On Approximation in Meshless Methods

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Abstract. We analyze the approximation properties of some meshless methods. Three types of functions systems are discussed: systems of functions that reproduce polynomials, a class of radial basis functions, and functions that are adapted to a differential operator. Additionally, we survey techniques for the enforcement of essential boundary conditions in meshless methods.

1 Introduction

The classical finite element method (FEM) is a well-established tool for numerically solving partial differential equations. New, non-standard methods, that are broadly covered by the term *meshless methods* or *meshfree methods* have recently emerged. A few examples frequently mentioned in this context are the diffuse element method, [87], the *element-free Galerkin* (EFG, [13,14,11]), the X-FEM (extended FEM), [84,29,98], the RKPM (reproducing kernel particle method, [72–75,70]), the generalized FEM/partition of unity method ([7,78,79,82,9]), the *hp*-cloud method, [89], the particle partition of unity particle method of [47–51,96], the finite point method [91], and the method of finite spheres [30]; also the use of radial basis functions, [65,66,108,44,61,110] and the older generalized finite difference method of [71] fall into this category. This list is by no means exhaustive, and surveys of such methods include [12,6,60]. Two of the reasons given for introducing such methods are:

- The cost of creating good quality meshes can be high. This is particularly true for three-dimensional problems and for problems where the standard FEM requires frequent remeshing such as time-dependent problems and crack propagation problems.
- For some non-standard problems, the standard FEM performs poorly. Here, it is attractive to create custom-tailored methods designed for a particular problem at hand.

A main aim of these notes is to illustrate some of the mechanisms of approximation that underlie meshless methods. In view of the multitude of methods and applications it is impossible to be exhaustive, and a selection had to be made concerning the approximation spaces and the type of approximation results. With respect to the approximation spaces, we have selected three types: an example of function systems that reproduce polynomials, a class of radial basis functions, and some examples of systems that are tailored to a particular differential operator. The type of approximation results that we obtain are mostly formulated with a view to an application in projection methods for second order elliptic problems. Since the natural setting of such problems is that of the Hilbert space H^1 (or subspaces thereof), most approximation results are formulated in this norm.

1.1 Notation

General Notation We write $\mathbb{N} = \{1, 2, \ldots, \}$ for the positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ represents the non-negative integers. $\mathbb{R}^+$ stands for the positive real numbers, $\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$ for the non-negative real numbers. We will denote by $\mathcal{P}_p$ the space of polynomials of degree p in d variables, i.e., $\mathcal{P}_p = \operatorname{span}\{\prod_{i=1}^d x_i^{\alpha_i} \,|\, \alpha_i \in \mathbb{N}_0 \text{ with } \sum_{i=1}^d \alpha_i \le p\}$. The Euclidean norm on $\mathbb{R}^d$ will be denoted by $\|\cdot\|_2$. Balls of radius r centered at x_0 are denoted by $B_r(x_0)$.

Spaces and Domains For domains $\Omega \subset \mathbb{R}^d$, integers $k \in \mathbb{N}_0$ and $q \in [1, \infty]$ the Sobolev spaces $W^{k,q}(\Omega)$ are defined in the usual way (see, e.g., [23, Chap. 1]). Also for values of $k \notin \mathbb{N}_0$ and $q \in [1, \infty)$, the Sobolev spaces $W^{k,q}(\Omega)$ are defined in the usual way, [23]; they can be equipped with the so-called Sobolev-Slobodeckij norm as follows: we write $k = \tilde{k} + \kappa$, where $\tilde{k} \in \mathbb{N}_0$ and $\kappa \in (0, 1)$, and we define

$$\|u\|^q_{W^{k,q}(\Omega)} = \|u\|^q_{W^{\tilde{k},q}(\Omega)} + |u|^q_{W^{k,q}(\Omega)},$$

where the semi-norm $|\cdot|_{W^{k,q}(\Omega)}$ is given by

$$|u|^q_{W^{k,q}(\Omega)} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = \tilde{k}}} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^q}{\|x - y\|_2^{d + q\kappa}} \, dx \, dy. \tag{1.1}$$

We remark in passing that an equivalent definition of the fractional order Sobolev spaces $W^{k,q}(\Omega)$ based on the interpolation of spaces using the K-method is possible, [15,104]. The case $q = 2$ is special in that the spaces $W^{k,2}(\Omega)$ are Hilbert spaces; it is customary to write $H^k(\Omega) = W^{k,2}(\Omega)$.
We denote by $H^1_0(\Omega) = \{u \in H^1(\Omega) \,|\, u|_{\partial\Omega} = 0\}$ the space of functions of $H^1(\Omega)$ that vanish on the boundary of Ω.
For $\xi \in \mathbb{R}^d$ with $\|\xi\|_2 = 1$, $x \in \mathbb{R}^d$, $r > 0$, and $\theta \in (0, \pi)$ we define the cone

$$C(x, \xi, \theta, r) := B_r(x) \cap \{y \in \mathbb{R}^d \,|\, (y - x)^\top \xi > \|y - x\| \cos\theta\}. \tag{1.2}$$

A domain Ω is said to satisfy a cone condition with angle θ and radius r if for each $x \in \Omega$ there exists a $\xi \in \mathbb{R}^d$ with $\|\xi\|_2 = 1$ such that $C(x, \xi, \theta, r) \subset \Omega$.

Notation for Particle Methods In these notes, the approximation spaces V_N will have the form

$$V_N = \operatorname{span}\{\varphi_i \,|\, i = 1, \dots, N\};$$

as is customary in FEM, the functions φ_i, $i = 1, \dots, N$, will be called shape functions. We furthermore introduce the *patches* Ω_i, which are the interior of the supports of the shape functions, and the diameters h_i of the patches by

$$\Omega_i := (\operatorname{supp} \varphi_i)^\circ, \qquad h_i := \operatorname{diam} \Omega_i \leq 1.$$

Remark 1.1. The assumption $h_i \leq 1$ is made for convenience only and could be replaced by boundedness of the patch diameters. ∎

Frequently, a shape function φ_i will be associated with a *particle* $x_i \in \Omega_i$. The particles are collected in the set

$$X_N := \{x_i \,|\, i = 1, \dots, N\},$$

which throughout these notes will be assumed to consist of N distinct points $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$. In the parlance of classical FEM the "connectivity" of the shape functions will be important. We therefore define

$$n(x) := \{i \in \mathbb{N} \,|\, x \in \Omega_i\}, \tag{1.3}$$

$$n(i) := \{j \in \mathbb{N} \,|\, \Omega_j \cap \Omega_i \neq \emptyset\}; \tag{1.4}$$

the notation $n(\cdot)$ is reminiscent of "neighbor."

