transfer equations that are widely used for energy balance, conduction and convection which are defined as,

Conduction heat transfer; $\dot{q}=-k.\ \frac{\partial T}{\partial x}$
Convection heat transfer; $\dot{q}=\rho.\overrightarrow{u}.H$

where k is thermal conductivity and H is enthalpy.
Therefore, the energy at each point can be written as:

$$\dot{e}=\left(-k.\frac{\partial T}{\partial x}+\rho.\overrightarrow{u}.H\right)\times Cross\ section\ area \tag{2.34}$$

Similar to mass balance, energy balance can also be driven by applying equation 2.33 and equation 2.34 on a single element like the one shown in figure 2.8.

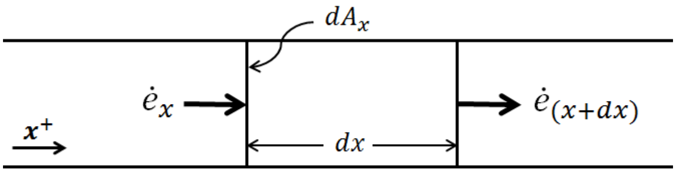

Figure 2.8: Energy balance element

$$\dot{e}_x=\dot{e}_{ix}=\left(-k.\frac{\partial T}{\partial x}+\rho.u_x.H\right)dydz \tag{2.35}$$

$$\dot{e}_{(x+dx)} = \dot{e}_{ox} = \left[\left(-k.\frac{\partial T}{\partial x} + \rho.u_x.H\right) + \frac{\partial}{\partial x}\left(-k.\frac{\partial T}{\partial x} + \rho.u_x.H\right).dx\right] dydz \tag{2.36}$$

$$\dot{e}_{ix} - \dot{e}_{ox} = \left(k\frac{\partial^2 T}{\partial x^2} - \frac{\partial}{\partial x}\left(\rho.u_x.H\right)\right) dxdydz = \left(k\frac{\partial^2 T}{\partial x^2}\right) dxdydz + \frac{\partial}{\partial x}\left(\frac{\rho}{\mu}.K.H.\frac{\partial P}{\partial x}\right) dxdydz \tag{2.37}$$

For wells, the heat transfer can divided into conduction and convection based on bottom hole temperature and fluid enthalpy, as below:

$$\dot{e}_w = \dot{e}_{cons} - \dot{e}_{gen} = \rho\left(q_{inj} - q_{prod}\right) H - (2\pi k h) r_w \left(\frac{\partial T}{\partial r}\right)_{r=r_w} \tag{2.38}$$

For the accumulation term in equation 2.33, both the rock and fluid must be taken into account as they both have heat capacities and are able to store energy in themselves. Therefore the accumulation term will be:

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left((\rho_o U_o S_o + \rho_w U_w S_w).\varphi.dxdydz + \rho_r U_r\left(1-\varphi\right) dxdydz\right) \tag{2.39}$$

Now all the above equations must be combined to obtain an energy balance equation for one dimensional flow in an oil and water system:

$$\left(\sum_{\alpha}^{o,w,r} k_\alpha\right).\frac{\partial^2 T}{\partial^2 x} + \frac{\partial}{\partial x}\left(\sum_{\alpha}^{o,w} \frac{\rho_\alpha}{\mu_\alpha}.K_\alpha.H_\alpha.\frac{\partial P}{\partial x}\right) + \frac{\dot{e}_w}{V_b} = \frac{\partial}{\partial t}\left(\varphi\sum_{\alpha}^{o,w} \rho_\alpha.U_\alpha.S_\alpha + (1-\varphi)\rho_r U_r\right) \tag{2.40}$$

where U_α, U_r, H_α and ρ_r are internal energy for each phase, rock internal energy, enthalpy for each phase and Rock density [8].
As it is mentioned before there is a condition which states that the fluid fills the rock pore volume. This condition gives the very helpful relation

$$S_o + S_w = 1 \tag{2.41}$$

Equations 2.23, 2.32 (for oil and water phases) and 2.40 are non-linear partial differential equations with coefficients that are complex functions of temperature, pressure and saturation. No techniques exist to solve these types of equations analytically. In this dissertation two numerical techniques are chosen to model these equations in order to find pressure, temperature and saturation distributions which are explained in detail in the following chapters.

2.4 Initial and Boundary Conditions

In this one dimensional reservoir, it is supposed that initially the reservoir rock is filed with oil and irreducible water under uniform and constant pressure and temperature. It is also supposed that two wells are located at two sides or boundaries of the reservoir. An injection well with constant rate (or total rate) and injection temperature (T_{inj}) is placed at the left boundary and water saturation at this boundary equals to $(1-S_{or})$. By using Darcy's equation in oil field units (refer to nomenclature). A constant total rate provides the pressure gradient (Neumann boundary condition)

$$q_t = u_x A_x \tag{2.42}$$

$$q_t = q_{winj} = -1.127 \ K_{abs} \frac{K_{rw}}{\mu_w} A_x \left(\frac{\partial P}{\partial x}\right)_{bL} \tag{2.43}$$

At the right boundary, oil and water are produced from a production well. It is considered that the total rate is constant from the producer, so the pressure gradient can be calculated at this boundary using Darcy's equation in following way;

$$q_t = q_{oprod} + q_{wprod} = -1.127 K_{abs} \ A_x \left(\frac{K_{rw}}{\mu_w} + \frac{K_{ro}}{\mu_o}\right) \left(\frac{\partial P}{\partial x}\right)_{bR} \tag{2.44}$$

2.5 Heat losses

The heat losses in a system begin at the thermal unit or heat source, with subsequent heat losses ocurring in the surface injection lines, the injection wellbore, the wellbore and the formation itself and the adjacent strata (see figure 2.9). Theoretical and laboratory studies have shown that the rate of heat loss to adjacent strata is the most important factor which determines the economic feasibility of a heat injection project[5]. The heated area of the reservoir rock is quite large, and the heat must be sustained for a long period of time. Therefore, the cumulative heat loss to adjacent strata is also large, in spite of the fact that the thermal conductivity of earth material is very small [5]. In this project, heat losses to wellbore and surface facilities are neglected and heat loss to adjacent layers which is the most significant one is modelled by thermal conduction. The other way of heat loss which is considered is through the producing fluids from the production well which is modeled by thermal convection.

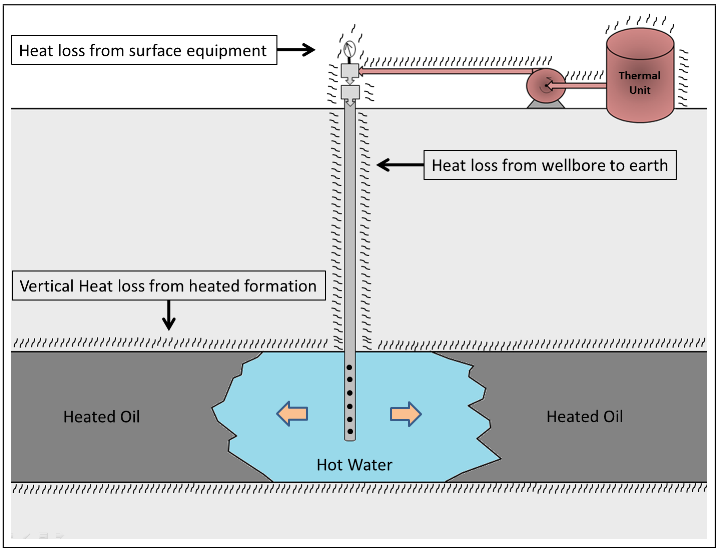

Figure 2.9: Illustration of heat losses which occur in a heat injection system

3 First Model

In this model, the Buckley-Leverett equation is used to find saturation profiles and then its results are used in the pressure equation which is solved by a fully implicit numerical technique. In order to find the temperature profile the saturation and pressure results are applied to a fully implicit energy equation. Finally, the mass balance equation (pressure equation) and the energy balance equation (temperature equation) are coupled to find the optimal result for pressure and temperature distributions, since these equations are highly nonlinear.

3.1 Buckley-Leverett Discretization

In this model, the Buckley- Leverett equation is used to find the saturation distribution. The numerical scheme used to solve this hyperbolic equation is the Lax-Wendroff scheme. The scheme is a second order finite difference method where the derivatives are approximated by differences of discrete values. An important requirement of numerical methods for such non-linear hyperbolic equations is to be in conservative form to maintain the conservation property of the equation. To derive the numerical method in conservative form we use standard finite difference discretization of the conservative form of the partial differential saturation equation, not the quasilinear form of the equation.

For a numerical scheme to be in conservation form [9] it must have the form

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle X}\{F(U_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - F(U_{j-\frac{1}{2}}^{n+\frac{1}{2}})\}, \tag{3.1}$$

where $\mathrm{U_j^n}$ is an approximation to the cell average of the analytic function, $\mathrm{F(U_{j+\frac{1}{2}}^{n+\frac{1}{2}})}$ is the numerical flux function, $\triangle \mathrm{t} = \mathrm{t^{n+1}} - \mathrm{t^n}$, and $\triangle \mathrm{x} = \mathrm{x_{j+\frac{1}{2}}} - \mathrm{x_{j-\frac{1}{2}}}$.

By using oil field units (refer to nomenclature), the Buckley-Leverett equation will be,

$$\left(\frac{A\varphi}{5.615\, q_t}\right)\frac{\partial S}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{3.2}$$

By choosing,

$$\begin{cases} X = \frac{x}{L} \Rightarrow dX = \frac{1}{L} dx \Rightarrow \frac{\partial ...}{\partial x} = \frac{1}{L} \frac{\partial ...}{\partial X} \\ \\ t_D = \frac{5.615\, q_t}{A.\varphi.L} t \Rightarrow dt_D = \frac{5.615}{A.\varphi.L}\, dt \\ \\ \qquad\qquad \Rightarrow\ \frac{\partial ...}{\partial t} = \frac{5.615 q_t}{A.\varphi.L} \frac{\partial ...}{\partial t_D} \end{cases} \tag{3.3}$$

the BL equation converts into the dimensionless equation,

$$\frac{\partial S}{\partial t_D} + \frac{\partial f}{\partial X} = 0 \tag{3.4}$$

In order to drive the Lax-Wendroff scheme applied to above equation to be in conservative form, for all intermediate blocks ($i = 2, \ldots, N_x - 1$), we start with;

$$\frac{\partial S}{\partial t_D} + \frac{\partial f}{\partial X} = 0 \ \ \Rightarrow\ \ \frac{\partial S}{\partial t_D} = -\frac{\partial f}{\partial X} \tag{3.5}$$

and using Taylor-series expansion about t_D

$$S\left(X_i,\, t_D + \Delta t_D\right) = S\left(X_i, t_D\right) + \Delta t_D\, \frac{\partial S\ \left(X_i, t_D\right)}{\partial t_D} + \frac{\Delta t_D^2}{2}\, \frac{\partial^2 S\left(X_i, t_D\right)}{\partial t_D^2} + O\left(\Delta t_D\right)^3 \tag{3.6}$$

$$\begin{cases} \frac{\partial S}{\partial t_D} = -\frac{\partial f}{\partial X} \\ \\ \frac{\partial^2 S}{\partial t_D^2} = -\left(f_s\right)_{t_D} = -\left(f_{t_D}\right)_X = -\left(\frac{\partial f}{\partial S} \cdot \frac{\partial S}{\partial t_D}\right)_X = -\left(\frac{\partial f}{\partial S}\left(-f_X\right)\right)_X = \left(\frac{\partial f}{\partial S} \cdot \frac{\partial f}{\partial X}\right)_X \end{cases} \tag{3.7}$$

By substituting central differences for space derivatives

$$\begin{cases} S_i^{n+1} = S_i^n - \Delta t_D \left(\frac{f_{i+1} - f_{i-1}}{2\Delta X}\right) + \frac{\Delta t_D^2}{2} \left[\frac{\left(\frac{\partial f}{\partial S} \cdot \frac{\partial f}{\partial X}\right)_{i+\frac{1}{2}} - \left(\frac{\partial f}{\partial S} \cdot \frac{\partial S}{\partial X}\right)_{i-\frac{1}{2}}}{\Delta X}\right] \\ S_i^{n+1} = S_i^n - \frac{\Delta t_D}{\Delta x} \left[h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}\right]\ :\ Conservative\ form \end{cases} \tag{3.8}$$

⇒ By comparison:

$$h_{i+\frac{1}{2}} = \frac{1}{2}\left(f_{i+1} + f_i\right) - \frac{\Delta t_D}{2}\left(\frac{\partial f}{\partial S} \cdot \frac{\partial f}{\partial X}\right)_{i+\frac{1}{2}} = \frac{1}{2}\left(f_{i+1} + f_i\right) - \frac{1}{2}\, \nu_{i+\frac{1}{2}} \cdot \left(f_{i+1} - f_i\right) \tag{3.9}$$

where

$$\nu_{i+\frac{1}{2}} = \begin{cases} \frac{\Delta t_D}{\Delta X}\left(\frac{f_{i+1} - f_i}{S_{i+1} - S_i}\right) & if\, S_i \neq S_{i+1} \\ \\ \frac{\Delta t_D}{\Delta X}\left(\frac{\partial f}{\partial S}\right)_i & if\ \, S_i = S_{i+1} \end{cases} \tag{3.10}$$

Similarly $h_{i-\frac{1}{2}} = \frac{1}{2}\left(f_i + f_{i-1}\right) - \nu_{i-\frac{1}{2}}\left(f_i - f_{i-1}\right)$

$$\nu_{i-\frac{1}{2}} = \begin{cases} \frac{\Delta t_D}{\Delta X}\left(\frac{f_i - f_{i-1}}{S_i - S_{i-1}}\right) & if\ \ S_i \neq S_{i-1} \\ \\ \frac{\Delta t_D}{\Delta X}\left(\frac{\partial f}{\partial S}\right)_{i-1} & if\ \ S_i = S_{i-1} \end{cases} \tag{3.11}$$

3.1.1 Effect of boundary conditions

For the first cell ($i = 1$), it is supposed that there is a known value of S at the boundary ($S = 1 - S_{or}$). At this point the Lax-Wendroff scheme is derived based on the unequal spacing (see figure 3.1)

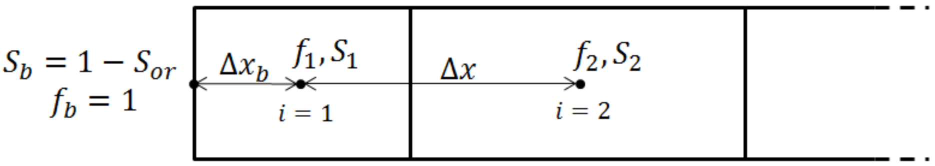

Figure 3.1: Unequal spacing for the left boundary cell after introducing an imaginary node

$$\begin{cases} f\ (X_1 + \Delta X) = f\ (X_1) + \Delta X.\ \left(\frac{\partial f}{\partial X}\right)_{X_1} \\ f\ (X_1 - \Delta X_b) = f\ (X_1) - \Delta X_b.\ \left(\frac{\partial f}{\partial X}\right)_{X_1} \end{cases} \Rightarrow \left(\frac{\partial f}{\partial X}\right)_{X_1} = \frac{f_2 - f_b}{\Delta X + \Delta X_b} \tag{3.12}$$

$$S_1^{n+1} = S_1^n - \Delta t_D\left(\frac{f_2 - f_b}{\Delta X + \Delta X_b}\right) + \left(\frac{\Delta t_D^2}{2}\right)\left[\frac{\left(\frac{\partial f}{\partial S}.\frac{\partial f}{\partial X}\right)_{1+\frac{\Delta X}{2}} - \left(\frac{\partial f}{\partial S}.\frac{\partial f}{\partial X}\right)_{1-\frac{\Delta X_b}{2}}}{\Delta X/2 + \Delta X_b/2}\right] \tag{3.13}$$

$$= S_1^n - \frac{\Delta t_D}{(\Delta X + \Delta X_b)}\left\{(f_2 - f_b) - \Delta t_D\left[\left(\frac{\partial f}{\partial S}.\frac{\partial f}{\partial X}\right)_{1+\frac{\Delta X}{2}} - \left(\frac{\partial f}{\partial S}.\frac{\partial f}{\partial X}\right)_{1-\frac{\Delta X_b}{2}}\right]\right. \tag{3.14}$$

$$\Rightarrow \begin{cases} S_1^{n+1} = S_1^n - \frac{\Delta t_D}{(\Delta X + \Delta X_b)}.\ \left(h_{1+\frac{\Delta X}{2}} - h_{1-\frac{\Delta X_b}{2}}\right) \\ \\ h_{1+\frac{\Delta X}{2}} = (f_2 + f_1) - \nu_{1+\frac{\Delta X}{2}}.\ (f_2 - f_1) \\ \\ h_{1-\frac{\Delta X_b}{2}} = (f_1 + f_b) - \nu_{1-\frac{\Delta X_b}{2}}\ (f_1 - f_b) \end{cases} \tag{3.15}$$

where

$$\begin{aligned}
\nu_{1+\frac{\Delta X}{2}} &= \begin{cases} \frac{\Delta t_D}{\Delta X}\cdot\frac{f_2-f_1}{S_2-S_1} & S_2 \neq S_1 \\ \frac{\Delta t_D}{\Delta X}\cdot\left(\frac{\partial f}{\partial S}\right)_1 & S_2 = S_1 \end{cases} \\
\nu_{1-\frac{\Delta X_b}{2}} &= \begin{cases} \frac{\Delta t_D}{\Delta X_b}\cdot\frac{f_1-f_b}{S_1-S_b} & S_b \neq S_1 \\ \frac{\Delta t_D}{\Delta X_b}\cdot\left(\frac{\partial f}{\partial S}\right)_b & S_b = S_1 \end{cases}
\end{aligned} \tag{3.16}$$

All the fractional water values and its derivatives in above formulas are calculated from equations 2.24 and 2.25

For last cell ($\mathrm{i} = \mathrm{N_x}$), the equation 3.4 is discretized based on forward in time and backward in space as follows,

$$\frac{S_i^{n+1} - S_i^n}{\Delta t_D} + \frac{f\left(S_i^n\right) - f\left(S_{i-1}^n\right)}{\Delta X} = 0 \tag{3.17}$$

$$\Rightarrow S_i^{n+1} = S_i^n - \left(\frac{\Delta t_D}{\Delta X}\right)\left[f\left(S_i^n\right) - f\left(S_{i-1}^n\right)\right] \tag{3.18}$$

Finally, the result for all grid points is a linear system, which we denote it by

$$A.S = b \tag{3.19}$$

where A is a tridiagonal matrix and S is a vector of unknowns (saturations), which is solved to find the saturation profile.