FEM and Projection Methods Techniques and terminology of the classical FEM will pervade much of these notes, and we refer to [27,23,94] for general reference on the topic. We will, for example, employ the notion of shape-regular affine triangulations $\mathcal{T}$ of a domain Ω. Based on such a triangulation of Ω, one can define the space $S^{p,1}(\mathcal{T}) \subset H^1(\Omega)$ of piecewise polynomials of degree p. We refer to [94] for a precise definition of $S^{p,1}(\mathcal{T})$. We will write $S_0^{p,1}(\mathcal{T})$ for the space $S_0^{p,1}(\mathcal{T}) := S^{p,1}(\mathcal{T}) \cap H_0^1(\Omega)$.
Many of the results of the presentation are obtained with a view to an application in projection methods such as the Galerkin method. An example of such as setting is the following: Let X be a Hilbert space, $a : X \times X \to \mathbb{R}$ be a continuous bilinear form, $l \in X'$ be a continuous linear form, and $u \in X$ solve

$$a(u, v) = l(v) \qquad \forall v \in X. \tag{1.5}$$

If $V_N \subset X$ is a subspace, then one can define an approximation $u_N \in V_N$ by:

$$\text{Find } u_N \in V_N \text{ such that} \quad a(u_N, v) = l(v) \qquad \forall v \in V_N. \tag{1.6}$$

Once a basis of V_N is chosen, the problem (1.6) represents a linear system of equations that has to be solved. Under suitable assumptions on the bilinear form a, one has existence and uniqueness of u_N together with a quasi-optimality result, i.e.,

$$\|u - u_N\|_X \leq C \inf_{v \in V_N} \|u - v\|_X, \tag{1.7}$$

where the constant $C > 0$ is independent of critical parameters (e.g., N). In this situation it is very important to understand the approximation properties of the space V_N employed so as to be able to be give bounds on the infimum in (1.7).

1.2 The notion of optimality

When discussing the approximation properties of a space V_N, it is instructive to have a notion of optimality so as to be able to compare this space V_N with the best possible choice. One notion of optimality that is common in approximation theory is that of n-width (see, e.g., [92]): For a normed space X with norm $\|\cdot\|_X$ and a subset $Y \subset X$ one defines for $n \in \mathbb{N}$

$$d_n := \inf_{\substack{E_n \subset X \\ \dim E_n \leq n}} \sup_{u \in Y} \inf_{v \in E_n} \|u - v\|_X;$$

here, the spaces E_n appearing in the first infimum are arbitrary linear subspaces of dimension n. The quantity d_n thus measures how well functions of the set Y can be approximated from linear spaces E_n of dimension n. Clearly, d_n depends on the error measure $\|\cdot\|_X$ and the set Y. For Sobolev spaces we have [62]:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $k \geq 1$. Then there exists $C > 0$ such that*

$$\inf_{\substack{V_N \subset H^1(\Omega) \\ \dim V_N \leq N}} \sup_{\substack{u \in H^k(\Omega) \\ \|u\|_{H^k(\Omega)} = 1}} \inf_{v \in V_N} \|u - v\|_{L^2(\Omega)} \geq N^{-(k-1)/d}.$$

The converse of Theorem 1.2 is well-known in classical FEM (see, e.g., [23]):

Theorem 1.3. *Let $\mathcal{T}$ be a quasi-uniform triangulation of a domain $\Omega \subset \mathbb{R}^d$ with maximum element size h. Then for $k \geq 1$ and the classical H^1-conforming space $S^{p,1}(\mathcal{T})$ of piecewise polynomials of degree p we have*

$$\inf_{v \in S^{p,1}(\mathcal{T})} \|u - v\|_{H^1(\Omega)} \leq C N^{-(\min\{p+1,k\}-1)/d} \|u\|_{H^k(\Omega)},$$

where $N = \dim S^{p,1}(\mathcal{T}) \sim h^{-d}$.

Theorems 1.2, 1.3 show that the classical FEM attains already the best possible rate of convergence if the only information available about the function to be approximated is membership in some Sobolev space $H^k(\Omega)$. In this setting, the use of approximation spaces V_N different from the classical FEM spaces is mainly justified by algorithmic considerations.

Remark 1.4. The approximation results of these notes are obtained with a view to an application in classical projection methods such as the Galerkin scheme (1.6). We will not cover non-linear approximation techniques, for which we refer to [32]. ▪

2 Polynomial Reproducing Systems

The fist class of approximation spaces V_N that we analyze is one where the space V_N reproduces polynomials of degree p. We will see that the approximation properties of such spaces are very similar to the classical FEM spaces. Such spaces can be constructed in different ways. One possibility is based on the moving least squares technique and will be illustrated in Section 2.3.

2.1 Motivation

Let $\Omega \subset \mathbb{R}^d$ be a domain, let $X_N = \{x_i \,|\, i = 1, \ldots, N\}$ be a set of particles, and let $V_N = \operatorname{span}\{\varphi_i \,|\, i = 1, \ldots, N\}$ be a space of functions defined on Ω. In this chapter, we will make the following assumptions:

Assumption 2.1 (finite overlap). There exists a constant $M \in \mathbb{N}$ such that for every $x \in \Omega$ the cardinality $n(x)$ of the set $n(x)$ satisfies $1 \le \operatorname{card} n(x) \le M$.

Assumption 2.2 (polynomial reproduction property). $\sum_{i=1}^{N} \pi(x_i)\varphi_i(x) = \pi(x)$ for all $x \in \Omega$ and all $\pi \in \mathcal{P}_p$.

Assumption 2.3 (stability). There exist $C_{stab} \ge 1$, $r_{stab} \in \mathbb{N}_0$ such that $\|D^\alpha \varphi_i\|_{L^\infty(\Omega)} \le C_{stab} h_i^{-|\alpha|}$ for all $i \in \{1, \ldots, N\}$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le r_{stab}$.

Assumption 2.4 (local comparability of patches). There exists $C_{comp} > 0$ such that $C_{comp}^{-1} h_i \le h_j \le C_{comp} h_i$ for all $i \in \{1, \ldots, N\}$ and $j \in n(i)$.