3.1.2 The CFL condition

In order to have stability when using explicit numerical schemes, we are required to apply the necessary condition known as the Courant-Friedrichs-Lewy condition. It is often referred to as the CFL or Courant condition, [9] and [10], and is

$$\mu = \left|\mathrm{a}\frac{\triangle \mathrm{t}}{\triangle \mathrm{x}}\right| \leq \mu_{max} \tag{3.20}$$

Where in this context, $\mathrm{a} = \mathrm{a}(\mathrm{S}) = \frac{\partial \mathrm{f}}{\partial \mathrm{S}}$. Here $\triangle \mathrm{t}$ and $\triangle \mathrm{x}$ are the time and space steps, respectively. The value of μ_{max} changes with the method used to solve the discretized equation. This condition is not sufficient for stability, as it is only a necessary condition for scheme to be stable. The Lax-Wendroff scheme is known to be stable for the region $\mu = \left|\mathrm{a}\frac{\triangle \mathrm{t}}{\triangle \mathrm{x}}\right| \leq 1$.

The picture on the right of figure 3.2 shows the domain of dependence for this numerical scheme. If $\mathrm{a}\frac{\triangle \mathrm{t}}{\triangle \mathrm{x}}$ is the slope of AB then the CFL condition is satisfied because AB lies in the stencil of the scheme, whilst the line AC violates the CFL condition, by lying outside the domain of dependence.

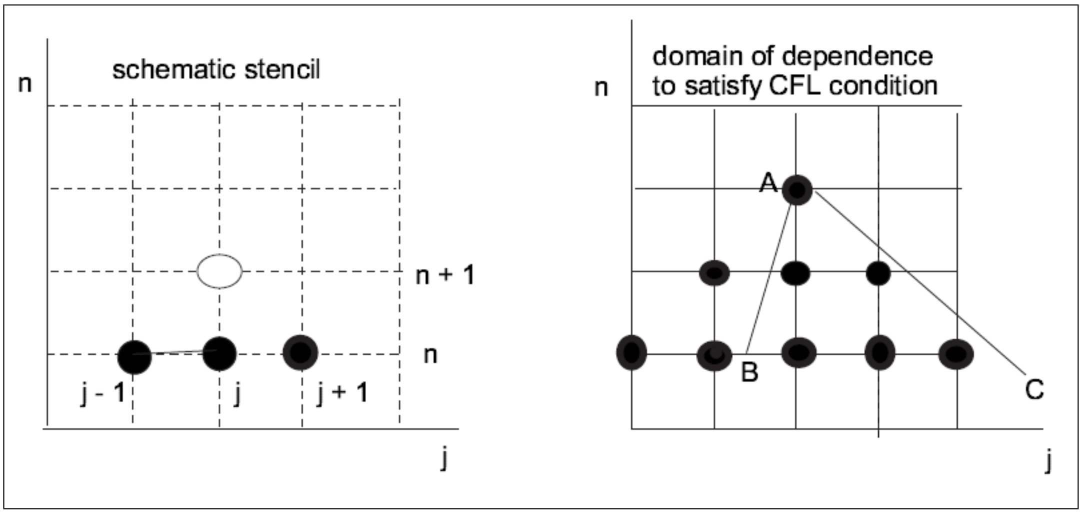

Figure 3.2: Stencils for the Lax-Wendroff scheme

3.2 Discretization of the Mass balance equation

The equation 2.32 in oil field units (refer to nomenclature), for oil phase and substituting $S_o = 1 - S$ results in

$$\frac{\partial}{\partial x}\left(\frac{\rho_o}{\mu_o}K_{abs}K_{ro}\frac{\partial p}{\partial x}\right) - \frac{\rho_o}{1.127V_b}\left(q_p - q_i\right) = \frac{1}{6.328}\frac{\partial}{\partial t}\left(\rho_o.(1-S).\varphi\right) \tag{3.21}$$

In order to discretize this equation, a fully implicit scheme is used. As mentioned before, the coefficients of this equation depend on pressure, temperature and saturation. The results of the saturation from the Buckley-leverett equation are used directly and indirectly (through the relative permeability) in this equation.
In order to expand the right hand side of equation 3.21, we need to remember that density is a function of pressure and temperature ($\rho = \rho(p(t), T(t))$), so we have

$$\frac{\partial \rho_o}{\partial t} = \frac{\partial \rho_o}{\partial p}.\frac{\partial p}{\partial t} + \frac{\partial \rho_o}{\partial T}.\frac{\partial T}{\partial t} = \rho'_{op}.\left(\frac{\partial p}{\partial t}\right) + \rho'_{oT}\left(\frac{\partial T}{\partial t}\right) \tag{3.22}$$

Hence, expansion of the equation 3.21 results in

$$\frac{\partial}{\partial x}\left(\rho_o\frac{K_o}{\mu_o}\frac{\partial \rho}{\partial x}\right) + \frac{\rho_o \Sigma q_o}{1.127\,V_b} = \frac{\varphi}{6.328}\left[(1-S)\;\rho'_{op}.\frac{\partial p}{\partial t} + (1-S)\;\rho'_{oT}\frac{\partial T}{\partial t} - \rho_0\frac{\partial S}{\partial t}\right] \tag{3.23}$$

Figure 3.3 shows the grid indexing scheme used for the method. The above equation is discretized as,

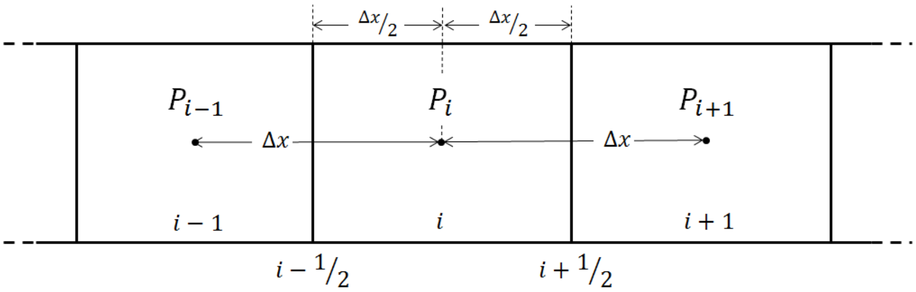

Figure 3.3: Grid indexing scheme in material balance equation

$$\frac{\left(\rho_o \frac{K_o}{\mu_o} \frac{\partial p}{\partial x}\right)_{i+\frac{1}{2}}^{n+1} - \left(\rho_o \frac{K_o}{\mu_o} \frac{\partial p}{\partial x}\right)_{i-\frac{1}{2}}^{n+1}}{\Delta x} + \frac{\rho_{oi} \Sigma q_o}{1.127\, V_{bi}}$$
$$= \frac{\varphi}{6.328\, \Delta t} \left[\left(1 - S_i^{n+1}\right) \rho'_{opi} \left(p_i^{n+1} - p_i^n\right) + \left(1 - S_i^{n+1}\right) \rho'_{oTi} \left(T_i^{n+1} - T_i^n\right) - \rho_{oi} \left(S_i^{n+1} - S_i^n\right) \right] \tag{3.24}$$

By using first order central difference scheme (with equal spacing) for the pressure gradient at interfaces,

$$\Rightarrow \frac{1}{\Delta x} \left(\rho_o \frac{K_o}{\mu_o}\right)_{i+\frac{1}{2}}^{n+1} \left(\frac{P_{i+1}^{n+1} - P_i^{n+1}}{\Delta x}\right) - \frac{1}{\Delta x} \left(\rho_o \frac{K_o}{\mu_o}\right)_{i-\frac{1}{2}}^{n+1} \left(\frac{P_i^{n+1} - P_{i-1}^{n+1}}{\Delta x}\right) + \frac{\rho_{oi} \Sigma q_o}{1.127\, V_{bi}}$$
$$= \frac{\varphi}{6.328\, \Delta t} \left[\left(1 - S_i^{n+1}\right) \rho'_{opi} \left(p_i^{n+1} - p_i^n\right) + \left(1 - S_i^{n+1}\right) \rho'_{oTi} \left(T_i^{n+1} - T_i^n\right) - \rho_{oi} \left(S_i^{n+1} - S_i^n\right) \right] \tag{3.25}$$

Oil properties, density and viscosity, are calculated from equations 2.9 and 2.12 at the interfaces, based on averages of pressure and temperature of two neighbouring blocks. Oil relative permeability is defined from average oil saturations.
By defining

$$D_{oi}^{n+1} = \frac{1}{\Delta x^2} \left(\rho_0 \frac{K_0}{\mu_0}\right)_{i-\frac{1}{2}}^{n+1} \quad , \quad i = 1, ..., \; N_x + 1 \tag{3.26}$$

Equation 3.25 can be rewritten in a simpler way for internal cells supposing no generation and no consumption in these cells,

$$\begin{aligned}&D_{oi+1}^{n+1}\left(p_{i+1}^{n+1}-p_i^{n+1}\right)-D_{oi}^{n+1}\left(p_i^{n+1}-p_{i-1}^{n+1}\right)\\&=\frac{\varphi}{6.328\,\Delta t}\left[\left(1-S_i^{n+1}\right)\rho'_{opi}\left(p_i^{n+1}-p_i^{n}\right)+\left(1-S_i^{n+1}\right)\rho'_{oTi}\left(T_i^{n+1}-T_i^{n}\right)-\rho_{oi}\left(S_i^{n+1}-S_i^{n}\right)\right]\end{aligned}\tag{3.27}$$

So, general oil pressure equation for each middle cell (i=2,...,N$_{x-1}$) is defined as a function of three variable p_{i+1}, p_i and p_{i-1},

$$\begin{aligned}&D_{oi}^{n+1}.p_{i-1}^{n+1}-\left(D_{oi}^{n+1}+D_{oi+1}^{n+1}+\frac{\varphi}{6.328\,\Delta t}\left(1-S_i^{n+1}\right)\rho'_{opi}\right)p_i^{n+1}+D_{oi+1}^{n+1}p_{i+1}^{n+1}=\\&-\frac{\varphi}{6.328\,\Delta t}\left(1-S_i^{n+1}\right)\rho'_{opi}.p_i^{n}+\frac{\varphi}{6.328\,\Delta t}\left(1-S_i^{n+1}\right)\rho'_{oTi}\left(T_i^{n+1}\right.\\&\left.-T_i^{n}\right)-\frac{\varphi}{6.328\,\Delta t}\rho_{oi}\left(S_i^{n+1}-S_i^{n}\right)\end{aligned}\tag{3.28}$$

Treating the equation for boundary cells is slightly different. For the first cell ($i=1$), the equation 3.23 is discretized as

$$\begin{aligned}&\frac{\left(\rho_o\frac{K_o}{\mu_o}\frac{\partial P}{\partial x}\right)_{i+\frac{1}{2}}^{n+1}-\left(\rho_o\frac{K_o}{\mu_o}\frac{\partial P}{\partial x}\right)_{bl}^{n+1}}{\Delta x}+\frac{\rho_{oi}\Sigma q_o}{1.127\,V_{bi}}\\&=\frac{\varphi}{6.328\,\Delta t}\left[\left(1-S_i^{n+1}\right)\rho'_{opi}\left(p_i^{n+1}-p_i^{n}\right)+\left(1-S_i^{n+1}\right)\rho'_{oTi}\left(T_i^{n+1}-T_i^{n}\right)-\rho_{oi}\left(S_i^{n+1}-S_i^{n}\right)\right]\end{aligned}\tag{3.29}$$

Using equation 2.43 and boundary conditions help to find water properties and boundary pressure gradient in above equation,

$$\begin{aligned}&S_b=1-S_{or}\,,\ \ T_b=T_{inj}\ ,\ \ p_b=p_{bl}\\&K_{wbl}=K_{abs}.K_{rw}\left(1-S_{or}\right)=K_{abs}.K_{rw}^{\max}\\&\mu_{wb}=\mu_w\left(P_{bl},T_{inj}\right)\\&\frac{P_i-P_{bl}}{\left(\Delta x/_2\right)}=\left(\frac{\partial P}{\partial x}\right)_{bl}=-\frac{q_t.\mu_{wbl}}{1.127\,K_{wbl}A_n}=-GPI\end{aligned}\tag{3.30}$$

Note that the injection well is located on the boundary, so the effect of generation terms is considered in the boundary pressure gradient. Hence, the discretized oil pressure equation for the first cell ($i=1$) is

$$\begin{aligned}&-\left(D_{oi+1}^{n+1}+\frac{\varphi\left(1-S_i^{n+1}\right)\rho'_{opi}}{6.328\,\Delta t}\right)p_i^{n+1}+D_{oi+1}^{n+1}.p_{i+1}^{n+1}=-D_{oi}^{n+1}.\Delta x.GPI\\&-\frac{\varphi\left(1-S_i^{n+1}\right)\rho'_{opi}}{6.328\,\Delta t}P_i^{n}+\frac{\varphi\left(1-S_i\right)\rho'_{oTi}}{6.328\,\Delta t}\left(T_i^{n+1}-T_i^{n}\right)-\frac{\varphi\,\rho_{oi}}{6.328\,\Delta t}\left(S_i^{n+1}-S_i^{n}\right)\end{aligned}\tag{3.31}$$

Condition at the right boundary cell is different. Using equation 2.44 and defining a new parameter, λ_{tb}, the pressure gradient for this boundary is calculated as

$$\lambda_{tb} = \frac{K_{rob}}{\mu_{ob}} + \frac{K_{rwb}}{\mu_{wb}} \qquad \Rightarrow \qquad \left(\frac{\partial P}{\partial x}\right)_b = -\frac{q_t}{1.127\,K_{abs}A_n\,\lambda_{tb}} = -GPO \tag{3.32}$$

$$\begin{matrix} \mathit{Boundary} \\ \mathit{conditions} \\ (i = Nx) \end{matrix} \quad \begin{cases} S_b = \frac{3}{2}\,S_{Nx} - \frac{1}{2}\,S_{Nx-1} \\ p_b = p_{br} \\ T_b = \frac{3}{2}\,T_{Nx} - \frac{1}{2}T_{Nx-1_{x-1}} \end{cases} \quad \Rightarrow \quad \begin{cases} K_{rob} = K_{ro}\,\left(1 - S_b\right) \\ K_{rwb} = K_{rw}\,\left(S_b\right) \\ \mu_{ob} = \mu_o\,\left(p_b, T_b\right) \\ \mu_{wb} = \mu_w\,\left(p_b, T_b\right) \end{cases}$$

By substituting this term into the equation below (discretized pressure equation for the last cell),

$$\frac{\left(\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x}\right)_{br}^{n+1} - \left(\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x}\right)_{i-\frac{1}{2}}^{n+1}}{\Delta x} \\ = \frac{\varphi}{6.328\,\Delta t}\,\left[\left(1 - S_i^{n+1}\right)\,\rho'_{opi}\,\left(p_i^{n+1} - p_i^n\right) + \left(1 - S_i^{n+1}\right)\,\rho'_{oTi}\,\left(T_i^{n+1} - T_i^n\right) - \rho_{oi}\,\left(S_i^{n+1} - S_i^n\right)\right] \tag{3.33}$$

And noting that the generation and consumption term in the main equation 3.21 is replaced by the effect of the boundary condition, we have the equation below for $= N_x$,

$$D_{oi}^{n+1}P_{i-1}^{n+1} - \left(D_{oi}^{n+1} + \frac{\varphi\left(1 - S_i^{n+1}\right)\rho'_{opi}}{6.328\,\Delta t}\right)\,P_i^{n+1} = -D_{oi}^{n+1}.\Delta x.GPO \\ - \frac{\varphi\left(1 - S_i^{n+1}\right)\rho'_{opi}}{6.328\,\Delta t}\,P_i^n + \frac{\varphi\left(1 - S_i^{n+1}\right)\rho'_{oTi}}{6.328\,\Delta t}\,\left(T_i^{n+1} - T_i^n\right) - \frac{\varphi\rho_{oi}}{6.328\Delta t}\,\left(S_i^{n+1} - S_i^n\right) \tag{3.34}$$

Finally, writing these equations for all grid blocks, we obtain the nonlinear system of $AS = b$ (where A is a tridiagonal matrix in which all its elements depend on the unknowns).Such a system is not easy to solve, but in the following section we explained how to deal with this difficulty.

3.3 Newton's Method for Nonlinear Systems of Equations

To find the solution of a system ($Ax = b$) of N_x nonlinear equations in N_x unknowns [13], the system can be written in the homogenous form

$$F\left(x\right)=Ax-B=0 \tag{3.35}$$

Consider the Taylor-Series expansion of $F(x)$ about $x=x_0$. Using only the first two terms of the expansion, a first approximation to the root of $F(x)$ can be obtained from

$$F\left(x\right)=F\ \left(x_0+\Delta x\right)=F\ \left(x_0\right)+\left(x-x_0\right)\left.\frac{\partial F}{\partial x}\right|_{x_0}$$

Let $\left.\frac{\partial F}{\partial x}\right|_{x_0}=J\left(x_0\right)$ giving $F\ \left(x\right)=F\left(x_0\right)+J\ \left(x_0\right)\ \left(x-x_0\right)=0$

$\Rightarrow \left(Ax_0-B\right)+J\ \left(x_0\right).\ \left(x-x_0\right)=0$

$\Rightarrow\ \left(Ax_0-B\right)+J\ \left(x_0\right).x-J\ \left(x_0\right).x_0=0$

$\Rightarrow\ J\left(x_0\right).x=\ J\ \left(x_0\right).x_0-\ \left(Ax_0-B\right)$

If $A_C=J\ \left(x_0\right)$, $C=J\ \left(x_0\right).x_0-\left(Ax_0-B\right)$
and $x=x_0$ (a vector) represents the first guess of the solution, successive approximation to the solution are obtained from

$$A_C.x\ =C \tag{3.36}$$

This is the Newton (Newton-Raphson) method for solving the system. It requires the evaluation of the Jacobian matrix of the system which is defined as:

$$\left.\frac{\partial F}{\partial x}\right|_{x}=J\ \left(x\right)=\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \cdots & \frac{\partial F_1}{\partial x_N} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \cdots & \frac{\partial F_2}{\partial x_N} \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \\ \frac{\partial F_N}{\partial x_1} & \frac{\partial F_N}{\partial x_2} & \cdots & \cdots & \frac{\partial F_N}{\partial x_N} \end{bmatrix} \tag{3.37}$$

Different convergence criteria can be applied to the system, to find the solution. In this project, the maximum of the modulus difference of the between consecutive vectors is used to be less than a certain tolerance (ε). In mathematical terms this is expressed as

$$max\left|x^{n+1}-x^{n}\right|<\varepsilon \tag{3.38}$$

The main complication with using Newton-Raphson to solve such a system of non-linear equations is having to define all the functions $\frac{\partial F_i}{\partial x_j}$, for $i,j=1,2,\ldots,\ N_x,$ included in the

Jacobian. As the number of equations and unknowns, N_x , increases, so do the number of elements in the Jacobian.

The convergence of Newton's method is quadratic when the Jacobian matrix is non-singular and the initial guess is close enough.

3.4 Jacobian Matrix Definition for Mass Balance Equation

In order to solve the nonlinear system resulted from applying fully implicit method to oil mass conservation equation (oil pressure equation), the Jacobian is defined as follow,

For internal equations $(i = 2, \ldots,\ N_x - 1)$

$$\begin{aligned} F_i(P_{i-1}, P_i, P_{i+1}) = {} & D_{oi}^{n+1}.p_{i-1}^{n+1} - \left(D_{oi}^{n+1} + D_{oi+1}^{n+1} + \frac{\varphi}{6.328\,\Delta t}\,\left(1 - S_i^{n+1}\right)\,\rho'_{opi}\right)\,p_i^{n+1} + D_{oi+1}^{n+1}p_{i+1}^{n+1} \\ & + \frac{\varphi}{6.328\,\Delta t}\,\left(1 - S_i^{n+1}\right)\,\rho'_{opi}.\,p_i^{n} - \frac{\varphi}{6.328\,\Delta t}\,\left(1 - S_i^{n+1}\right)\,\rho'_{oTi}\,\left(T_i^{n+1} - T_i^{n}\right) \\ & + \frac{\varphi}{6.328\,\Delta t}\,\rho_{oi}\,\left(S_i^{n+1} - S_i^{n}\right) = 0 \end{aligned} \tag{3.39}$$

$$\begin{aligned} \frac{\partial F_i}{\partial P_i} = {} & \frac{1}{2}D_{oi}^{n+1}\,\eta_{opi}^{n+1}P_{i-1}^{n+1} - \left(D_{oi}^{n+1} + D_{oi+1}^{n+1} + \frac{\varphi}{6.328\,\Delta t}\,\left(1 - S_i^{n+1}\right)\,\rho'_{opi}\right)\eta \\ & \left(\frac{D_{oi}^{n+1}}{2}\,\eta_{opi}^{n+1} + \frac{D_{oi+1}^{n+1}}{2}\eta_{opi+1}^{n+1} + \frac{\varphi}{6.328\,\Delta t}\,\left(1 - S_i^{n+1}\right)\,\rho''_{oppi}\right)\,P_i^{n+1} \\ & + \frac{1}{2}\,D_{oi+1}^{n+1}\,\eta_{opi+1}^{n+1}\,P_{i+1}^{n+1} + \frac{\varphi}{6.328\Delta t}\left(1 - S_i^{n+1}\right)\,\rho''_{oppi}\,P_i^{n} \\ & - \frac{\varphi}{6.328\,\Delta t}\left(1 - S_i^{n+1}\right)\,\rho''_{oPTi}\,\left(T_i^{n+1} - T_i^{n}\right) + \frac{\varphi}{6.328\,\Delta t}\,\rho'_{opi}\,\left(S_i^{n+1} - S_i^{n}\right) \end{aligned} \tag{3.40}$$

$$\frac{\partial F_i}{\partial P_{i-1}} = D_{oi}^{n+1} + \frac{1}{2}D_{oi}^{n+1}\,\eta_{opi}^{n+1}\,P_{i-1}^{n+1} - \frac{1}{2}\,D_{oi}^{n+1}\,\eta_{opi}^{n+1}P_i^{n+1} = D_{oi}^{n+1}\,\left[1 + \frac{1}{2}\,\eta_{oi}^{n+1}\,\left(P_{i-1}^{n+1} - P_i^{n+1}\right)\right] \tag{3.41}$$

$$\frac{\partial F_i}{\partial P_{i+1}} = D_{oi+1}^{n+1}\,\left[1 + \frac{1}{2}\,\eta_{opi+1}^{n+1}\left(P_{i+1}^{n+1} - P_i^{n+1}\right)\right] \tag{3.42}$$

For the first equation $(i = 1)$

$$\begin{aligned} \frac{\partial F_i}{\partial P_i} = {} & -\left(D_{oi+1}^{n+1} + \frac{\varphi\,\left(1 - S_i^{n+1}\right)\,\rho'_{opi}}{6.328\,\Delta t}\right) - \left(\frac{1}{2}\,\eta_{opi+1}^{n+1}\,D_{oi+1}^{n+1} + \frac{\varphi\left(1 - S_i^{n+1}\right)\,\rho''_{oppi}}{6.328\,\Delta t}\right)\,P_i^{n+1} \\ & + \frac{1}{2}\,\eta_{opi+1}^{n+1}\,D_{oi+1}^{n+1}\,P_{i+1}^{n+1} + \frac{3}{2}\,GPI.\Delta x\,\eta_{opi}^{n+1}\,D_{oi}^{n+1} + \frac{3}{2}\,GPI.\Delta x.D_{oi}^{n+1}.\frac{\mu'_{wpb}}{\mu_{wb}} \\ & + \frac{\varphi\left(1 - S_i^{n+1}\right)\,\rho''_{oppi}}{6.328\,\Delta t}\,P_i^{n} - \frac{\varphi\,\left(1 - S_i^{n+1}\right)\,\rho''_{opTi}}{6.328\,\Delta t}\,\left(T_i^{n+1} - T_i^{n}\right) + \frac{\varphi\,\rho'_{opi}}{6.328\,\Delta t}\left(S_i^{n+1} - S_i^{n}\right) \end{aligned}$$

(3.43)

$$
\begin{aligned}
\frac{\partial F_i}{\partial P_{i+1}} &= \frac{1}{2}\,\eta^{n+1}_{opi+1}\,D^{n+1}_{oi+1}\,\left(P^{n+1}_{i+1}-P^{n+1}_{i}\right)+D^{n+1}_{oi+1} \\
&\quad -\frac{1}{2}\,GPI.\Delta x.\,\eta^{n+1}_{opi}D^{n+1}_{oi}-\frac{1}{2}\,GPI.\Delta x.D^{n+1}_{oi}.\frac{\mu'_{wpb}}{\mu_{wb}}
\end{aligned}
\tag{3.44}
$$

For the last equation ($i = N_x$)

$$
\begin{aligned}
\frac{\partial F_i}{\partial P_i} &= \frac{1}{2}\,D^{n+1}_{oi}\,\eta^{n+1}_{opi}P^{n+1}_{i-1}-\left(D^{n+1}_{oi}+\frac{\varphi\left(1-S^{n+1}_{i}\right)\,\rho'_{opi}}{6.328\Delta t}\right) \\
&\quad -\left(\frac{1}{2}\,D^{n+1}_{oi}\,\eta^{n+1}_{opi}+\frac{\varphi\left(1-S^{n+1}_{i}\right)\,\rho''_{oppi}}{6.328\Delta t}\right)P^{n+1}_{i}-\frac{3}{2}D^{n+1}_{oi}\,\eta^{n+1}_{opi}\Delta x.GPO \\
&\qquad +\frac{\varphi\left(1-S^{n+1}_{i}\right)\,\rho''_{oppi}}{6.328\Delta t}P^{n}_{i}-\frac{\varphi\left(1-S^{n+1}_{i}\right)\,\rho''_{opTi}}{6.328\Delta t}\left(T^{n+1}_{i}-T^{n}_{i}\right) \\
&\quad +\frac{\varphi\rho'_{opi}}{6.328\Delta t}\left(S^{n+1}_{i}-S^{n}_{i}\right)
\end{aligned}
\tag{3.45}
$$

$$
\frac{\partial F_i}{\partial P_{i-1}} = D^{n+1}_{oi}\left(1+\frac{1}{2}\eta_{opi}\left(P^{n+1}_{i-1}-P^{n+1}_{i}\right)\right)+\frac{1}{2}D^{n+1}_{oi}\,\eta^{n+1}_{opi}\Delta x.GPO \tag{3.46}
$$

where

$$
\eta^{n+1}_{opi} = \left(\frac{\rho'_{op}}{\rho_o}-\frac{\mu'_{op}}{\mu_o}\right)^{n+1}_{i-\frac{1}{2}} \qquad i=1,...,N_x+1 \tag{3.47}
$$

Derivatives of oil and water densities and viscosities to pressure and temperature are defined using the equations 2.9 - 2.12.

3.5 Well Coupling

In order to find bottom hole pressures at the left and right boundaries, coupling between the block and well pressures is performed. It starts by taking initial guesses for boundary (well) pressures. Grid pressures are then solved based on these guesses. Because block and well pressures are correlated, new well pressures can be calculated from equations 3.48 and 3.49 and these new values used in a new iteration to calculate new grid pressures. Iterations continue to find fixed block and well pressures. These two equations are nonlinear for the well pressures, the Newton-Raphson method is used to find the roots.
The left boundary well pressure equation is

$$
q_t = -\frac{1.127K_{abs}K^{max}_{rw}A_n}{\mu_w(P_{wl})}\frac{P_B-P_{wl}}{\Delta x/2} \tag{3.48}
$$

The right boundary well pressure equation is

$$q_t=\frac{1.127K_{abs}A_n}{\Delta x/2}\left\{\frac{K_{rw}}{\mu_w(P_{wr})}+\frac{K_{ro}}{\mu_o(P_{wr})}\right\}(P_{wr}-P_B) \tag{3.49}$$

3.6 Energy Balance Equation Discretization

The energy equation 2.40 in oil field units (refer to nomenclature) is rewritten as

$$\begin{aligned}\frac{\partial}{\partial x}.(24K_H\frac{\partial T}{\partial x})+6.328\frac{\partial}{\partial x}.\{(\rho_o\frac{K_o}{\mu_o}H_o+\rho_w\frac{K_w}{\mu_w}H_w)\frac{\partial P}{\partial x}\}+\frac{\Sigma e}{Vb}\\ =\frac{\partial}{\partial t}\{\varphi(\rho_oS_oU_o+\rho_wS_wU_w)+(1-\varphi)\rho_rU_r\}\end{aligned} \tag{3.50}$$

$$\begin{aligned}H_\alpha&=H_\alpha^{ref}+C_{P\alpha}(T-T_{ref}) && \alpha=o,w\\ U_\alpha&=U_\alpha^{ref}+C_{V\alpha}(T-T_{ref}) && \alpha=o,w,r\end{aligned} \tag{3.51}$$

A fully implicit central finite difference scheme is used to discretize the equation. First, the equation is expanded by substituting equation 3.51 and then new equation is shortened by defining some parameters,

$$\begin{aligned}&\frac{\partial}{\partial x}.(24K_H\frac{\partial T}{\partial x})+6.328\frac{\partial}{\partial x}.\{(\rho_o\frac{K_o}{\mu_o}(H_o^{ref}+C_{Po}(T-T_{ref}))\frac{\partial P}{\partial x}\}\\ &+6.328\frac{\partial}{\partial x}\{\rho_w\frac{K_w}{\mu_w}(H_w^{ref}+C_{Pw}(T-T_{ref}))\frac{\partial P}{\partial x}\}+\frac{\Sigma e}{Vb}=\\ &\quad\frac{\partial}{\partial t}\{\varphi(\rho_oS_o(U_o^{ref}+C_{Vo}(T-T_{ref}))+\rho_wS_w(U_w^{ref}+C_{Vw}(T-T_{ref})))\\ &\quad+(1-\varphi)\rho_r(U_r^{ref}+C_{Vr}(T-T_{ref}))\}\end{aligned} \tag{3.52}$$

By defining,

$$HR=\rho_o\frac{K_o}{\mu_o}(H_o^{ref}-C_{Po}T_{ref})+\rho_w\frac{K_w}{\mu_w}(H_w^{ref}-C_{Pw}T_{ref}) \tag{3.53}$$

$$HB=\rho_o\frac{K_o}{\mu_o}C_{Po}+\rho_w\frac{K_w}{\mu_w}C_{Pw} \tag{3.54}$$

$$UR=\varphi\,(\rho_oS_o(U_o^{ref}-C_{Vo}T_{ref})+\rho_wS_w(U_w^{ref}-C_{Vw}T_{ref}))+(1-\varphi)\rho_r(U_r^{ref}-C_{Vr}T_{ref}) \tag{3.55}$$

$$UB=\varphi\,(\rho_oS_oC_{Vo}+\rho_wS_wC_{Vw})+(1-\varphi)\rho_rC_{Vr} \tag{3.56}$$

Equation 3.52 is simplified to

$$\frac{\partial}{\partial x}.(24K_H\frac{\partial T}{\partial x}) + 6.328\frac{\partial}{\partial x}((HR + HB.T)\frac{\partial P}{\partial x}) + \frac{\sum e}{V_b} = \frac{\partial}{\partial t}(UR + UB \times T) \tag{3.57}$$

3.6.1 Discretization of Right Hand Side of the Energy Equation 3.57

Replace the derivatives as follow

$$\begin{aligned} \frac{\partial}{\partial t}(UR + UB.T) &= \frac{1}{\Delta t}\{(UR + UB.T)^{n+1} - (UR + UB.T)^n\} \\ &= (\frac{UR^{n+1} - UR^n}{\Delta t}) + (\frac{UB^{n+1}}{\Delta t}).T^{n+1} - (\frac{UB^n}{\Delta t})T^n \end{aligned} \tag{3.58}$$

3.6.2 Discretization of Left Hand Side of Energy Equation 3.57 for Middle Cells ($\mathbf{i=2,\dots,N_x}$)

This side of the equation is divided into two terms; conductive and convective heat transfer. Conduction is the transfer of heat energy by diffusion due to the temperature gradient. In this project, conduction takes place in both rock and fluids. While convective heat transfer takes place through advection mostly, in which heat is transferred by the motion of currents in the fluid.

3.6.2.1 Conduction Term

In this dissertation, conductive heat is considered to transfer in two dimensions in order to model heat loss to adjacent strata [11]. Figure 3.4 shows the schematic diagram of the model.

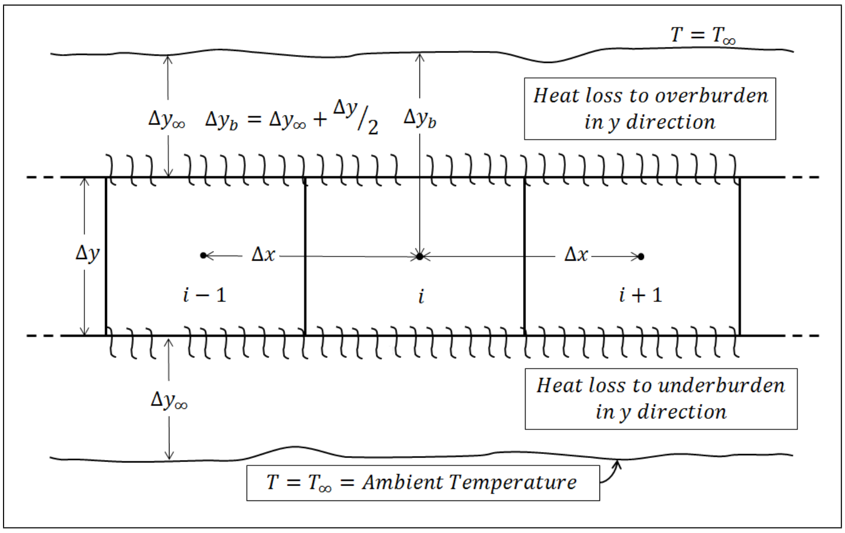

Figure 3.4: Heat transfer in the x and y direction by conduction

Using a central difference scheme with equal spacing in the x direction and unequal spacing in the y direction results in

$$\begin{gathered}\frac{\partial}{\partial x}(24K_H\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(24K_r\frac{\partial T}{\partial Y}) = \\ \frac{(24K_H\frac{\partial T}{\partial x})^{n+1}_{i+\frac{1}{2}} - (24K_H\frac{\partial T}{\partial x})^{n+1}_{i-\frac{1}{2}}}{\Delta x} + \frac{(24k_r\frac{\partial T}{\partial y})^{n+1}_{UP} - (24K_r\frac{\partial T}{\partial y})^{n+1}_{Down}}{y_\infty + \frac{\Delta y}{2}} = \\ (\frac{24K_H}{\Delta x^2})T^{n+1}_{i-1} - (\frac{48K_H}{\Delta x^2} + \frac{48k_r}{\Delta y_b^2})T_i^{n+1} + (\frac{24K_H}{\Delta x^2})T^{n+1}_{i+1} + (\frac{48K_r}{\Delta y_b^2})T_\infty\end{gathered} \tag{3.59}$$

where $\Delta y_b = y_\infty + \frac{\Delta y}{2}$.