These assumptions are a generalization of certain properties of the classical FEM. For $p = 1$ and shape-regular affine meshes $\mathcal{T}$, the classical piecewise linear FEM shape functions satisfy the above assumptions. For $p > 1$, the shape functions employed in the FEM are not as standardized; nevertheless, a basis of $S^{p,1}(\mathcal{T})$ satisfying Assumptions 2.1–2.4 can be constructed as the following exercise shows.

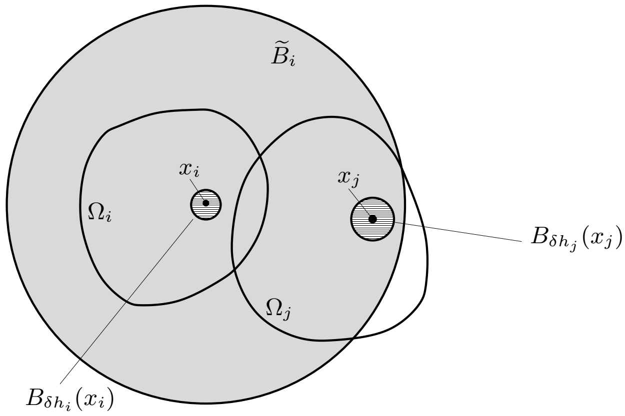

Fig. 2.1. Notation of Theorem 2.6.

Exercise 2.5. Let $\mathcal{T}$ be a mesh on $\Omega = (0,1)$ determined by the points $0 = x_0 < x_1 < \cdots < x_n = 1$. Assume that the element sizes are locally comparable, i.e., $C^{-1} \leq \frac{x_{i+1}-x_i}{x_i-x_{i-1}} \leq C$ for $i = 1, \dots, n-1$. Construct a basis of $S^{p,1}(\mathcal{T}) = \{u \in C([0,1]) \,|\, u|_{(x_i,x_{i+1})} \in \mathcal{P}_p$ for $i = 0, \dots, n-1\}$ such that Assumptions 2.1–2.4 are satisfied. ∎

The construction of shape functions φ_i that satisfy Assumptions 2.1–2.4 will be the topic of Section 2.3.

2.2 Approximation properties of systems reproducing polynomials

Spaces V_N that satisfy Assumptions 2.1–2.4 inherit the local approximation properties of polynomials:

Theorem 2.6. *Suppose Assumptions 2.1–2.4 hold. Let* δ, $C > 0$ *be given. Choose for each* x_i *a ball* $\widetilde{B}_i$ *with radius* $r_i \leq C h_i$ *such that* $B_{\delta h_j}(x_j) \subset \widetilde{B}_i$ *for all* $j \in n(i)$ *and* $\widetilde{B}_i \subset \Omega_i$ *(see Fig. 2.1).*
Then there exists a linear operator $Q_N : L^1(\mathbb{R}^d) \to V_N$ *with the following approximation property: For* $u \in H^k(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, *with* $\sum_{i=1}^N \|u\|^2_{H^k(\widetilde{B}_i)} < \infty$ *we have for* $s = 0, \dots, \min\{k, r_{stab}\}$

$$\|u - Q_N u\|^2_{H^s(\Omega)} \leq C \sum_{i=1}^N h_i^{2(\min\{p+1,k\}-s)} \|u\|^2_{H^k(\widetilde{B}_i)}.$$

Remark 2.7. Theorem 2.6 could be generalized to approximation in the space $W^{k,q}(\Omega)$. Additionally, the proof shows that the balls $\widetilde{B}_i$ could be replaced with other set, e.g., squares, rectangles.
Inspection of the proof also shows that it is sufficient to have u defined on $\cup_{i=1}^N \widetilde{B}_i$ instead of $\mathbb{R}^d$. ∎

Proof of Theorem 2.6. We abbreviate $\mu := \min\{k, p+1\}$ and denote by χ_j the characteristic function of the patch Ω_j, i.e., $\chi_j(x) = 1$ if $x \in \Omega_j$ and $\chi_j(x) = 0$ if $x \notin \Omega_j$. We note that Assumption 2.1 gives

$$1 \leq \sum_{j=1}^{N} \chi_j(x) \leq M \qquad \forall x \in \Omega. \tag{2.1}$$

For each patch Ω_i we choose with the aid of the polynomial approximation result Theorem B.1 (and, for the case $\min\{k, p+1\} < \min\{k, r_{stab}\}$ the inverse estimate Theorem B.3 together with the assumption $h_i \leq 1$) a polynomial $\pi_i \in \mathcal{P}_p$ such that

$$\|u - \pi_i\|_{H^s(\widetilde{B}_i)} \leq C r_i^{\mu - s} \|u\|_{H^k(\widetilde{B}_i)}, \qquad s = 0, \ldots, \min\{k, r_{stab}\}. \tag{2.2}$$

We then define the desired approximation $Q_N u$ by

$$Q_N u := \sum_{i=1}^{N} \pi_i(x_i) \varphi_i. \tag{2.3}$$

Note that the map $u \mapsto Q_N u$ is linear since the maps $u|_{\widetilde{B}_i} \mapsto \pi_i$, whose existence is ascertained in Theorem B.1, is linear. By Assumption 2.2 we have for each $i \in \{1, \ldots, N\}$

$$\pi_i(x) = \sum_{j=1}^{N} \pi_i(x_j) \varphi_j(x) \qquad \forall x \in \Omega. \tag{2.4}$$

For each $i \in \{1, \ldots, N\}$ we can write

$$\begin{aligned} u - Q_N u &= u - \sum_{j=1}^{N} \pi_j(x_j) \varphi_j \\ &= (u - \pi_i) + \sum_{j=1}^{N} \left[\pi_i(x_j) - \pi_j(x_j)\right] \varphi_j =: T_{1,i} + T_{2,i}. \end{aligned}$$

Since the patches Ω_i, $i = 1, \ldots, N$, cover Ω by Assumption 2.1, we get for each $s = 0, \ldots, \min\{k, r_{stab}\}$

$$\begin{aligned} \|u - Q_N u\|^2_{H^s(\Omega)} &\leq \sum_{i=1}^{N} \|u - Q_N u\|^2_{H^s(\Omega_i \cap \Omega)} \\ &\leq 2 \sum_{i=1}^{N} \|T_{1,i}\|^2_{H^s(\Omega_i \cap \Omega)} + \|T_{2,i}\|^2_{H^s(\Omega_i \cap \Omega)}. \end{aligned}$$