3.6.2.2 Convection Term Discretization

Using a first order central scheme in space, this term will be discretized as,

$$6.328\frac{\partial}{\partial x}((HR{+}HB.T)\frac{\partial P}{\partial x}) = \frac{6.328}{\Delta x}\{(HR + HB.T) \times \frac{\partial P}{\partial x}\}^{n+1}_{i+\frac{1}{2}} - \frac{6.328}{\Delta x}\{(HR + HB.T)\frac{\partial P}{\partial x}\}^{n+1}_{i-\frac{1}{2}} \tag{3.60}$$

and by defining GP_i (pressure gradients) on the interfaces of blocks and taking an average temperature on interfaces, between two neighbouring blocks,

$$(\tfrac{\partial P}{\partial x})^{n+1}_{i-\frac{1}{2}} = GP_i^{n+1}$$

$$i = 1, ..., N_x + 1$$

$$T_{i+\frac{1}{2}} = (T_i + T_{i+1})/2 \quad , \quad T_{i-\frac{1}{2}} = (T_i + T_{i-1})/2$$

Convection term is sorted out as

$$\begin{aligned}6.328\frac{\partial}{\partial x}((HR+HB.T)\frac{\partial P}{\partial x}) =& \\ &\frac{6.328}{\Delta x}(HR_{i+1}^{n+1}.GP_{i+1}^{n+1}-HR_{i}^{n+1}.GP_{i}^{n+1})-\frac{6.328}{2\Delta x}(HB_{i}^{n+1}.GP_{i}^{n+1})\,T_{i}^{n+1}+ \\ &\frac{6.328}{2\Delta x}(HB_{i+1}^{n+1}.GP_{i+1}^{n+1}-HB_{i}^{n+1}.GP_{i}^{n+1})\,T_{i}^{n+1}+\frac{6.328}{2\Delta x}(HB_{i+1}^{n+1}.GP_{i+1}^{n+1})\,T_{i}^{n+1}\end{aligned} \tag{3.61}$$

By combining equations 3.57, 3.58, 3.59 and 3.61 together, the general energy equation for middle cells is obtained as($i = 2, \ldots, N_x$);

$$\begin{aligned}&\{\frac{24K_H}{\Delta x^2}-\frac{6.328}{2\Delta x}(HB_{i}^{n+1}.GP_{i}^{n+1})\}\,T_{i}^{n+1}\{\tfrac{24K_H}{\Delta x^2}-\tfrac{6.328}{2\Delta x}(HB_{i}^{n+1}.GP_{i}^{n+1})\}\,T_{i}^{n+1} \\ &\quad+\{-\frac{48K_H}{\Delta x^2}-\frac{48k_r}{\Delta y_b^2}+\frac{6.328}{2\Delta x}(HB_{i+1}^{n+1}.GP_{i+1}^{n+1}-HB_{i}^{n+1}.GP_{i}^{n+1})-\frac{UB_{i}^{n+1}}{\Delta t}\}\,T_{i}^{n+1} \\ &\quad+\{\frac{24K_H}{\Delta x^2}+\frac{6.328}{2\Delta x}(HB_{i+1}^{n+1}.GP_{i+1}^{n+1}\}\,T_{i+1}^{n+1}=(\frac{UR_{i}^{n+1}-UR_{i}^{n}}{\Delta t}) \\ &\quad-(\frac{UB_{i}^{n}}{\Delta t})\,T_{i}^{n}-(\frac{48K_r}{\Delta y_b^2})T_\infty-\frac{6.328}{\Delta x}(HR_{i+1}^{n+1}.GP_{i+1}^{n+1}-HR_{i}^{n+1}.GP_{i}^{n+1})\end{aligned} \tag{3.62}$$

3.6.3 Calculations for the Left Boundary Cell (i=1)

For this cell, the conduction and convection terms are treated differently because of the effect of boundary conditions.

3.6.3.1 Conduction Term

Applying a central difference scheme to the conduction term in the x direction (with unequal spacing) gives;

$$\frac{\partial}{\partial x}(24K_H\frac{\partial T}{\partial x})=\frac{24K_H(\frac{T_{i+1}^{n+1}-T_{i}^{n}}{\Delta x})-24K_H(\frac{T_{i}^{n+1}-T_{inj}}{\Delta x/2})}{(\frac{3\Delta x}{2})/2} \tag{3.63}$$

and in the y direction

$$\frac{\partial}{\partial y}(24K_r\frac{\partial T}{\partial y})=\frac{24k_r(\frac{T_\infty-T_{i}^{n+1}}{\Delta y_\infty+\Delta y/2})-24K_r(\frac{T_{i}^{n+1}-T_\infty}{\Delta y_\infty+\Delta y/2})}{(\Delta y_\infty+\Delta y/2)} \tag{3.64}$$

By adding these two together and rearranging the terms, the conduction term is written as

$$\begin{aligned}&\frac{\partial}{\partial x}(24K_H\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(24K_r\frac{\partial T}{\partial y}) = \tfrac{\partial}{\partial x}(24K_H\tfrac{\partial T}{\partial x}) + \tfrac{\partial}{\partial y}(24K_r\tfrac{\partial T}{\partial y}) = \\ &\quad (-\frac{96K_H}{\Delta x^2} - \frac{48K_r}{\Delta y_b^2})\,T_i^{n+1} + (\frac{32K_H}{\Delta x^2})\,T_{i+1}^{n+1} + (\frac{64K_H}{\Delta x^2})T_{inj} + (\frac{48K_r}{\Delta y_b^2})T_\infty\end{aligned} \tag{3.65}$$

3.6.3.2 Convection Term

By expanding the convection term on the first cell and substituting the injection temperature,

$$\begin{aligned}&6.328\frac{\partial}{\partial x}((HR + HB.T)\frac{\partial P}{\partial x}) = \\ &\quad \frac{6.328}{\Delta x}\,\{HR.GP + HB.GP.T\}_{i+\frac{1}{2}}^{n+1} - \frac{6.328}{\Delta x}\{HR.GP + HB.GP.T\}_b^{n+1} = \\ &\quad (\frac{3.164}{\Delta x})\,(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1})\,(T_i^{n+1} + T_{i+1}^{n+1}) + (\frac{6.328}{\Delta x})\,\{HR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} - HR_i^{n+1}\,.\,GP_i^{n+1}\} \\ &\quad - (\frac{6.328}{\Delta x})(HB_i^{n+1}\,.\,GP_i^n)\,T_{inj}\end{aligned} \tag{3.66}$$

Hence, the discretized energy equation is rearranged based on main variables (temperatures) as below;

$$\begin{aligned}&-\{\frac{96K_H}{\Delta x^2} + \frac{48K_r}{\Delta y_b^2} - \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}) \\ &\quad + \frac{UB_i^{n+1}}{\Delta t}\}.\,T_i^{n+1} + \{\frac{32K_H}{\Delta x^2} + \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1})\}.\,T_{i+1}^{n+1} = \\ &\quad \frac{1}{\Delta t}(UR_i^{n+1} - UR_i^n) - (\frac{UB_i^n}{\Delta t})\,T_i^n + (\frac{24K_r}{\Delta y_b^2})(T_{inj} - T_\infty) - (\frac{64K_H}{\Delta x^2})T_{inj} - (\frac{48K_r}{\Delta y_b^2})T_\infty \\ &\quad - (\frac{6.328}{\Delta x})\,\{HR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} - HR_i^{n+1}\,.\,GP_i^{n+1}\} + (\frac{6.328}{\Delta x})(HB_i^{n+1}\,.\,GP_i^{n+1})\,T_{inj}\end{aligned} \tag{3.67}$$

3.6.4 Calculations for the Right Boundary Cell (i=$\mathbf{N_x}$)

Considering the right boundary conditions defined in the previous chapter, the conduction term in the x direction is zero for this cell. In the y direction is defined as in the other blocks from equation 3.64.

The convection term is derived in a similar way; the difference being to substitute $T_b = \frac{1}{2}(3T_i - T_{i-1})$ for T_{i+1} so, it will be

$$\begin{aligned}6.328\frac{\partial}{\partial x}((HR+HB.T)\frac{\partial P}{\partial x}) &= \frac{6.328}{\Delta x}(HR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}-HR_{i}^{n+1}\,.\,GP_{i}^{n+1})\\ &+(\frac{3.164}{\Delta x})(3.\,HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}-HB_{i}^{n+1}\,.\,GP_{i}^{n+1}).\,T_{i}^{n+1}\\ &-(\frac{3.164}{\Delta x})(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}+HB_{i}^{n+1}\,.\,GP_{i}^{n+1}).\,T_{i-1}^{n+1}\end{aligned} \tag{3.68}$$

Heat loss is included in the energy term,

$$\frac{\sum e}{V_b}=-(\frac{24K_H}{\Delta x.\Delta x_{\infty}})(\frac{3}{2}\,T_{i}^{n+1}-\frac{1}{2}\,T_{i-1}^{n+1}-T_{\infty}) \tag{3.69}$$

Hence, the energy equation in its discretized form for the last cell ($i=N_x$) is

$$\begin{aligned}&\{(\frac{-3.164}{\Delta x})(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}+HB_{i}^{n+1}\,.\,GP_{i}^{n+1})+(\frac{12K_r}{\Delta x.\Delta x_{\infty}})\}\,T_{i-1}^{n+1}\\ &+\{(\frac{3.164}{\Delta x})(3\,HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}-HB_{i}^{n+1}\,.\,GP_{i}^{n+1})-(\frac{36K_r}{\Delta x.\Delta x_{\infty}})-(\frac{48K_r}{\Delta y_b^2})-\frac{UB_i^{n+1}}{\Delta t}\}\,T_{i}^{n+1}=\\ &(\frac{-6.328}{\Delta x})\,\{HR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}-HR_{i}^{n+1}\,.\,GP_{i}^{n+1}\}+\frac{1}{\Delta t}(UR_{i}^{n+1}-UR_{i}^{n})\\ &-(\frac{UB_i^{n}}{\Delta t})\,T_{i}^{n}-(\frac{24K_r}{\Delta x.\Delta x_{\infty}})T_{\infty}-(\frac{48K_r}{\Delta y_b^2})T_{\infty}\end{aligned} \tag{3.70}$$

Writing these equations for all grid blocks will result in a nonlinear system. To solve this system, it is necessary to define the Jacobian. For simplicity, before calculating the Jacobian , new parameters are defined as the derivatives of HB_i, HR_i ,$UB_i$and UR_i.
All these terms can be redefined using D_{oi}(equation 3.26) so,

$$DHB_i=D_{oi}\eta_{oTi}C_{Po}+D_{wi}\eta_{wTi}C_{Pw} \tag{3.71}$$

$$DHR_i=D_{oi}\eta_{oTi}(H_o^{ref}-C_{Po}.T_{ref})+D_{wi}\eta_{wTi}(H_w^{ref}-C_{Pw}.T_{ref}) \tag{3.72}$$

$$DUB_i=\varphi\,\rho'_{oTi}\,S_{oi}.C_{Vo}+\varphi\,\rho'_{wTi}\,S_{wi}C_{Vw} \tag{3.73}$$

$$DUR_i=\varphi\,\rho'_{oTi}\,S_{oi}.(U_o^{ref}-C_{Vo}T_{ref})+\varphi\,\rho'_{wTi}\,S_{wi}(U_w^{ref}-C_{Vw}T_{ref}) \tag{3.74}$$

where

$$\eta_{\alpha Ti}^{n+1}=\left(\frac{\rho'_{\alpha T}}{\rho_\alpha}-\frac{\mu'_{\alpha T}}{\mu_\alpha}\right)_{i-\frac{1}{2}}^{n+1} \qquad i=1,...,N_x+1\ and\ \alpha=oil\,and\,water \tag{3.75}$$

Note that $C_{P\alpha}$and $C_{V\alpha}$ are considered to be independent of temperature (constant) and the derivatives of density and viscosity with respect to temperature and pressure are defined using the equations 2.9 - 2.12.

Based on above definitions and equation 3.62, the Jacobian calculations for $i = 2, \ldots, N_x - 1$ give

$$\begin{aligned}\frac{\partial F_i}{\partial T_{i-1}} &= (\frac{24K_H}{\Delta x^2} - \frac{3.164}{\Delta x}(HB_i^{n+1}\,.\,GP_i^{n+1})) - 3.164\Delta x.\,DHR_i^{n+1}\,.\,GP_i^{n+1} \\ &- 3.164(\frac{\Delta x}{2}).\,DHB_i^{n+1}\,.\,GP_i^{n+1}\,.(T_i^{n+1} + T_{i-1}^{n+1})\end{aligned} \tag{3.76}$$

$$\begin{aligned}\frac{\partial F_i}{\partial T_i} &= -3.164(\frac{\Delta x}{2}).\,DHB_i^{n+1}\,.\,GP_i^{n+1}\,.(T_i^{n+1} + T_{i-1}^{n+1}) \\ &+ 3.164(\frac{\Delta x}{2}).\,DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\,.(T_i^{n+1} + T_{i+1}^{n+1}) - \frac{DUB_i^{n+1}}{\Delta t}.\,T_i^{n+1} - \frac{DUR_i^{n+1}}{\Delta t} \\ &+ \{-\frac{48K_H}{\Delta x^2} - \frac{48K_r}{\Delta y_b^2} + \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} - HB_i^{n+1}\,.\,GP_i^{n+1}) - \frac{UB_i^{n+1}}{\Delta t}\} \\ &+ 6.328(\frac{\Delta x}{2})(DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} - DHR_i^{n+1}\,.\,GP_i^{n+1}\end{aligned} \tag{3.77}$$

$$\begin{aligned}\frac{\partial F_i}{\partial T_{i+1}} &= (\frac{24K_H}{\Delta x^2} + \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1})) + 3.164.\Delta x.\,DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\tfrac{\partial F_i}{\partial T_{i+1}} + \\ &(\frac{24K_H}{\Delta x^2} + \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1})) + 3.164.\Delta x.\,DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} \\ &3.164(\frac{\Delta x}{2}).\,DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\,.(T_i^{n+1} + T_{i+1}^{n+1})\end{aligned} \tag{3.78}$$

In the case of the first cell, the Jacobian elements for the energy equation 3.67 are obtained as

$$\begin{aligned}\frac{\partial F_i}{\partial T_i} &= -\{\frac{96K_H}{\Delta x^2} + \frac{48K_r}{\Delta y_b^2} - \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}) + \frac{UB_i^{n+1}}{\Delta t}\} \\ &+ 3.164(\frac{\Delta x}{2}).\,DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\,.(T_i^{n+1} + T_{i+1}^{n+1}) \\ &+ \frac{DUB_i^{n+1}}{\Delta t}.\,T_i^{n+1} - \frac{DUR_i^{n+1}}{\Delta t} + 3.164.\Delta x.(DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\end{aligned} \tag{3.79}$$

$$\begin{aligned}\frac{\partial F_i}{\partial T_{i+1}} &= +\{\frac{32K_H}{\Delta x^2} + \frac{3.164}{\Delta x}(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1})\} \\ &+ 3.164(\frac{\Delta x}{2}).\,DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\,.(T_i^{n+1} + T_{i+1}^{n+1}) + 3.164.\Delta x.\,DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1}\end{aligned} \tag{3.80}$$

and finally, the Jacobian calculations for the last cell ($i = N_x$) will be

$$\begin{aligned}\frac{\partial F_i}{\partial T_{i-1}} &= (\frac{-3.164}{\Delta x})(HB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} + HB_i^{n+1}\,.\,GP_i^{n+1}) + (\frac{12K_r}{\Delta x.\Delta x_\infty}) \\ &+ 3.164(\frac{\Delta x}{2})(DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} - DHB_i^{n+1}\,.\,GP_i^{n+1})\,T_{i-1}^{n+1} \\ &- 3.164(\frac{\Delta x}{2})(3\,DHB_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} + DHB_i^{n+1}\,.\,GP_i^{n+1}).\,T_i^{n+1} \\ &- 3.164.\Delta x.(DHR_{i+1}^{n+1}\,.\,GP_{i+1}^{n+1} + DHR_i^{n+1}\,.\,GP_i^{n+1})\end{aligned} \tag{3.81}$$

$$
\begin{aligned}
\frac{\partial F_i}{\partial Ti} &= \{(\frac{3.164}{\Delta x})(3HB_{i+1}^{n+1}.GP_{i+1}^{n+1} - HB_i^{n+1}.GP_i^{n+1}) - (\frac{36K_r}{\Delta x.\Delta x_\infty}) - (\frac{48K_r}{\Delta y_b^2}) - \frac{UB_i^{n+1}}{\Delta t}\} \\
&- 3.164(\frac{\Delta x}{2})(3\,DHB_{i+1}^{n+1}.GP_{i+1}^{n+1} + DHB_i^{n+1}.GP_i^{n+1}).T_{i-1}^{n+1} \\
&+ \{3.164(\frac{\Delta x}{2})(9\,DHB_{i+1}^{n+1}.GP_{i+1}^{n+1} - DHB_i^{n+1}.GP_i^{n+1}) - \frac{DUB_i^{n+1}}{\Delta t}\}.T_i^{n+1} \\
&+ 3.164.\Delta x.(3\,DHR_{i+1}^{n+1}.GP_{i+1}^{n+1} - DHR_i^{n+1}.GP_i^{n+1}) - \frac{DUR_i^{n+1}}{\Delta t}
\end{aligned}
\tag{3.82}
$$