Using (2.2) we can estimate $\|T_{1,i}\|_{H^s(\Omega_i \cap \Omega)}$ by

$$\|T_{1,i}\|_{H^s(\Omega_i \cap \Omega)} \leq C h_i^{\mu - s} \|u\|_{H^k(\widetilde{B}_i)} \quad s = 0, \ldots, \min\{k, r_{stab}\}. \tag{2.5}$$

Hence, $\sum_{i=1}^N \|T_{1,i}\|^2_{H^s(\Omega_i\cap\Omega)}$ can be estimated in the desired fashion. For the term involving the functions $T_{2,i}$, we use Assumptions 2.3 to get for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = s \in \{0,\dots,\min\{k, r_{stab}\}\}$

$$|D^\alpha T_{2,i}(x)| \le C \sum_{j=1}^N |\pi_i(x_j) - \pi_j(x_j)| h_j^{-s} \chi_j(x).$$

Thus, we get for the H^s-semi norm of $T_{2,i}$ on $\Omega_i \cap \Omega$:

$$\begin{aligned}
|T_{2,i}|^2_{H^s(\Omega)} &\le C \int_{\Omega_i\cap\Omega} \left| \sum_{j=1}^N |\pi_i(x_j) - \pi_j(x_j)| h_j^{-s}\chi_j \right|^2 \\
&\le CM \int_{\Omega\cap\Omega_i} \sum_{j=1}^N |\pi_i(x_j) - \pi_j(x_j)|^2 h_j^{-2s}\chi_j \\
&\le CM \int_{\Omega} \sum_{j\in n(i)} |\pi_i(x_j) - \pi_j(x_j)|^2 h_j^{-2s}\chi_j\chi_i, \qquad (2.6)
\end{aligned}$$

where we exploited (2.1) in the second bound and, in the last bound, we used the observation that $\chi_j(x)\chi_i(x) \neq 0$ can only happen if $j \in n(i)$. For $j \in n(i)$ we bound $|\pi_i(x_j) - \pi_j(x_j)| \le \|\pi_i - \pi_j\|_{L^\infty(B_{\delta h_j}(x_j))}$, note that $\pi_i - \pi_j \in \mathcal{P}_p$, and use the polynomial inverse estimate Theorem B.3 to get

$$\begin{aligned}
&\|\pi_i - \pi_j\|_{L^\infty(B_{\delta h_j}(x_j))} \le C h_j^{-d/2} \|\pi_i - \pi_j\|_{L^2(B_{\delta h_j}(x_j))} \\
&\le C h_j^{-d/2} \left[\|u - \pi_i\|_{L^2(B_{\delta h_j}(x_j))} + \|u - \pi_j\|_{L^2(B_{\delta h_j}(x_j))} \right].
\end{aligned}$$

Using $B_{\delta h_j}(x_j) \subset \widetilde{B}_j \cap \widetilde{B}_i$, we then get from (2.2) and Assumption 2.4

$$\|\pi_i - \pi_j\|_{L^\infty(B_{\delta h_j}(x_j))} \le C h_j^{-d/2} \left[h_j^\mu \|u\|_{H^\mu(\widetilde{B}_j)} + h_i^\mu \|u\|_{H^\mu(\widetilde{B}_i)} \right].$$

Inserting this in (2.6) and using Assumption 2.4 gives

$$\begin{aligned}
&|T_2|^2_{H^s(\Omega_i\cap\Omega)} \le \\
&\qquad CM \int_\Omega \sum_{j=1}^N \left[h_j^{2(\mu-s)-d} \|u\|^2_{H^k(\widetilde{B}_j)} + h_i^{2(\mu-s)-d} \|u\|^2_{H^k(\widetilde{B}_i)} \right] \chi_j\chi_i.
\end{aligned}$$

The sum $\sum_{i=1}^N |T_{1,i}|^2_{H^s(\Omega\cap\Omega_i)}$ can then be bounded by using again (2.1)

$$\begin{aligned}
&\sum_{i=1}^N |T_{2,i}|^2_{H^s(\Omega_i\cap\Omega)} \le CM \int_\Omega \sum_{j=1}^N \sum_{i=1}^N h_i^{2(\mu-s)-d} \|u\|^2_{H^k(\widetilde{B}_i)} \chi_i\chi_j \\
&\le CM^2 \sum_{i=1}^N h_i^{2(\mu-s)-d} \|u\|^2_{H^k(\widetilde{B}_i)} \int_\Omega \chi_i \le CM^2 \sum_{i=1}^N h_i^{2(\mu-s)} \|u\|^2_{H^k(\widetilde{B}_i)}.
\end{aligned}$$

This concludes the proof of the theorem. □

Theorem 2.6 assumes u to be defined on $\mathbb{R}^d$. An extension result, e.g., Theorem A.1, allows us to treat the case of bounded domains:

Corollary 2.8. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Assume that the balls $\widetilde{B}_i$ of Theorem 2.6 satisfy additionally an overlap condition, i.e., for some $M \in \mathbb{N}$ we have*

$$\sup_{x \in \mathbb{R}^d} \operatorname{card}\{i \in \mathbb{N} \,|\, x \in \widetilde{B}_i\} \leq M.$$

Then there exists a linear map $Q_N : L^1(\Omega) \to V_N$ such that for each $k \in \mathbb{N}_0$ there exists $C > 0$ with

$$\|u - Q_N u\|_{H^s(\Omega)} \leq h^{\min\{p+1,k\}-s} \|u\|_{H^k(\Omega)}, \quad s = 0, \ldots, \min\{p+1, r_{stab}\},$$

where $h := \max_{i=1,\ldots,N} h_i$.

Proof. Let $\widetilde{Q}_N$ be the linear operator of Theorem 2.6 and let $E : L^1(\Omega) \to L^1(\mathbb{R}^d)$ be the extension operator of Theorem A.1. Set $Q_N := \widetilde{Q}_N \circ E$. Then by abbreviating $\mu := \min\{p+1, k\}$ we get from Theorem 2.6 for $s = 0, \ldots, \min\{r_{stab}, k\}$

$$\begin{aligned} \|u - Q_N u\|^2_{H^s(\Omega)} = \|Eu - \widetilde{Q}_N E u\|^2_{H^s(\Omega)} &\leq C \sum_{i=1}^N h_i^{2(\mu-s)} \|Eu\|^2_{H^k(\widetilde{B}_i)} \\ &\leq C h^{2(\mu-s)} \sum_{i=1}^N \|Eu\|^2_{H^k(\widetilde{B}_i)} \leq C h^{2(\mu-s)} M^2 \|Eu\|^2_{H^k(\mathbb{R}^d)}; \end{aligned}$$

here, the last step followed from arguments analogous to those employed in the proof of Theorem 2.6. The extension operator E finally has the property $\|Eu\|_{H^k(\mathbb{R}^d)} \leq C\|u\|_{H^k(\Omega)}$, which allows us to conclude the proof. □