3.7 Summary of The First Model

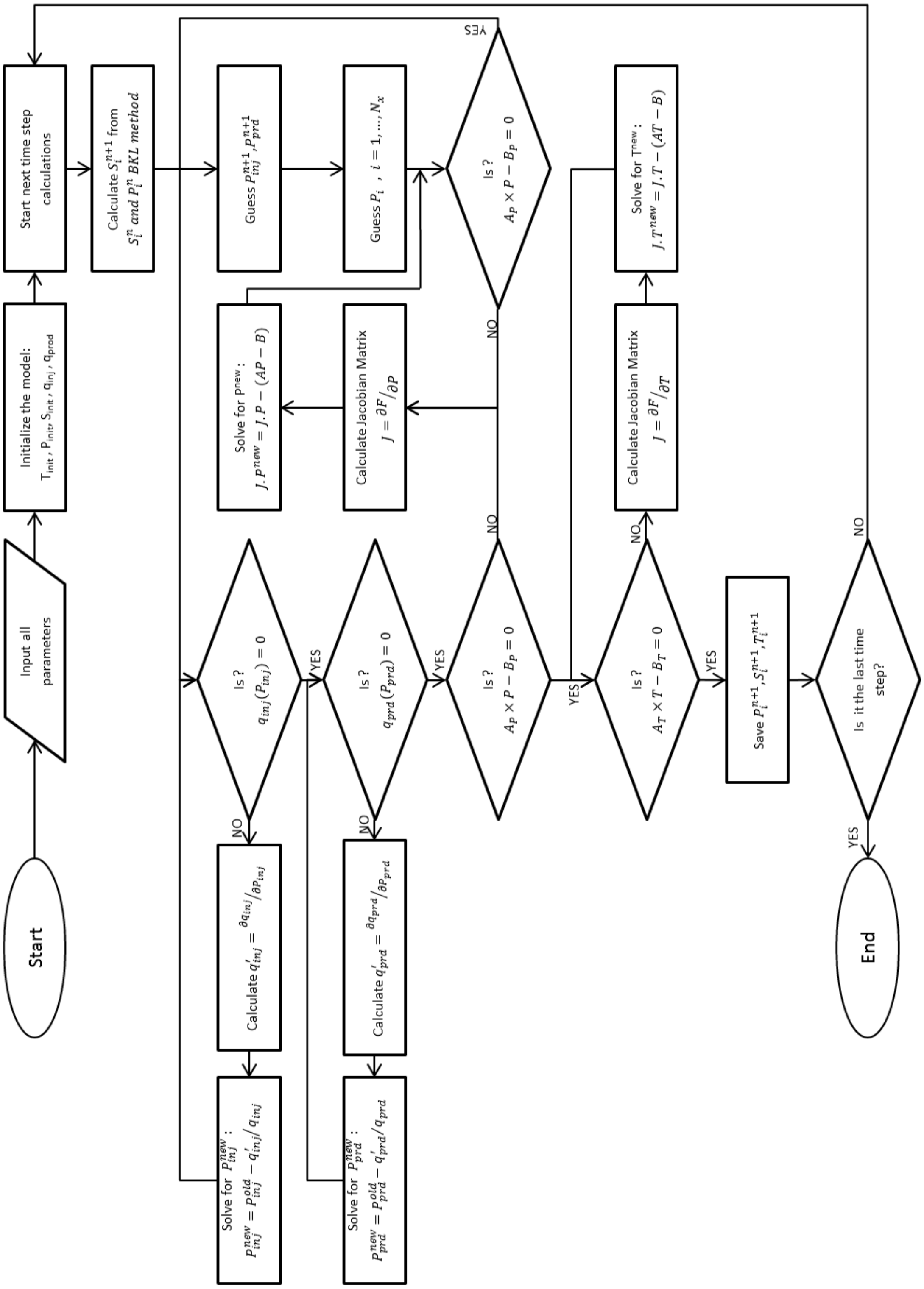

Figure 3.5: The First Model Calculations Flow Chart

4 Second Model

In this model, the pressure will be solved implicitly and after finding the pressure solution, saturation values can be determined explicitly. This technique is called IMPES and is much used in the oil industry [8]. During one time step the results of IMPES are used in the temperature equation which is solved fully implicitly, and finally there will a coupling between the IMPES technique and the fully implicit temperature equation in order to find the final pressure, saturation and temperature distribution results.

4.1 IMPES Technique

In the oil and water system, the general 3D equations for oil and water are

$$\begin{cases} \vec{\nabla} \cdot \left(\rho_o \frac{K_o}{\mu_o} \vec{\nabla} P_o \right) + \frac{\rho_o \Sigma q_o}{1.127 V_b} = \frac{1}{6.328} \frac{\partial}{\partial t} \left(\rho_o \varphi \, S_o \right) \\ \vec{\nabla} \cdot \left(\rho_w \frac{K_w}{\mu_w} \vec{\nabla} P_w \right) + \frac{\rho_w \Sigma q_w}{1.127 V_b} = \frac{1}{6.328} \frac{\partial}{\partial t} \left(\rho_w \varphi \, S_w \right) \end{cases} \tag{4.1}$$

By considering the assumptions made in section 2.1 , and considering that generation and consumption terms are replaced by boundary conditions and expansion of right hand side of the equations, we have

$$\begin{cases} \frac{\partial}{\partial x} \left(\rho_o \frac{K_o}{\mu_o} \frac{\partial P}{\partial x} \right) = \frac{\varphi}{6.328} \left[(1-S) \, \rho'_{op} \frac{\partial P}{\partial t} + (1-S) \, \rho'_{oT} \frac{\partial T}{\partial t} - \rho_o \frac{\partial S}{\partial t} \right] \\ \frac{\partial}{\partial x} \left(\rho_w \frac{K_w}{\mu_w} \frac{\partial P}{\partial x} \right) = \frac{\varphi}{6.328} \left[S \, \rho'_{wp} \frac{\partial P}{\partial t} + S \, \rho'_{wT} \frac{\partial T}{\partial t} - \rho_w \frac{\partial S}{\partial t} \right] \end{cases} \tag{4.2}$$

Divide oil pressure equation by ρ_o and water pressure equation by ρ_w , then add them together, and rearrange the final equation to find the discretized pressure equation for internal cells ($i = 2, \ldots, \ N_x - 1$), giving

$$\begin{aligned} &\frac{1}{\rho_o} \frac{\partial}{\partial x} \left(\rho_o \frac{K_o}{\mu_o} \frac{\partial P}{\partial x} \right) + \frac{1}{\rho_w} \frac{\partial}{\partial x} \left(\rho_w \frac{K_w}{\mu_w} \frac{\partial P}{\partial x} \right) = \\ &\qquad \frac{\varphi}{6.328} \left[(1-S) \frac{\rho'_{op}}{\rho_o} + S \frac{\rho'_{wp}}{\rho_w} \right] \frac{\partial P}{\partial t} + \frac{\varphi}{6.328} \left[(1-S) \frac{\rho'_{oT}}{\rho_o} + S \frac{\rho'_{wT}}{\rho_w} \right] \frac{\partial T}{\partial t} \end{aligned} \tag{4.3}$$

For simplicity we define the equations parameters and terms to be similar to those defined in the first model (refer to section 3.2). The above equation 4.3 is discretized using a fully implicit scheme and by considering pressures as the main variables as follows,

$$\begin{aligned}
&(\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}})\, P_{i-1}^{n+1} - \{\frac{D_{oi}^{n+1} + D_{oi+1}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1} + D_{wi+1}^{n+1}}{\rho_{wi}} + \frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{opi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'_{wpi}}{\rho_{wi}})\}\, P_i^{n+1} \\
&\quad + (\frac{D_{oi+1}^{n+1}}{\rho_{oi}} + \frac{D_{wi+1}^{n+1}}{\rho_{wi}})\, P_{i+1}^{n+1} = -\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{opi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'_{wpi}}{\rho_{wi}})\}\, P_i^n + \\
&\quad \frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{oTi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'_{wTi}}{\rho_{wi}})\}(T_i^{n+1} - T_i^n)
\end{aligned} \tag{4.4}$$

Different treatments are required to obtain the pressure equation for the left boundary cell ($i = 1$), by considering its boundary conditions (refer to section 2.4).

$$\begin{aligned}
&\frac{1}{\rho_{oi}.\Delta x}\{(\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x})_{i+\frac{1}{2}} - (\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x})_{bl}\} + \frac{1}{\rho_{wi}\Delta x}\{(\rho_w \frac{K_w}{\mu_w}\frac{\partial P}{\partial x})_{i+\frac{1}{2}} - (\rho_w \frac{K_w}{\mu_w}\frac{\partial P}{\partial x})_{bl}\} \\
&\quad = \frac{\varphi}{6.328}\left[(1-S)\frac{\rho'_{op}}{\rho_o} + S\frac{\rho'_{wp}}{\rho_w}\right]\frac{\partial P}{\partial t} + \frac{\varphi}{6.328}\left[(1-S)\frac{\rho'_{oT}}{\rho_o} + S\frac{\rho'_{wT}}{\rho_w}\right]\frac{\partial T}{\partial t}
\end{aligned} \tag{4.5}$$

By using equation 3.30 for the pressure gradient on this boundary,

$$\begin{aligned}
&-\{\frac{D_{oi+1}^{n+1}}{\rho_{oi}} + \frac{D_{wi+1}^{n+1}}{\rho_{wi}} + \frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'{}_{opi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'{}_{wpi}}{\rho_{wi}})\}\, P_i^{n+1} \\
&\quad + (\frac{D_{wi+1}^{n+1}}{\rho_{wi}} + \frac{D_{oi+1}^{n+1}}{\rho_{oi}})\, P_{i+1}^{n+1} = -GPI.\Delta x.(\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}}) \\
&\quad -\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'{}_{opi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'{}_{wpi}}{\rho_{wi}})\}\, P_i^n + \\
&\quad \{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'{}_{oTi}}{\rho_{oi}}) + \frac{\varphi\, S_i^n}{6.328\Delta t}(\frac{\rho'{}_{wTi}}{\rho_{wi}})\}(T_i^{n+1} - T_i^n)
\end{aligned} \tag{4.6}$$

The investigation for the right boundary cell ($i = N_x$) shows that

$$\begin{aligned}
&\frac{1}{\rho_{oi}\Delta x}\{(\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x})_{br}^{n+1} - (\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x})_{i-\frac{1}{2}}^{n+1}\} + \frac{1}{\rho_{wi}\Delta x}\{(\rho_w \frac{K_w}{\mu_w}\frac{\partial P}{\partial x})_{br}^{n+1} - (\rho_w \frac{K_w}{\mu_w}\frac{\partial P}{\partial x})_{i-\frac{1}{2}}^{n+1} = \\
&\quad \frac{\varphi}{6.328}\left[(1-S)\frac{\rho'_{op}}{\rho_o} + S\frac{\rho'_{wp}}{\rho_w}\right]\frac{\partial P}{\partial t} + \frac{\varphi}{6.328}\left[(1-S)\frac{\rho'_{oT}}{\rho_o} + S\frac{\rho'_{wT}}{\rho_w}\right]\frac{\partial T}{\partial t}
\end{aligned} \tag{4.7}$$

By using equation 3.32 as the pressure gradient on this boundary and the oil and water properties at boundary which are calculated from extrapolated temperature and saturation, and using all these definitions, the discretized pressure equation for $i = N_x$ is

$$
\begin{aligned}
&(\frac{D_{oi}^{n+1}}{\rho_{oi}}+\frac{D_{wi}^{n+1}}{\rho_{wi}})\,P_i^{n+1}-\{\frac{D_{oi}^{n+1}}{\rho_{oi}}+\frac{D_{wi}^{n+1}}{\rho_{wi}}+\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{opi}}{\rho_{oi}})+\frac{\varphi\,S_i^n}{6.328\Delta t}(\frac{\rho'_{wpi}}{\rho_{wi}})\}\,P_i^{n+1}=\\
&\qquad GPO.\Delta x.(\frac{D_{oi+1}^{n+1}}{\rho_{oi}}+\frac{D_{wi+1}^{n+1}}{\rho_{wi}})-\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{opi}}{\rho_{oi}})+\frac{\varphi\,S_i^n}{6.328\Delta t}(\frac{\rho'_{wpi}}{\rho_{wi}})\}\,P_i^n\\
&\qquad -\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{oTi}}{\rho_{oi}})+\frac{\varphi\,S_i^n}{6.328\Delta t}(\frac{\rho'_{wTi}}{\rho_{wi}})\}(T_i^{n+1}-T_i^n)\\
&\qquad -\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}(\frac{\rho'_{oTi}}{\rho_{oi}})+\frac{\varphi\,S_i^n}{6.328\Delta t}(\frac{\rho'_{wTi}}{\rho_{wi}})\}(T_i^{n+1}-T_i^n)
\end{aligned}
\tag{4.8}
$$

The result of writing the pressure equation for all blocks is also a nonlinear system to solve, so the definition of the Jacobian for this system is required.

4.2 Jacobian Calculations for the Pressure Equation

The general equation 4.4 is a function of three variables; P_{i-1}, P_i and P_{i+1} , therefore the Jacobian matrix is defined as follow

$$
\frac{\partial F_i}{\partial P_{i-1}}=\frac{1}{2}(\frac{D_{wi}^{n+1}}{\rho_{wi}}\eta_{wpi}^{n+1}+\frac{D_{oi}^{n+1}}{\rho_{oi}}\eta_{opi}^{n+1})(P_{i-1}^{n+1}-P_i^{n+1})+(\frac{D_{oi}^{n+1}}{\rho_{oi}}+\frac{D_{wi}^{n+1}}{\rho_{wi}})
\tag{4.9}
$$

$$
\begin{aligned}
\frac{\partial F_i}{\partial P_i}&=\{D_{oi+1}^{n+1}(\frac{1}{2}\frac{\eta_{opi+1}^{n+1}}{\rho_{oi}}-\frac{\rho'_{opi}}{\rho_{oi}^2})+D_{wi+1}^{n+1}(\frac{1}{2}\frac{\eta_{wpi+1}^{n+1}}{\rho_{wi}}-\frac{\rho'_{wpi}}{\rho_{wi}^2})\}\,P_{i+1}^{n+1}\\
&\quad-\{\frac{D_{oi}^{n+1}+D_{oi+1}^{n+1}}{\rho_{oi}}+\frac{D_{wi}^{n+1}+D_{wi+1}^{n+1}}{\rho_{wi}}+\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho'_{opi}}{\rho_{oi}}+\frac{\varphi\,S_i^n}{6.328\Delta t}.\frac{\rho'_{wpi}}{\rho_{wi}}\}\\
&\quad-\{\frac{D_{oi}^{n+1}\,\eta_{opi}^{n+1}+D_{oi+1}^{n+1}\,\eta_{opi+1}^{n+1}}{2\,\rho_{oi}}+\frac{D_{wi}^{n+1}\,\eta_{wpi}^{n+1}+D_{wi+1}^{n+1}\,\eta_{wpi+1}^{n+1}}{2\,\rho_{wi}}\\
&\quad-(D_{oi}^{n+1}+D_{oi+1}^{n+1})(\frac{\rho'_{opi}}{\rho_{oi}})-(D_{wi}^{n+1}+D_{wi+1}^{n+1})(\frac{\rho'_{wpi}}{\rho_{wi}})+\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{oppi}\,\rho_{oi}-(\rho'_{opi})^2}{\rho_{oi}^2}\\
&\quad+\frac{\varphi\,S_i^n}{6.328\Delta t}.\frac{\rho''_{wppi}\,\rho_{wi}-(\rho'_{wpi})^2}{\rho_{wi}^2}\}P_i+\{D_{oi}^{n+1}(\frac{1}{2}\frac{\eta_{opi}^{n+1}}{\rho_{oi}}-\frac{\rho'_{opi}}{\rho_{oi}^2})+D_{wi}^{n+1}(\frac{1}{2}\frac{\eta_{wpi}^{n+1}}{\rho_{wi}}-\frac{\rho'_{wpi}}{\rho_{wi}^2})\}\,P_{i-1}^{n+1}\\
&\quad+\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{oppi}\,\rho_{oi}-(\rho'_{opi})^2}{\rho_{oi}^2}+\frac{\varphi\,S_i^n}{6.328\Delta t}.\frac{\rho''_{wppi}\,\rho_{wi}-(\rho'_{wpi})^2}{\rho_{wi}^2}\}\,P_i^n\\
&\quad-\{\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{opTi}\,\rho_{oi}-\rho'_{opi}\,\rho'_{oTi}}{\rho_{oi}^2}+\frac{\varphi\,S_i^n}{6.328\Delta t}.\frac{\rho''_{wpTi}\,\rho_{wi}-\rho'_{wpi}\,\rho'_{wTi}}{\rho_{wi}^2}\}(T_i^{n+1}-T_i^n)
\end{aligned}
\tag{4.10}
$$

$$
\frac{\partial F_i}{\partial P_{i+1}}=\frac{1}{2}(\frac{D_{wi+1}^{n+1}}{\rho_{wi}}\eta_{wpi+1}^{n+1}+\frac{D_{oi+1}^{n+1}}{\rho_{oi}}\eta_{opi+1}^{n+1})(P_{i+1}^{n+1}-P_i^{n+1})+(\frac{D_{oi+1}^{n+1}}{\rho_{oi}}+\frac{D_{wi+1}^{n+1}}{\rho_{wi}})
\tag{4.11}
$$

The first cell ($i = 1$) pressure equation depends on two variables; p_i and p_{i+1} , so

$$\begin{aligned}\frac{\partial F_i}{\partial P_i} &= \{D_{oi+1}^{n+1}(\frac{1}{2}\frac{\eta_{opi+1}^{n+1}}{\rho_{oi}} - \frac{\rho'_{opi}}{\rho_{oi}^2}) + D_{wi+1}^{n+1}(\frac{1}{2}\frac{\eta_{wpi+1}^{n+1}}{\rho_{wi}} - \frac{\rho'_{wpi}}{\rho_{wi}^2})\}(P_{i+1}^{n+1} - P_i^{n+1}) \\ &\quad - \{\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{oppi}\,\rho_{oi} - (\rho'_{opi})^2}{\rho_{oi}^2} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho''_{wppi}\,\rho_{wi} - (\rho'_{wpi})^2}{\rho_{wi}^2}\}(P_i^{n+1} - P_i^n) \\ &- \{\frac{D_{oi+1}^{n+1}}{\rho_{oi}} + \frac{D_{wi+1}^{n+1}}{\rho_{wi}} + \frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho'_{opi}}{\rho_{oi}} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho'_{wpi}}{\rho_{wi}}\} \\ &+ GPI.\Delta x.\{D_{oi}^{n+1}.(\frac{3}{2}\frac{\eta_{opi}^{n+1}}{\rho_{oi}} - \frac{\rho'{}_{opi}}{\rho_{oi}^2}) + D_{wi}^{n+1}.(\frac{3}{2}\frac{\eta_{wpi}^{n+1}}{\rho_{wi}} - \frac{\rho'{}_{wpi}}{\rho_{wi}^2})\} \\ &+ \frac{3}{2}.GPI.\Delta x.(\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}})(\frac{\mu'{}_{wpb}}{\mu_{wb}}) \\ &- \{\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{opTi}\,\rho_{oi} - \rho'_{opi}\,\rho'_{oTi}}{\rho_{oi}^2} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho''_{wpTi}\,\rho_{wi} - \rho'_{wpi}\,\rho'_{wTi}}{\rho_{wi}^2}\}(T_i^{n+1} - T_i^n)\end{aligned} \tag{4.12}$$

$$\begin{aligned}\frac{\partial F_i}{\partial P_{i+1}} &= \frac{1}{2}(\frac{D_{wi+1}^{n+1}}{\rho_{wi}}\eta_{wpi+1}^{n+1} + \frac{D_{oi+1}^{n+1}}{\rho_{oi}}\eta_{opi+1}^{n+1})(P_{i+1}^{n+1} - P_i^{n+1}) + (\frac{D_{oi+1}^{n+1}}{\rho_{oi}} + \frac{D_{wi+1}^{n+1}}{\rho_{wi}}) \\ &- \frac{1}{2}GPI.\Delta x.\{D_{oi}^{n+1}.(\frac{\eta_{opi}^{n+1}}{\rho_{oi}}) + D_{wi}^{n+1}.(\frac{\eta_{wpi}^{n+1}}{\rho_{wi}}) - \frac{1}{2}.GPI.\Delta x.(\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}})(\frac{\mu'{}_{wpb}}{\mu_{wb}})\end{aligned} \tag{4.13}$$

and finally, for the last cell ($i = N_x$) pressure equation which is function of p_i and p_{i-1}, we have;

$$\begin{aligned}\frac{\partial F_i}{\partial P_{i-1}} &= \frac{1}{2}(\frac{D_{wi}^{n+1}}{\rho_{wi}}\eta_{wpi}^{n+1} + \frac{D_{oi}^{n+1}}{\rho_{oi}}\eta_{opi}^{n+1})(P_{i-1}^{n+1} - P_i^{n+1}) + (\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}}) \\ &+ \frac{1}{2}GPO.\Delta x(\frac{D_{wi+1}^{n+1}}{\rho_{wi}}\eta_{wpi+1}^{n+1} + \frac{D_{oi+1}^{n+1}}{\rho_{oi}}\eta_{opi+1}^{n+1})\end{aligned} \tag{4.14}$$

$$\begin{aligned}\frac{\partial F_i}{\partial P_i} &= \{D_{oi}^{n+1}(\frac{1}{2}\frac{\eta_{opi}^{n+1}}{\rho_{oi}} - \frac{\rho'_{opi}}{\rho_{oi}^2}) + D_{wi}^{n+1}(\frac{1}{2}\frac{\eta_{wpi}^{n+1}}{\rho_{wi}} - \frac{\rho'_{wpi}}{\rho_{wi}^2})\}\,(P_{i-1}^{n+1} - P_i^{n+1}) \\ &\quad \frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{oppi}\,\rho_{oi} - (\rho'_{opi})^2}{\rho_{oi}^2} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho''_{wppi}\,\rho_{wi} - (\rho'_{wpi})^2}{\rho_{wi}^2}\}(P_i^{n+1} - P_i^n) \\ &- \{\frac{D_{oi}^{n+1}}{\rho_{oi}} + \frac{D_{wi}^{n+1}}{\rho_{wi}} + \frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho'_{opi}}{\rho_{oi}} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho'_{wpi}}{\rho_{wi}}\} \\ &- GPO.\Delta x.\{D_{oi+1}^{n+1}.(\frac{3}{2}\frac{\eta_{opi+1}^{n+1}}{\rho_{oi}} - \frac{\rho'{}_{opi}}{\rho_{oi}^2}) + D_{wi+1}^{n+1}.(\frac{3}{2}\frac{\eta_{wpi+1}^{n+1}}{\rho_{wi}} - \frac{\rho'{}_{wpi}}{\rho_{wi}^2})\} \\ &- \{\frac{\varphi(1-S_i^n)}{6.328\Delta t}.\frac{\rho''_{opTi}\,\rho_{oi} - \rho'_{opi}\,\rho'_{oTi}}{\rho_{oi}^2} + \frac{\varphi\, S_i^n}{6.328\Delta t}.\frac{\rho''_{wpTi}\,\rho_{wi} - \rho'_{wpi}\,\rho'_{wTi}}{\rho_{wi}^2}\}(T_i^{n+1} - T_i^n)\end{aligned} \tag{4.15}$$

4.3 Saturation Calculations

Oil and water saturations are evaluated explicitly by using the results of the fully implicit pressure equation. The following equations are applied to find the saturations

$$\begin{aligned} &\frac{\partial}{\partial x}(\rho_o \frac{K_o}{\mu_o}\frac{\partial P}{\partial x}) = \frac{1}{6.328}\frac{\partial}{\partial t}(\varphi\,\rho_o S_o) \\ &\frac{\partial}{\partial x}(\rho_w \frac{K_w}{\mu_w}\frac{\partial P}{\partial x}) = \frac{1}{6.328}\frac{\partial}{\partial t}(\varphi\,\rho_w S_w) \end{aligned} \tag{4.16}$$

These equations are discretized as

$$\begin{aligned} &D_{oi+1}^{n+1} \cdot (P_{i+1}^{n+1} - P_i^{n+1}) - D_{oi}^{n+1}(P_i^{n+1} - P_i^{n+1}) = \frac{\varphi}{6.328}\{\rho_{oi}^{n+1} \cdot S_{oi}^{n+1} - \rho_{oi}^{n}\, S_{oi}^{n}\} \\ &D_{wi+1}^{n+1} \cdot (P_{i+1}^{n+1} - P_i^{n+1}) - D_{wi}^{n+1}(P_i^{n+1} - P_i^{n+1}) = \frac{\varphi}{6.328}\{\rho_{wi}^{n+1} \cdot S_{wi}^{n+1} - \rho_{wi}^{n}\, S_{wi}^{n}\} \end{aligned} \tag{4.17}$$

Hence, oil and water saturations are calculated for internal cells ($i = 2, \ldots, N_x$) from

$$\begin{aligned} S_{oi}^{n+1} &= (\frac{6.328\Delta t}{\varphi\,\rho_{oi}^{n+1}})\{D_{oi+1}^{n+1} \cdot P_{i+1}^{n+1} - (D_{oi}^{n+1} + D_{oi+1}^{n+1})\, P_i^{n+1} + D_{oi}^{n+1} \cdot P_i^{n+1}\} + (\frac{\rho_{oi}^{n}}{\rho_{oi}^{n+1}})\, S_{oi}^{n} \\ S_{wi}^{n+1} &= (\frac{6.328\Delta t}{\varphi\,\rho_{wi}^{n+1}})\{D_{wi+1}^{n+1} \cdot P_{i+1}^{n+1} - (D_{wi}^{n+1} + D_{wi+1}^{n+1})\, P_i^{n+1} + D_{wi}^{n+1} \cdot P_i^{n+1}\} + (\frac{\rho_{wi}^{n}}{\rho_{wi}^{n+1}})\, S_{wi}^{n} \end{aligned} \tag{4.18}$$

For the first cell ($i = 1$),

$$\begin{aligned} S_{oi}^{n+1} &= (\frac{6.328\Delta t}{\varphi\,\rho_{oi}^{n+1}})\{D_{oi+1}^{n+1}(P_{i+1}^{n+1} + P_i^{n+1}) + GPI.\Delta x.\, D_{oi}^{n+1}\} + (\frac{\rho_{oi}^{n}}{\rho_{oi}^{n+1}})\, S_{oi}^{n} \\ S_{wi}^{n+1} &= (\frac{6.328\Delta t}{\varphi\,\rho_{wi}^{n+1}})\{D_{wi+1}^{n+1}(P_{i+1}^{n+1} + P_i^{n+1}) + GPI.\Delta x.\, D_{wi}^{n+1}\} + (\frac{\rho_{wi}^{n}}{\rho_{wi}^{n+1}})\, S_{wi}^{n} \end{aligned} \tag{4.19}$$

and finally for the right boundary cell ($i = N_x$)

$$\begin{aligned} S_{oi}^{n+1} &= -(\frac{6.328\Delta t}{\varphi\,\rho_{oi}^{n+1}})\{D_{oi}^{n+1}(P_i^{n+1} - P_{i-1}^{n+1}) + GPO.\Delta x.\, D_{oi+1}^{n+1}\} + (\frac{\rho_{oi}^{n}}{\rho_{oi}^{n+1}})\, S_{oi}^{n} \\ S_{wi}^{n+1} &= -(\frac{6.328\Delta t}{\varphi\,\rho_{wi}^{n+1}})\{D_{wi}^{n+1}(P_i^{n+1} - P_{i-1}^{n+1}) + GPO.\Delta x.\, D_{wi+1}^{n+1}\} + (\frac{\rho_{wi}^{n}}{\rho_{wi}^{n+1}})\, S_{wi}^{n} \end{aligned} \tag{4.20}$$

This is the whole procedure of the IMPES method for this model.

In summary, in this model pressure and saturations are calculated from the IMPES technique, but a similar method (fully implicit method) with the first model is used to find the temperature distribution.

4.4 Summary of the Second Model

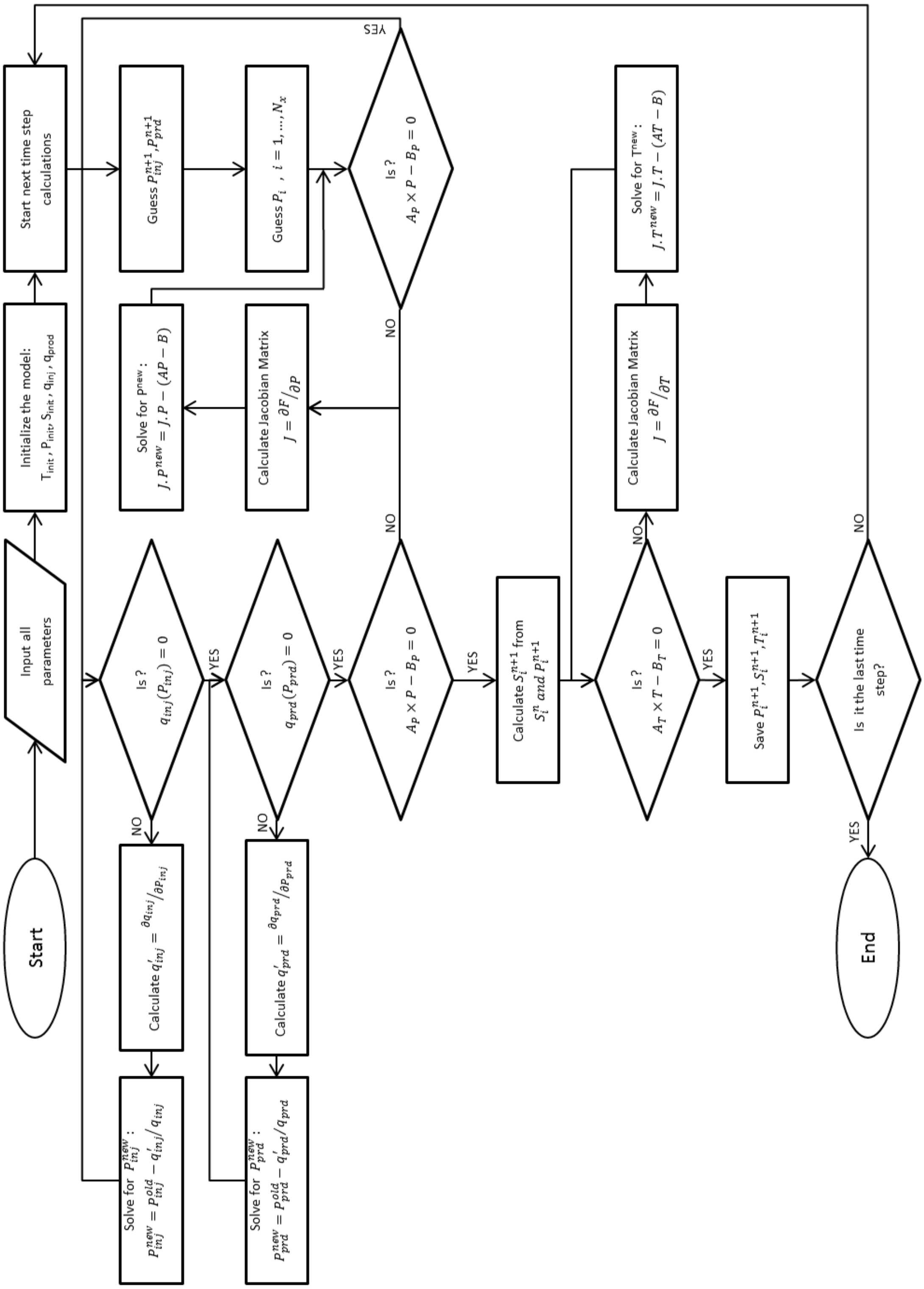

Figure 4.1: The Second Model Calculations Flow Chart

5 Results

It is interesting to see the results of the two different numerical techniques applied to a physical process and to see how choosing between these different techniques can change the results using the same inputs. Table 1 shows the values of the model parameters used in the two models.

5.1 First Model Results

In this model, the Buckley-Leverett equation and mass and energy balance equations are solved using the Lax-Wendroff scheme and a fully implicit central schemes, respectively, in order to find saturation, pressure and temperature distribution in the one dimensional hot water model. The results for pressure, saturation and temperature profiles are shown in figure 5.1 using a step size of $\triangle x = 5(ft)$ and a time step of $\triangle t = 0.025(day)$. Distributions are plotted after 100, 250 and 400 days. It can be seen from figure 5.1 (the water saturation profile) that initially there is no steep front in the system but later, due to injection a shock (water front) is created which moves to the right in time. The pressure profile changes based on the water and oil properties and shock position. Surprisingly, there are no oscillations around the discontinuity (water saturation front) although the second order accurate Lax-Wenderoff scheme is used to solve Buckley-Leverett equation. Figure 5.2 shows that with higher number of divisions the front is steeper, as expected, but still no oscillation is observed around this steep front. This behavior might be related to using Lax-Wendroff for unequal spacing and backward in space and forward in time schemes at the boundaries or front is not steep enough.

5.2 Second Model Results

The fully implicit pressure explicit saturation (IMPES) and fully implicit temperature techniques are used to find pressure, saturation and temperature profiles in this model. Results are presented in figure 5.3 by using similar step sizes and time steps to the first model. The results show that there are similar trend for outputs.

5.3 Comparing Two Models

By looking at the results of the two models it is clear that both have similar trends for P, S and T. Whereas, by looking at the results more closely, it can be seen that the position of steep front in the first model lag behind its corresponding front position in second model, which could be a result of using the incompressible fluid assumption in the Buckley-leverett equation for the saturation profile. Consequently lower pressure values result in the first model (lower injected fluid lower pressure increase). Therefore, the second model using IMPES technique for solving pressure and saturation and fully implicit method for temperature is a better numerical model. The only problem is that it has the numerical instability in the IMPES technique which comes from using an explicit saturation calculation. So, to make the model more applicable, the stability limits must be considered. In the next section some sensitivity analysis is performed to find these limits.

5.4 Sensitivity Analysis

Sensitivity analysis is performed on the second model to investigate the influence of grid and time step size. Figures 5.4 - 5.5 show the effect of grid sizes starting from 2.5 (ft) up to 10 (ft) respectively. As is clear, the larger the grid size, the faster the shock (front) reaches the right boundary. This is known as numerical dispersion and the best solution is to choose the optimum number of cells which reduces the dispersion enough and does not make our calculation very expensive. In this case, 30 cells looks to give a good result and the shock location is not very far from the case of 60 cells. More precise data about sensitivity analysis on number of grid in x direction (step sizes) is presented in Table 2. It is clear from the table information that CPU time grows exponentially as the number of division increases. So, it is essential to do the benchmarking to find the optimum number of divisions which preserves the front shape and does not have dispersion effect as well as not being very expensive in terms of calculation.