Approximation of singular functions The diameters of the balls $\widetilde{B}_i$ in Theorem 2.6 play the role of the local mesh size in the classical FEM approximation theorem. In the classical FEM, meshes that are locally refined are important, for example, for the treatment of elliptic boundary value problems in domains with piecewise smooth geometries. The solutions of such problems exhibit singularities (the functions S_{ji} of (6.2) are a typical example), which can be resolved in the classical FEM by the use of appropriately graded meshes, [93,8]. In fact, the optimal rate of convergence, as measured in error versus problem size, can be recovered. Meshless methods can mimic this mesh refinement of the classical FEM by an appropriate clustering of particles and a corresponding shrinking of the diameters of the balls $\widetilde{B}_i$. The following two Exercises 2.10, 2.11 illustrate this.

To stress the analogy of our approach in Exercises 2.10, 2.11 with the classical FEM situation and to motivate the distribution of the diameters of the balls $\widetilde{B}_i$, we first recall the following example (see, e.g., [94, Sec. 3.3.7]):

Example 2.9. Let $\Omega = (0,1)$ and $u(x) = x^\alpha$, $\alpha \in (1/2, 1)$. Fix $p \in \mathbb{N}$ and $\beta > \frac{p+1/2}{\alpha-1/2}$. Consider a mesh $\mathcal{T}$ consisting of N intervals I_i, $i = 0, \dots, N-1$, such that

$$\operatorname{diam} I_0 \leq Ch^\beta, \qquad \operatorname{diam} I_i \sim h \operatorname{dist}(I_i, 0)^{1-1/\beta}, \quad i = 1, \dots, N-1. \tag{2.7}$$

Then, for some $C > 0$ independent of N we have

$$\inf_{v \in S^{p,1}(\mathcal{T})} \|u - v\|_{H^1(\Omega)} \leq CN^{-p},$$

i.e., the optimal rate of convergence is recovered. A specific mesh $\mathcal{T}$ that satisfies (2.7) is determined by the nodes x_i, $i = 0, \dots, N$, where $x_i = \Phi(\widehat{x}_i)$, $\Phi(x) = x^\beta$, and $\widehat{x}_i = ih$ for $h = 1/N$. ∎

The function Φ of Example 2.9 maps a uniform node distribution to a highly non-uniform one that is suitable for the approximation of the function $x \mapsto x^\alpha$. We use this function Φ to create particle distributions, and we use (2.7) as a guideline for our choice of the diameters of the patches Ω_i and the balls $\widetilde{B}_i$ in the following Exercise 2.10. We will show there that this choice leads to patches that satisfy Assumptions 2.1, 2.4, and we will see that polynomials of degree p have good approximation properties on the balls $\widetilde{B}_i$. The construction of concrete shape functions associated with these patches that satisfy Assumptions 2.2, 2.3 is postponed until Exercise 2.22. Corresponding results exist for two-dimensional problems and are sketched in Exercises 2.10, 2.23.

Exercise 2.10. Let $\Omega = (0,1)$, $u(x) = x^\alpha$ for some $\alpha \in (1/2, 1)$. Fix $p \in \mathbb{N}_0$ and choose $\beta \geq \frac{p+1/2}{\alpha-1/2} > 1$. Define

$$\Phi(x) := x^\beta.$$

For $N \in \mathbb{N}$ set $h = 1/N$, $\widehat{x}_i := ih$, $i = 0, \dots, N$, and define the particles $X_N = \{x_i \,|\, i = 0, \dots, N\}$ by $x_i = \Phi(\widehat{x}_i)$. Let $\rho > 0$ be a parameter and choose for each particle x_i

$$\rho_i = \rho \begin{cases} h x_i^{1-1/\beta} & i \geq 1, \\ x_1 & i = 0. \end{cases}$$

Let a shape function φ_i be associated with particle x_i. Assume furthermore that $\Omega_i := (\operatorname{supp} \varphi_i)^\circ = B_{\rho_i}(x_i)$.

(a) Show: For each fixed M there holds $\rho_i \sim h^\beta$ for $i \in \{0, \dots, M\}$ (The constants of the $\sim$-notation depend on ρ, β, M).
(b) Show: There exist λ, λ' (depending only on ρ, β) such that

$$B_{\lambda h}(\widehat{x}_i) \cap \Omega \subset \Phi^{-1}(B_{\rho_i}(x_i) \cap \Omega) \subset B_{\lambda' h}(\widehat{x}_i) \cap \Omega \qquad i = 0, \dots, N.$$

Conclude that Assumption 2.1 is satisfied.

(c) Show: Assumption 2.4 is satisfied.

(d) Let C_{comp} be the constant of Assumption 2.4, whose existence was ascertained in (c). Set $\widetilde{B}_i := B_{\widetilde{\rho}_i}(x_i)$ with $\widetilde{\rho}_i := (1 + (1 + \delta)C_{comp})\rho_i$. Show: $B_{\rho_i}(x_i) \cap B_{\rho_j}(x_j) \neq \emptyset$ implies $B_{\delta\rho_j}(x_j) \subset \widetilde{B}_i$.

(e) Show: The balls $\widetilde{B}_i$, $i = 0, \ldots, N$ satisfy an overlap condition, i.e., there exists $M > 0$ (depending only on ρ, β) such that $\operatorname{card}\{j \mid \widetilde{B}_i \cap \widetilde{B}_j \neq \emptyset\} \leq M$ for all $i \in \{0, 1, \ldots, N\}$.