Time step size is very important for explicit calculations and its stability. As the saturation calculation is explicit in IMPES method, it is expected to see some degree of instability when the time step size grows. This effect can be seen in figures 5.6 - 5.7 where the time steps are increasing from 0.025 (Day) to 4.5 (Day). It is worth mentioning that, even in the case of dt=0.1 (day), although no oscillation / fluctuation is observed in saturation like the one in figure 5.4, the pressure values show a kind of oversaturated system with higher than normal values. So, the oscillation can not only be a sign of instability but also an over pressurized system which can show the effect of a wrong saturation calculation in the explicit scheme. Table 3 includes more information about stability of this scheme as time step size grows.

Table 1: Model Properties

Model properties	**Value (Units)**
Length (L)	150 (ft)
Width (dy)	20 (ft)
Height (dz)	30 (ft)
Distance to ambient temperature in x direction($\triangle x_{\infty}$)	100 (ft)
Distance to ambient temperature in x direction ($\triangle y_{\infty}$)	100 (ft)
Porosity (φ)	0.2
Absolute permeability (K_{abs})	3 (darcy)
Initial temperature (Tinit)	559.67 (^{o}R)
Initial pressure (Pinit)	4000 (psia)
Initial oil saturation ($S_o$init)	0.84
Total rate (q_t)	1 (bbl/day)
Reference temperature (Tref)	536.4 (^{o}R)
Ambient temperature (T_{∞})	559.67 (^{o}R)
Injection temperature (T_{inj})	800 (^{o}R)
Rock density (ρ_r)	145 (lb/ft^3)
Rock thermal conductivity (k_r)	0.9824 (Btu/ft.hr.^{o}F)
Average thermal conductivity, rock, oil and water (k_H)	0.4623 (Btu/ft.hr.^{o}F)
Oil specific gravity (Sgo)	0.9
Water specific heat capacity, constant pressure (C_{pw})	0.986 (Btu/lb.^{o}F)
Water specific heat capacity, constant volume (C_{Vw})	0.932 (Btu/lb.^{o}F)
Rock specific heat capacity, constant volume (C_{Vr})	0.22 (Btu/lb.^{o}F)

Table 2: Sensitivity Analysis on Nx (Nt=5000 , dt=0.01)

Nx	**CPU Time (sec)**	**Stability Condition**
5	108	Stable, very high numerical dispersion
10	156	Stable, high numerical dispersion
30	364	Stable, moderate numerical dispersion
60	712	Some oscillation, low numerical dispersion
150	2244	Unstable, very low numerical dispersion
300	5408	Unstable

Table 3: Sensitivity Analysis on dt (Nx=30 , Final time=50 Day)

Nx	**CPU Time (sec)**	**Stability Condition**
0.01	364	Stable
0.05	76	Stable
0.1	36	No oscillation, over-saturated
0.5	8	No oscillation, over-saturated
1	8	Some oscillation
2	8	More oscillation
5	–	Nearly singular matrix - no result

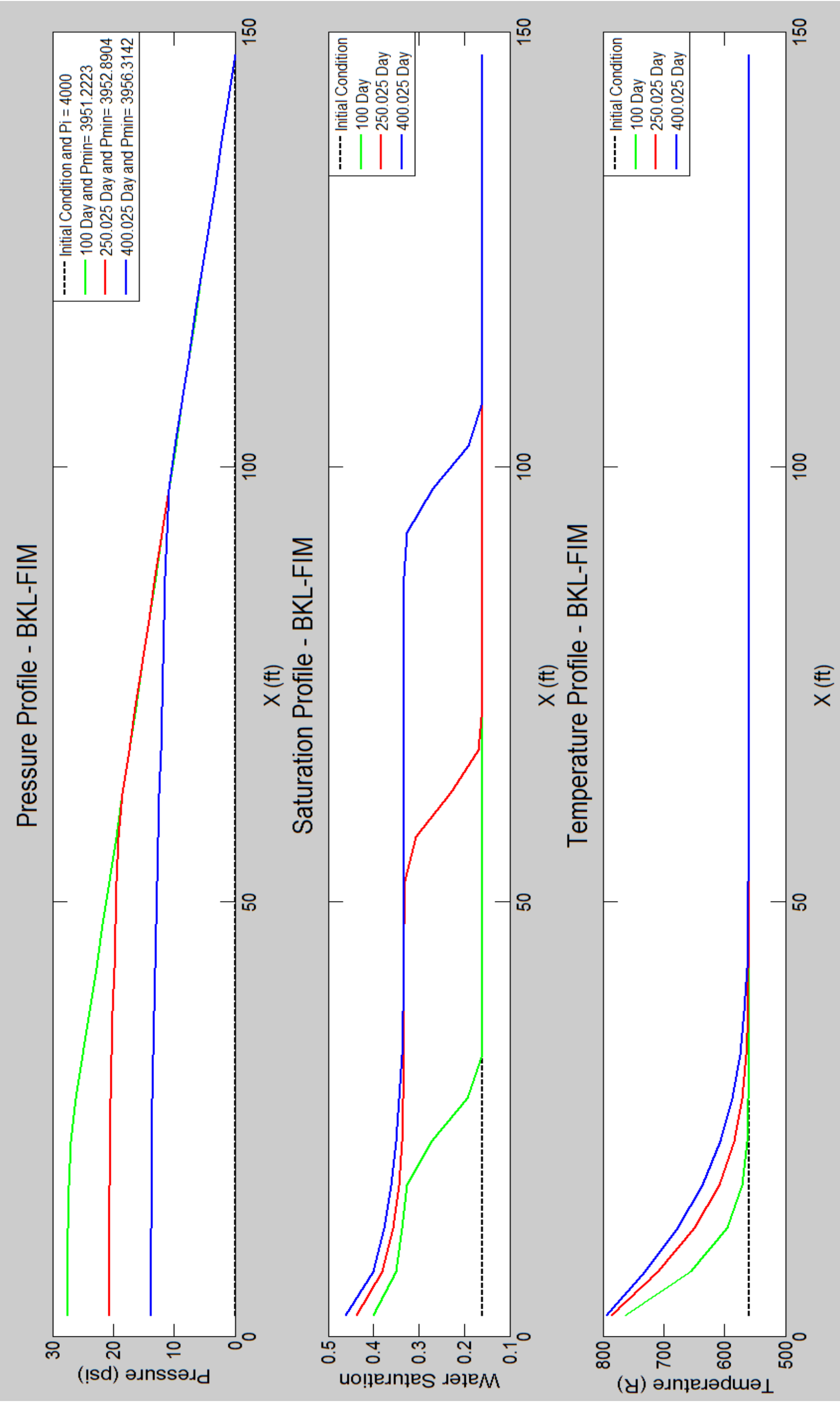

Figure 5.1: The First Model Pressure, Saturation and Temperature Results

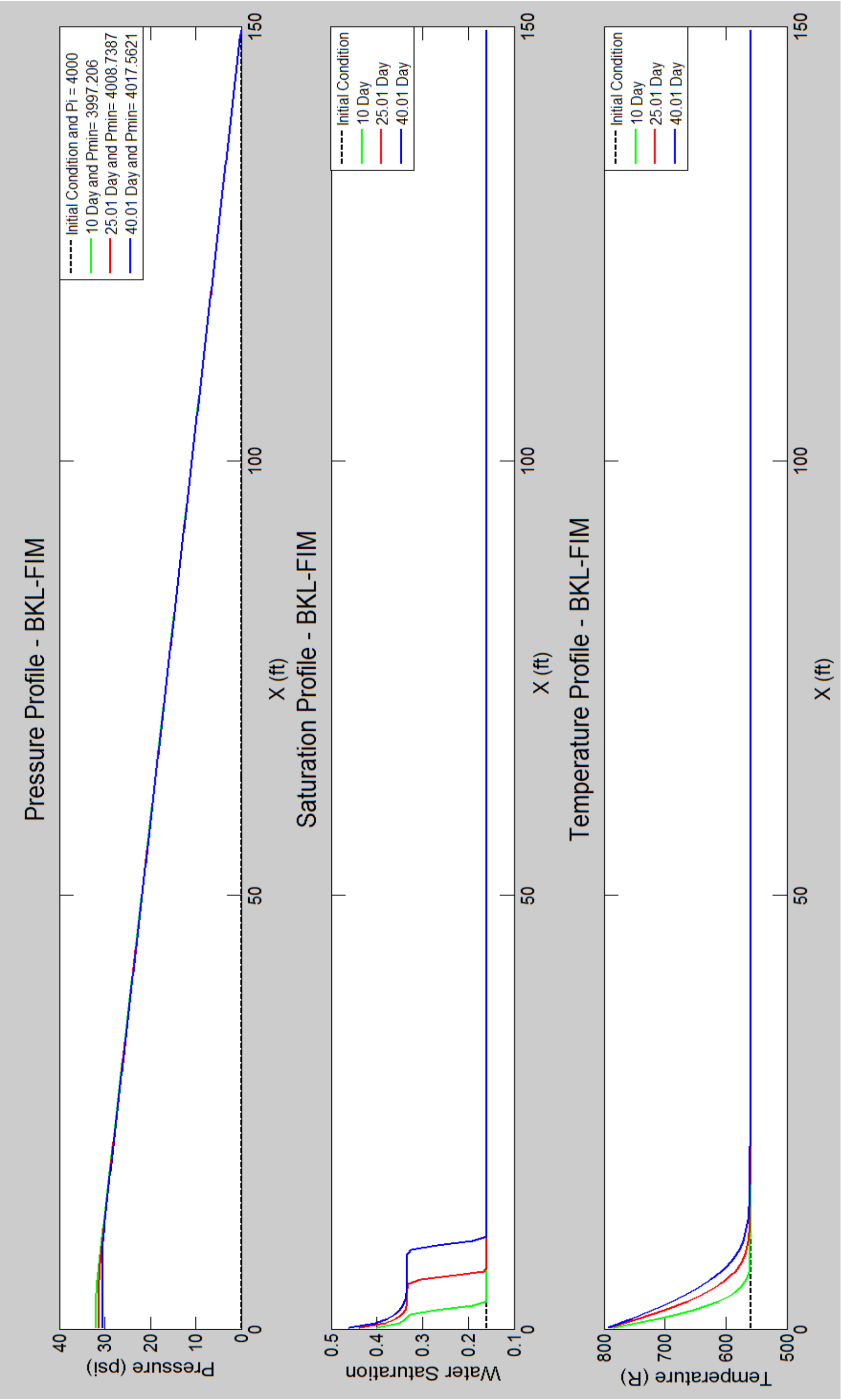

Figure 5.2: The First Model Pressure, Saturation and Temperature Results , Nx=300

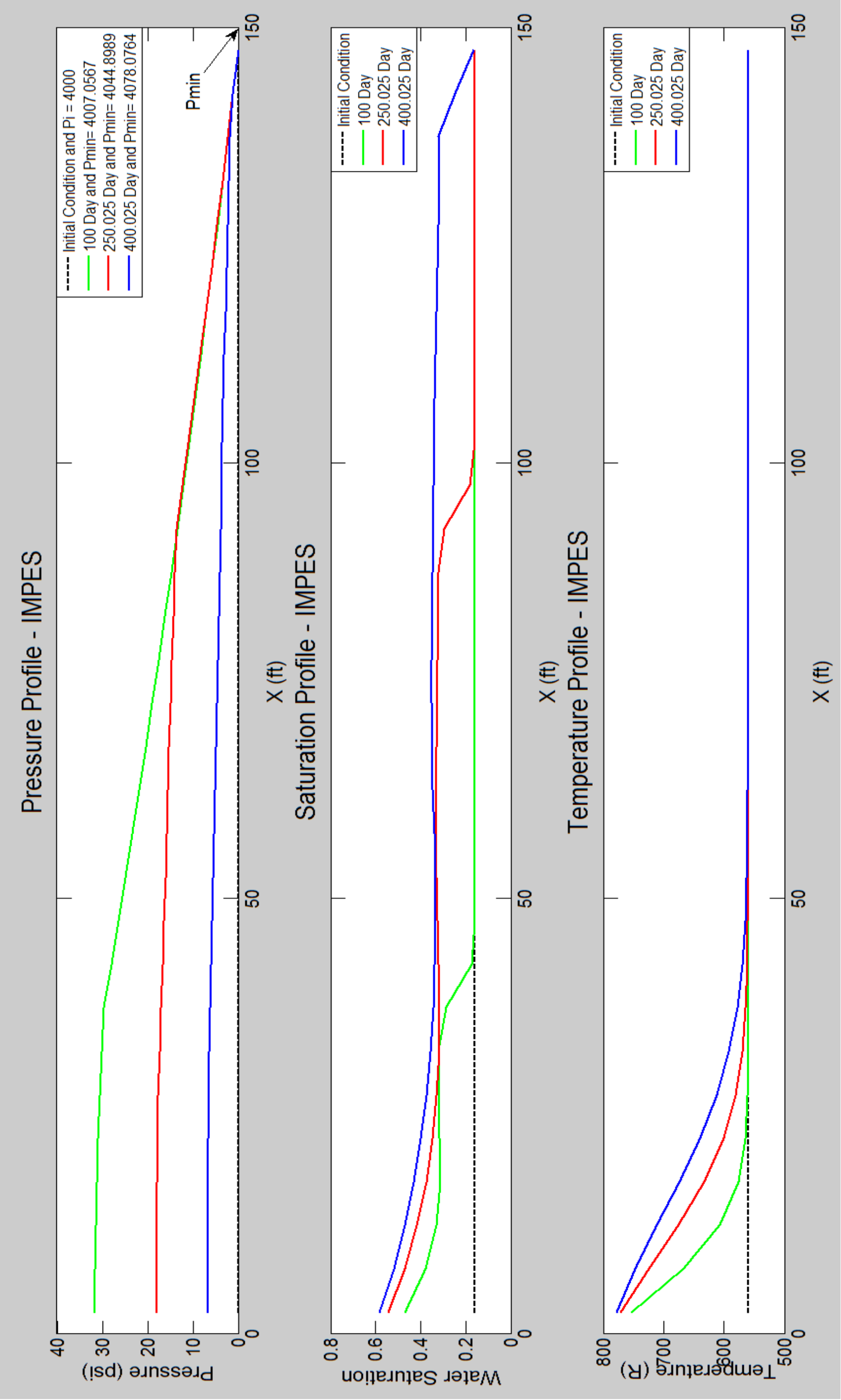

Figure 5.3: The Second Model Pressure, Saturation and Temperature Results

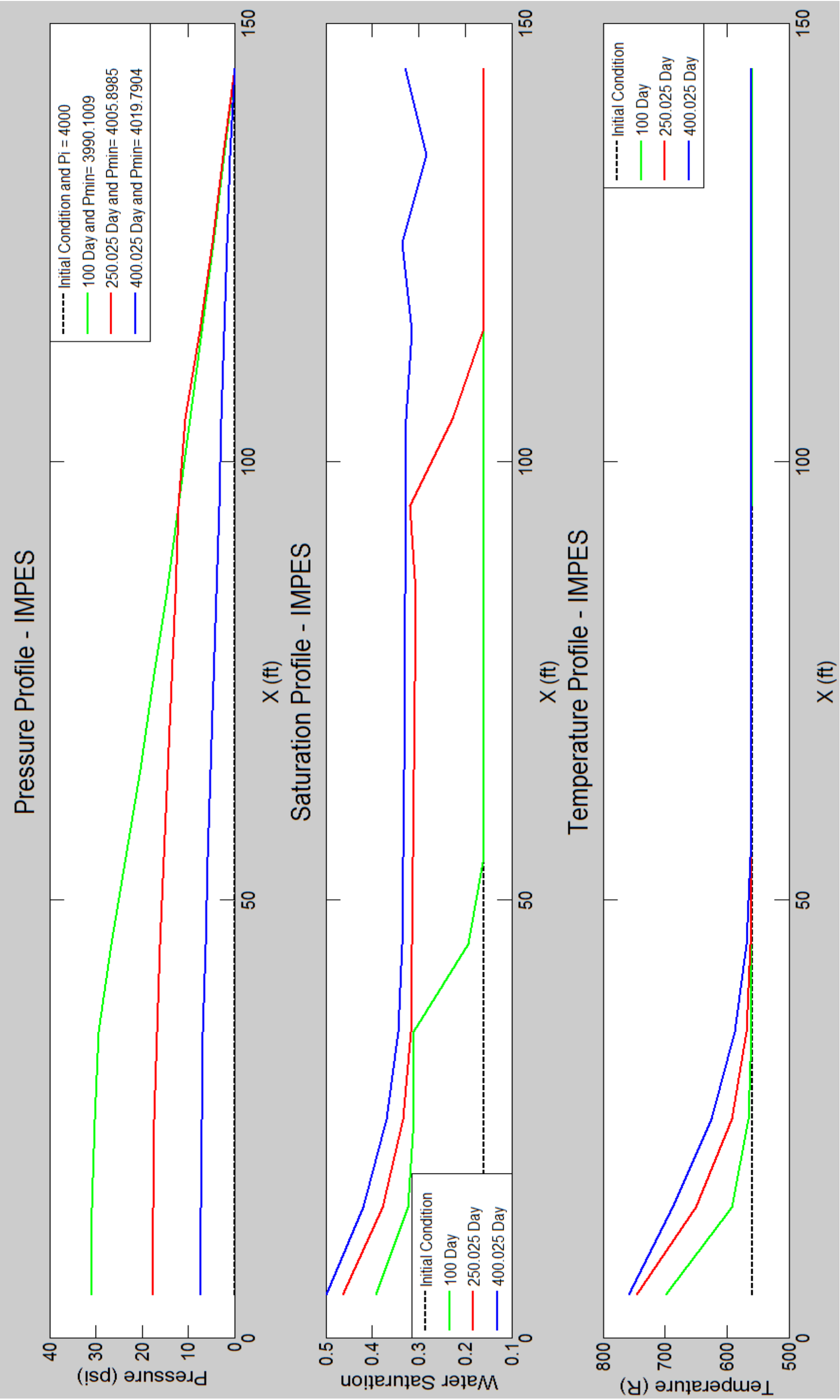

Figure 5.4: Pressure, Saturation and Temperature Profiles for the second model for Nx=15

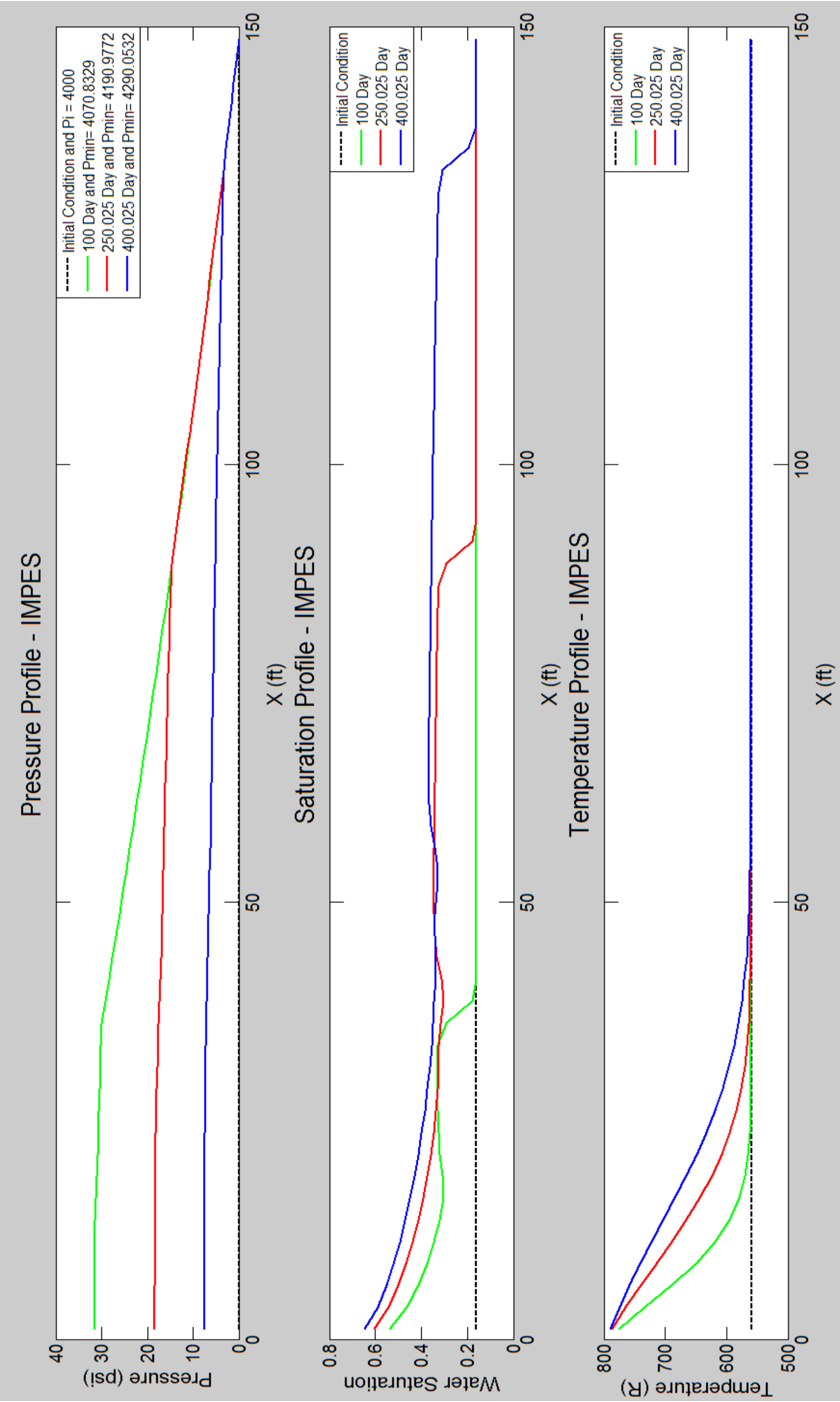

Figure 5.5: Pressure, Saturation and Temperature Profiles for the second model for Nx=60

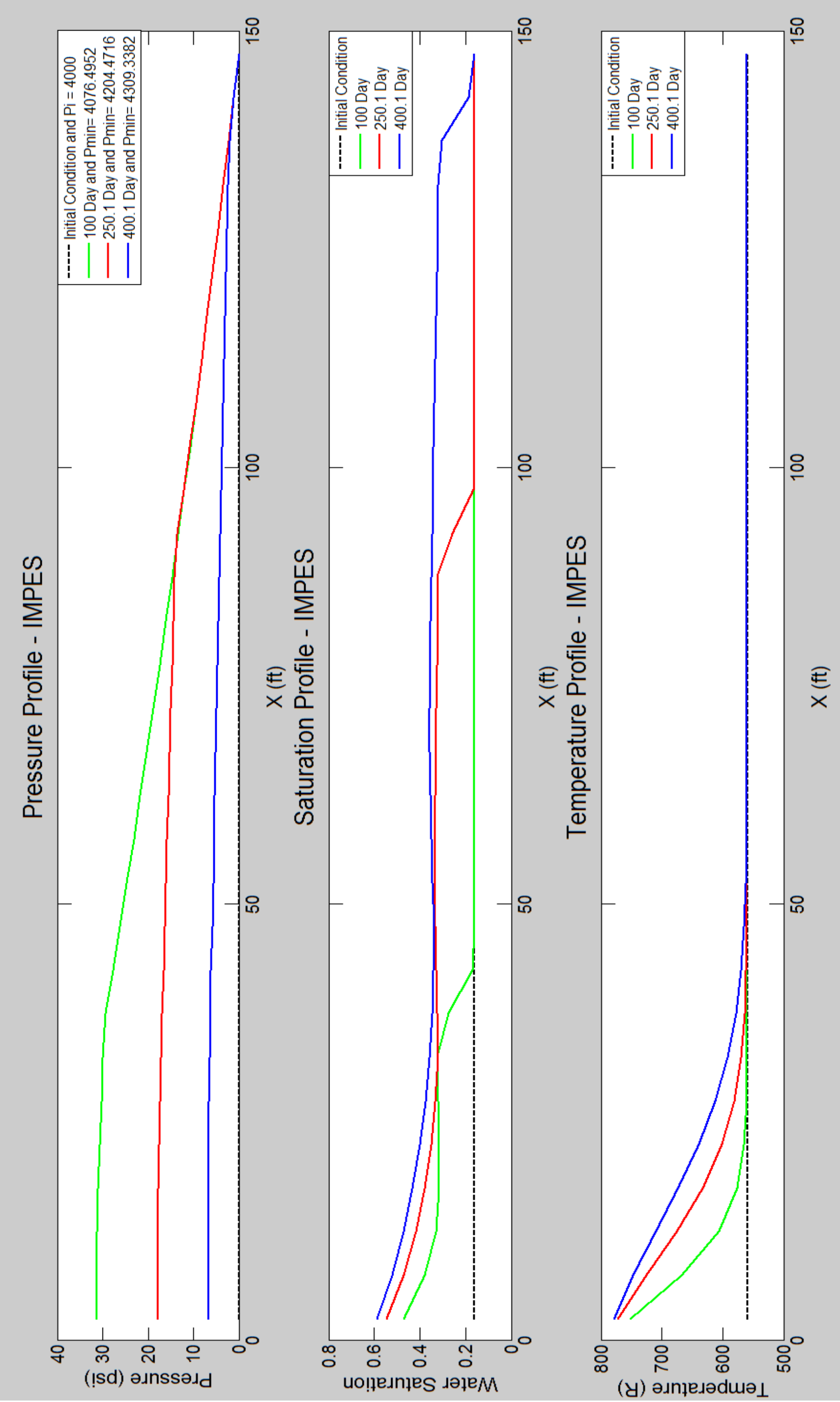

Figure 5.6: The Second Model Pressure, Saturation and Temperature Profiles (dt=0.1 Day)

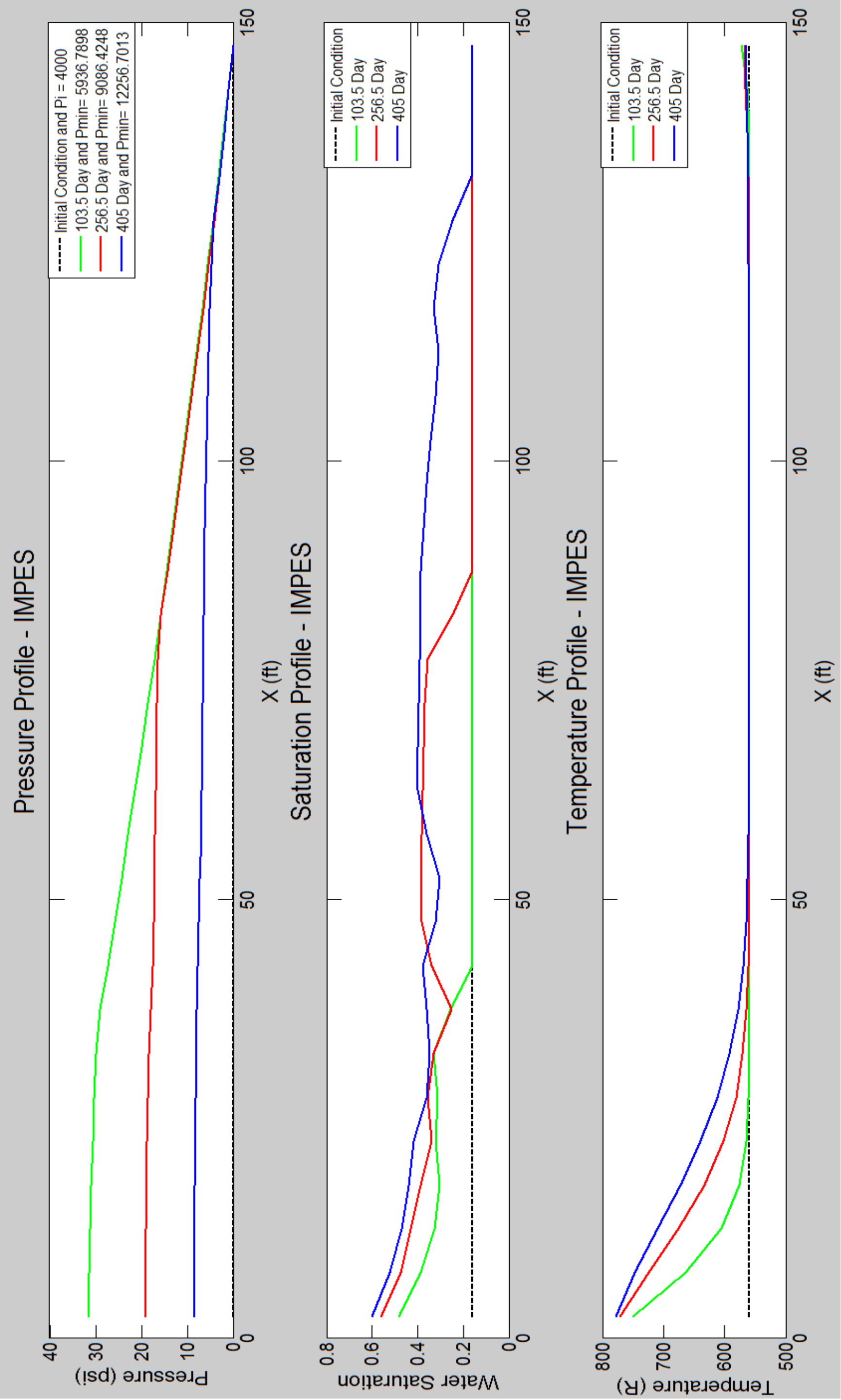

Figure 5.7: The Second Model Pressure, Saturation and Temperature Profiles (dt=4.5 Day)

6 Conclusion

In this dissertation two numerical models have been applied to a method of oil recovery, hot water injection. This physical process can be described using oil and water mass conservation equations, energy balance and Buckley-Leverett equations. Hot water injection is modeled by using these equations in order to find pressure, saturation and temperature profiles.

In first model Lax-Wendroff and fully implicit scheme have been used to solve Buckley-Leverett, oil mass and energy balance equations respectively. The model may be less reliable since one of the basic assumptions of Buckley-Leverett equation is considering fluids as incompressible which is not a reasonable assumption in this study. It is worth noticing that Lax-Wendroff method is a second order accurate scheme, and it is a well-known fact that second order accurate numerical schemes produce oscillations at discontinuities, but there are no oscillations in the saturation profile of this study and the results around shock are smooth. The reason may be related to, using backward in space and forward in time scheme at right boundary and Lax-Wendroff based on unequal step size at left boundary.

Implicit pressure explicit saturation (IMPES) and fully implicit energy equation have been implemented to obtain the distributions in the second model. This technique seems to be more reliable as it has been driven by considering fluid compressibility which is an effective factor in thermal processes. However, the explicit saturation calculations make this method sensitive to stability so, the stability of this model is under question and has to be treated carefully to get correct results. Sometimes, it seems there is no visual instability in water saturation profile but saturation can easily be overestimated resulting in over-pressurization of the system.

6.1 Future work

- The model can be easily extended to a two- or three-dimension. There would be similar trend but calculations would be more expensive.
- Other phenomena such as gravity, capillary pressure, and temperature dependence of relative permeability can be included in the models which cause higher degree of non-linearity.
- Other numerical methods can be applied to the set of equations and compare their

results with the result of this study. For instance, oil and water mass balance equations and energy balance equation can be modeled using fully implicit method for all of them and solve them simultaneously which is a well-known method but, it is really expensive technique in calculation.

- Using different boundary conditions (well models) to investigate their effect on the saturation, pressure and temperature distributions in the system.
- In first model, other numerical methods like Warming-Beam, Fromm, etc., can be used to find the saturation in Buckley-Leverett equation.

Nomenclature

P	Pressure (psia)
V	Volume (cu ft)
ρ	Density (lbm/cu ft)
u	Velocity (ft/days)
g	Gravitational acceleration
d	Reservoir depth (ft)
x	Distance along the x direction (ft)
$\triangle x$	Grid size in x-direction (ft)
$\triangle t$	Grid size in temporal direction (days)
K	Permeability (darcy)
K_r	Relative permeability (dimensionless)
μ	Viscosity (cp)
φ	Porosity (fraction)
S	Saturation (fraction)
API^o	Oil gravity measurement
q	Flow rate (bbl/days)
A	Area (sq ft)
f	Fractional fluid flow (dimensionless)
t	Time (days)
m^o	Mass flux (lbm/days)
M	Mass (lbm)
E	Total rock and fluid energy (Btu/cu ft)
e^o	Energy flux (Btu/days)
U	Internal energy (Btu/lbm)
H	Enthalpy (Btu/lbm)
Q	Amount of heat (Btu/lbm)
T	Temperature (^{0}R)
C_p	Heat capacity at constant pressure (Btu/lbm. ^{0}F)
C_V	Heat capacity at constant volume (Btu/lbm. ^{0}F)
k	Thermal conductivity (Btu/hr. ft. ^{0}F)
r_w	Well radius (ft)
h	Reservoir thickness (ft)
X	Dimensionless distance
L	Distance between injection and production wells (ft)

Subscripts and Superscripts

α	Phase index ($o = oil,\ w = water$)
w	Water phase
o	Oil phase
b	Boundary
c	Capillary pressure
abs	Absolute
p	Pore or Pressure
r	Relative or Rock
t	Total
$prod$	Production
inj	Injection
gen	Generation
$cons$	Consumption
V	Volume
or	Residual oil
wc	Connate water
av	Average
ref	Reference

References

[1] Tarek Ahmed, *Reservoir Engineering Handbook* , Second Edition, Gulf Professional Publishing, 2001, pp.184-228, 281-286.

[2] R. E. Ewing, *The Mathematics of Reservoir Simulation* , SIAM, Philadelphia, (1983)

[3] A. T.Corey,*The Interrelation Between Gas and Oil Relative Permeabilities* , Producers Monthly, 1954, pp.38-41

[4] *www.most.gov.mm/techuni/media/PE_05045_2.pdf.*

[5] M.Y. Soliman, W.E. Brigham, Raj Raghavan, *Numerical Simulation of Thermal Recovery Processes* , SPE 9942, 1981.

[6] Professor Jon Kleppe, *TPG4150 Reservoir Recovery Techniques* , Handout notes 4 : Buckley-Leverett Analysis, Norwegian University of Science and Technology, Department of Petroleum Engineering and Applied Geology, Sep. 2012.

[7] B.C.Craft, M.Hawkins, *Applied Petroleum Reservoir Engineering* , Second Edition, Prentice Hall PTR, 1991.

[8] Chen Z.,Huan G.,Ma Y, *Computational Methods for Multiphase Flow in Porous Media* , SIAM Publications, 2006, pp.10-14, 265-267, 382-384.

[9] P.K Sweby,*Numerical Solution of Conservation Laws* Lecture notes, University of Reading (2002).

[10] K.W. Morton and D.F. Mayers,*Numerical Solution of Partial Differential Equations* , Cambridge university press, (1994).

[11] H.G. Weinstein, J.A. Wheeler, E.G. Woods, *Numerical Model for Thermal Processes* , SPE 4759, 1974.

[12] Prats, Michael, *Thermal Recovery* , New York, (1986).

[13] C. T. Kelley, *Solving Nonlinear Equations with Newton's Method*, No. 1 in Fundamentals of Algorithms, SIAM, 2003.