(f) Let $I_1 := \{i \in \{0, \ldots, N\} \mid \operatorname{dist}(\widetilde{B}_i, 0) \geq 2\widetilde{\rho}_i\}$. Show: For $i \in I_1$ the point $\widetilde{x}_i := \inf\{x \mid x \in \widetilde{B}_i\}$ satisfies $\widetilde{x}_i \sim x_i$. Furthermore, there exist polynomials $\pi_i \in \mathcal{P}_p$ such that

$$\|u - \pi_i\|_{L^2(\widetilde{B}_i)} + \widetilde{\rho}_i \|(u - \pi_i)'\|_{L^2(\widetilde{B}_i)} \leq C \widetilde{\rho}_i^{\,p+3/2} \widetilde{x}_i^{\,\alpha-1-p}.$$

(g) Set $I_2 := \{1, \ldots, N\} \setminus I_1$. Show: $I_2 \subset \{1, \ldots, M\}$ for some $M > 0$ independent of N. Show: For each $i \in I_2$ one can find a $\pi_i \in \mathcal{P}_1$ such that

$$\begin{aligned} \|u - \pi_i\|_{L^2(\Omega\cap\widetilde{B}_i)} + \widetilde{\rho}_i \|(u - \pi_i)'\|_{L^2(\Omega\cap\widetilde{B}_i)} &\leq C \widetilde{\rho}_0^{\,\alpha+1/2}, \\ \|u - \pi_i\|_{L^\infty(\Omega\cap\widetilde{B}_i)} &\leq C \widetilde{\rho}_0^{\,\alpha}. \end{aligned}$$

(h) Assume that the shape functions φ_i satisfy Assumptions 2.2, 2.3. (We will see in Exercise 2.22 that such functions can be constructed with the moving least squares procedure if ρ is chosen sufficiently large). By adapting the proof of Theorem 2.6 show that the approximation space $V_N = \operatorname{span}\{\varphi_i \mid i = 0, \ldots, N\}$ satisfies

$$\inf_{v \in V_N} \|u - v\|_{H^1(\Omega)} \leq C h^p = C N^{-p}.$$

■

A similar idea leads to approximation results in two spatial dimensions:

Exercise 2.11. Define for $h = 1/n$ the uniform particle distribution $\widehat{X}_n = \{\widehat{x}_{ij} = (ih, jh) \mid 0 \leq i, j \leq n\}$. For some $\beta > 1$, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\Phi(x) = \|x\|_2^{\beta-1} x$. Define the particle distribution $X_n := \{x_{ij} = \Phi(\widehat{x}_{ij}) \mid 0 \leq i, j \leq n\}$. Associate with each particle x_{ij} a radius

$$\rho_{ij} = \rho \begin{cases} h \|x_{ij}\|_2^{1-1/\beta} & \text{if } (i,j) \neq (0,0) \\ h^\beta & \text{if } i = j = 0, \end{cases}$$

where $\rho > 0$ is a parameter. The patches Ω_{ij} are taken as $\Omega_{ij} := B_{\rho_{ij}}(x_{ij})$. Set $\Omega := (0, 1/2)^2$.

(a) Proceed as in Exercise 2.10 to show that Assumptions 2.1 and 2.4 hold.

(b) Assume that the shape functions φ_{ij}, $i, j = 0, \ldots, n$, that are associated with the nodes x_{ij} satisfy additionally Assumptions 2.2, 2.3. (We will show in Exercise 2.23 that this can be achieved by taking ρ sufficiently large). Consider a function u in polar coordinates (r, φ) of the form $u = r^\alpha \Theta(\varphi)$, where $\alpha > 0$ and $\Theta : (-\varepsilon, \pi/2 + \varepsilon) \to \mathbb{R}$ for some $\varepsilon > 0$ is smooth. Show: If $\beta > \frac{p}{\alpha}$, then

$$\inf_{v \in V_N} \|u - v\|_{H^1(\Omega)} \leq C h^p, \qquad h = \frac{1}{n} \sim \frac{1}{\sqrt{N}},$$

where N denotes the number of particles. Note that this is the optimal rate of convergence. ∎

2.3 Construction of shape functions with the moving least squares procedure

The approximation result Theorem 2.6 hinges on Assumptions 2.1–2.4. In the present section we construct shape functions that satisfy these requirements.

Motivation from scattered data fitting One approach to construct shape functions φ_i from a collection of particles X_N is based on the so-called moving least squares (MLS) technique that we describe in more detail in this section. The MLS technique was devised to fit a "smooth" function $x \mapsto If$ to a collection of given scattered data (x_i, f_i), $i = 1, \ldots, N$, obtained, for example, from measurements. Here, the points x_i, $i = 1, \ldots, N$, are N distinct points and the "smooth" function If that is sought should satisfy $If(x_i) \approx f_i$, $i = 1, \ldots, N$. The idea is to define the value $If(x)$ for a given x as a weighted average of the given data f_i. More specifically, one chooses a polynomial degree $p \in \mathbb{N}_0$ and for each $i \in \{1, \ldots, N\}$ a weight $w_i(x) \geq 0$ and then defines

$$If(x) := \pi(x), \tag{2.8}$$

where the polynomial $\pi \in \mathcal{P}_p$ is the solution of the minimization problem:

$$\text{Find } \pi \in \mathcal{P}_p \text{ s.t.} \quad \sum_{i=1}^{N} |f_i - \pi(x_i)|^2 w_i(x) \leq \sum_{i=1}^{N} |f_i - v(x_i)|^2 w_i(x) \qquad \forall v \in \mathcal{P}_p. \tag{2.9}$$

Remark 2.12. The choice of the weight functions $x \mapsto w_i(x)$ depends, of course, on the application. In practice, the weight function $x \mapsto w_i(x)$ is chosen to have small support or to decay rapidly as $\|x - x_i\| \to \infty$ so as to give the data points x_i close to x more weight than data points far from x. ∎

Under reasonable assumptions on the weight functions w_i, the minimization problem is uniquely solvable. As we will show in Theorem 2.13, this solution

If takes the form

$$If(x) = \sum_{i=1}^{N} f_i \varphi_i(x) \tag{2.10}$$

for some functions φ_i. Theorem 2.13 also provides an explicit formula for the functions φ_i. Their differentiability properties are then analyzed in Theorem 2.20. The goal of this section is to show that the functions shape functions φ_i, which are motivated by the above data fitting technique, satisfy the assumptions of the approximation result Theorem 2.6. Indeed, we will discover that Assumption 2.2 is ensured by construction and that Assumption 2.3 can be satisfied if, roughly speaking, each particle has sufficiently many neighbors. Assumptions 2.1, 2.4 have to be checked separately.

Construction of the shape functions The shape functions φ_i appearing in (2.10) are constructed in the following theorem.

Theorem 2.13. *Let particles $X_N = \{x_i \,|\, i = 1, \dots, N\}$ and weight functions $w_i \in C(\mathbb{R}^d)$ with $w_i \geq 0$, $i = 1, \dots, N$ be given. Set $\Omega_i := (\operatorname{supp} w_i)^\circ$. Assume that for each $x \in \Omega$ the set $X(x) := \{x_i \,|\, i \in n(x)\}$ is $\mathcal{P}_p$-unisolvent*[1]*. Then the approximant If of (2.8), (2.9) is well-defined, and there are unique functions φ_i, $i = 1, \dots, N$, depending solely on X_N and the weight functions w_i such that*

$$If(x) = \sum_{i=1}^{N} f_i \varphi_i(x).$$

Moreover, we have the representation formula

$$\varphi_i(x) = w_i(x) \sum_{k=1}^{Q} \lambda_k(x) \pi_k(x_i), \qquad i = 1, \dots, N, \tag{2.11}$$

where $\{\pi_k \,|\, k = 1, \dots, Q\}$ is an arbitrary basis of $\mathcal{P}_p$, and the values $\lambda_k(x)$ are the unique solution of the linear system

$$\sum_{k=1}^{Q} \sum_{i=1}^{N} w_i(x) \pi_k(x_i) \pi_l(x_i) \lambda_k(x) = \pi_l(x), \qquad l = 1, \dots, Q. \tag{2.12}$$

Proof. We follow the presentation of [109]. We fix $x_* \in \Omega$ and seek $\pi \in \mathcal{P}_p$ of (2.8) in the form $\pi = \sum_{l=1}^{Q} \widetilde{\lambda}_l \pi_l$. The minimization problem (2.9) then leads to the following system of equations: Find $\widetilde{\lambda}_l$, $l = 1, \dots, Q$, such that

$$\sum_{i=1}^{N} w_i(x_*) \left(f_i - \sum_{l=1}^{Q} \widetilde{\lambda}_l \pi_l(x_i) \right) \pi_k(x_i) = 0, \qquad k = 1, \dots, Q. \tag{2.13}$$

[1] A set $Y \subset \mathbb{R}^d$ is $\mathcal{P}_p$-unisolvent, if $\pi \in \mathcal{P}_p$ and $\pi(y) = 0$ for all $y \in Y$ implies $\pi \equiv 0$.

We prove unique solvability of this linear system of equations by proving that the symmetric matrix $\mathbf{G} \in \mathbb{R}^{Q\times Q}$ with entries $\mathbf{G}_{kl} = \sum_{i=1}^{N} w_i(x_*)\pi_l(x_i)\pi_k(x_i)$ is symmetric positive definite: For $\mathbf{a} \in \mathbb{R}^Q$ we compute

$$\mathbf{a}^\top \mathbf{G}\mathbf{a} = \sum_{i=1}^{N} w_i(x_*) \left| \sum_{k=1}^{Q} \mathbf{a}_k \pi_k(x_i) \right|^2 ;$$

in view of the assumption $w_i \geq 0$, we conclude that $\mathbf{G}$ is positive semi-definite. If $\mathbf{G}$ were not positive definite, then there existed a vector $\mathbf{a} \in \mathbb{R}^Q$ with $\mathbf{a} \neq 0$ such that $\mathbf{a}^\top \mathbf{G}\mathbf{a} = 0$. Hence, for the non-trivial polynomial $\widetilde{\pi} = \sum_{k=1}^{Q} \mathbf{a}_k \pi_k$, we would have $\widetilde{\pi}(x_i) = 0$ for all $x_i \in X(x_*)$, since $x_i \in X(x_*)$ implies $x_i \in (\operatorname{supp} w_i)^\circ$, i.e., by $w_i \in C(\mathbb{R}^d)$ we have $w_i(x_*) > 0$. But then $\widetilde{\pi} = 0$ by our assumption of unisolvence. We have thus arrived at a contradiction and conclude that $\mathbf{G}$ is positive definite.
We now evaluate $If(x_*) = \pi(x_*)$ (writing $w_i = w_i(x_*)$, $\lambda_k = \lambda_k(x_*)$)

$$\pi(x_*) = \sum_{l=1}^{Q} \widetilde{\lambda}_l \pi_l(x_*) \stackrel{(2.12)}{=} \sum_{i,k,l} \widetilde{\lambda}_l \lambda_k w_i \pi_k(x_i)\pi_l(x_i) \stackrel{(2.13)}{=} \sum_{i,k} f_i \lambda_k \pi_k(x_i),$$

which leads to the desired representation formula (2.11). □

Exercise 2.14. Show: For $p = 0$ the functions φ_i are given by

$$\varphi_i(x) = \frac{w_i(x)}{\sum_{j=1}^{N} w_j(x)} = \frac{w_i(x)}{\sum_{j\in n(i)} w_j(x)}. \tag{2.14}$$

These functions are called Shephard functions, [97]. ■

An important observation is that the functions φ_i constructed by the MLS procedure reproduce polynomials, i.e., they satisfy Assumption 2.2:

Exercise 2.15. Show that the functions φ_i satisfy Assumption 2.2, i.e.,

$$\sum_{i=1}^{N} \pi(x_i)\varphi_i(x) = \pi(x) \quad \forall x \in \Omega \qquad \forall \pi \in \mathcal{P}_p. \tag{2.15}$$

■

Remark 2.16. The representation formula (2.11) shows that the functions φ_i can be evaluated at a point $x \in \Omega$ by solving a $Q \times Q$ system of linear equations. Likewise, by differentiating the linear system (2.12), it is clear that also the values of derivatives of the functions $x \mapsto \lambda_k(x)$ can be obtained as solutions of linear systems; therefore, derivatives of the functions φ_i can be determined. The question of bounds of the derivatives of the functions φ_i will be discussed in more detail in Theorem 2.20. ■

The weight functions w_i have to be chosen by the user. A popular form is

$$w_i(x) = w\left(\frac{x - x_i}{\rho_i}\right), \tag{2.16}$$

where the *window function* w is of one of the following types:

1. w is *radial*, i.e., $w(z) = \widetilde{w}(\|z\|)$ for some $\widetilde{w} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$;
2. w has *tensor product* form, i.e., $w(z) = \prod_{j=1}^d \tilde{w}_j(z_j)$.

We note that if the window function w is compactly supported, then the parameter ρ_i in (2.16) is a measure for the support size and $\rho_i \sim h_i = \operatorname{diam} \Omega_i$. In this situation, the univariate functions $\widetilde{w}$ or $\widetilde{w}_j$ are often taken to be compactly supported splines, e.g., the symmetric part of the classical piecewise cubic C^2 B-spline given by

$$w(r) = \begin{cases} 4 - 6r^2 + 3r^3 & \text{for } 0 \le r \le 1, \\ (2 - r)^3 & \text{for } 1 < r \le 2, \\ 0 & \text{for } r > 2. \end{cases}$$

Remark 2.17. If the window function is a radial function and has compact support, then the norm $\|\cdot\|$ on $\mathbb{R}^d$ can be still be chosen. For example, the patches Ω_i can be balls (or, more generally, ellipsoids) if $\|\cdot\|$ is taken as the Euclidean norm; the patches Ω_i can be cubes if $\|\cdot\|_{l^\infty}$ is chosen. ■

Regularity of the shape functions Our analysis of the differentiability properties of the functions φ_i in Theorem 2.20 below will be based on the assumption that the weight functions w_i are determined by a window function w via (2.16). This window function w will be required to satisfy

Assumption 2.18. The window function $w \in C^k(\mathbb{R}^d)$ satisfies $w(x) \ge 0$ for all $x \in \mathbb{R}^d$, and $(\operatorname{supp} w)^\circ = B_1(0)$.

Remark 2.19. We take $B_1(0)$ as the unit ball with respect to the Euclidean norm. This is not essential, however, and results analogous to Theorem 2.20 below hold if we replace the Euclidean norm with another norm on $\mathbb{R}^d$. ■

The formula (2.14) for the special case $p = 0$ suggests that $\varphi_i \in C^k$ if the weights w_i are determined by a window function w satisfying Assumptions 2.18. Roughly speaking, if for every $x \in \Omega$ the number of particles in the vicinity of x, i.e., $\operatorname{card} n(x)$, is sufficiently large, then the shape functions φ_i are indeed as smooth as the window function. In order to prove this result in Theorem 2.20 below, we introduce the fill distance function h by

$$h(x) := \operatorname{dist}(x, X_N) \tag{2.17}$$

and can now formulate:

Theorem 2.20. *Let Ω satisfy a cone condition with angle θ and radius r. Let $\alpha \in (0,1)$, $X_N = \{x_i \,|\, i = 1,\dots,N\} \subset \mathbb{R}^d$ and $\{\rho_i \,|, i = 1,\dots,N\} \subset \mathbb{R}^+$. Set*

$$\hat\rho_i := \min\{\rho_i, r\}, \qquad i = 1,\dots,N,$$

and assume the covering condition

$$\Omega \subset \cup_{i=1}^N B_{\alpha\hat\rho_i}(x_i). \tag{2.18}$$

Let w satisfy Assumption 2.18, define the weight functions $w_i(x) := w(\frac{x-x_i}{\rho_i})$ with corresponding patches $\Omega_i = (\operatorname{supp} w_i)^\circ = B_{\rho_i}(x_i)$. Suppose that Assumption 2.4 is valid. Let $p \in \mathbb{N}_0$.
Then there exist $\delta > 0$ and $C > 0$ (depending only on θ, r, α, p, k, C_{comp}) such that if

$$\sup_{x \in B_{\hat\rho_i}(x_i)\cap\Omega} h(x) \le \delta\hat\rho_i \qquad \forall x_i \in X_N, \tag{2.19}$$

then the functions φ_i of (2.11) satisfy $\varphi_i \in C^k(\mathbb{R}^k)$, $\operatorname{supp}\varphi_i \subset \overline{B_{\rho_i}(x_i)}$, and

$$\|D^\alpha\varphi_i\|_{L^\infty(\Omega)} \le C\rho_i^{-|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^d, \quad |\alpha| \le k. \tag{2.20}$$

Before proving Theorem 2.20 it is instructive to check that the assumptions of Theorem 2.20 can be satisfied in simple circumstances.

Example 2.21. The assumption (2.19) is often formulated in a simpler, global way. If we define the *fill distance* $\overline{h} := \sup_{x\in\Omega} h(x)$ and use constant $\rho_i = \rho$ for all $i \in \{1,\dots,N\}$, then (2.19) merely requires that $\overline{h}$ be sufficiently small compared to ρ, the size of the supports of the patches Ω_i. ∎

We have seen Exercises 2.10, 2.11 two examples of highly non-uniform particle distributions and greatly varying patches sizes that are suitable for the approximation of singularity functions. The following two exercises show that the assumptions of Theorem 2.20 can be fulfilled in such circumstances as well.

Exercise 2.22. In Exercise 2.10 we constructed particles and patch sizes that were appropriate for the approximation of the singular function $x \mapsto x^\alpha$. We assumed, however, that the shape functions φ_i satisfied Assumptions 2.2 and 2.3. Show that by choosing ρ in Exercise 2.10 sufficiently large, the hypotheses of Theorem 2.20 are satisfied. Conclude that the shape functions obtained by the MLS technique yield the optimal approximation result of Exercise 2.10.
Hint: Show that the fill distance function h satisfies

$$h(x) \le C\left[hx^{1-1/\beta} + h^\beta\right]$$

for a constant $C > 0$ independent of ρ and N. ∎

Exercise 2.23. Assume the hypotheses of Exercise 2.11. Show: If ρ is chosen sufficiently large, then the hypotheses of Theorem 2.20 are satisfied.
Hint: Show that the fill distance function h satisfies $h(x) \leq C\big[h\|x\|_2^{1-1/\beta} + h^\beta\big]$ for a constant $C > 0$ independent of ρ and N. ∎

Proof of Theorem 2.20. The proof is broken up into several steps.
1. step: We notice that the representation formula (2.11) is independent of the choice of the basis of $\mathcal{P}_p$. In particular, we may chose for each $x_* \in \Omega$ a different basis. We will exploit this observation as follows: First, we fix a basis $\{\widetilde{\pi}_k \,|\, k = 1, \dots, Q\}$ of $\mathcal{P}_p$; then, for each fixed $x_* \in \Omega$, we define the basis $\{\pi_k \,|\, k = 1, \dots, Q\}$ by

$$\pi_k(x) := \widetilde{\pi}_k(\frac{x - x_*}{\rho_*}),$$

where, for some arbitrary (but fixed) $i_* \in n(x_*)$ we set

$$\rho_* := \rho_{i_*}.$$

(Note that the covering condition (2.18) guarantees that $n(x_*) \neq \emptyset$). Since $2\rho_i = h_i = \operatorname{diam} \Omega_i = \operatorname{diam} B_{\rho_i}(x_i)$, Assumption 2.4 guarantees that

$$\rho_* C_{comp}^{-1} \leq \rho_j \leq \rho_* C_{comp} \qquad \forall j \in n(x_*). \tag{2.21}$$

We next define the matrix $\mathbf{G}(x_*) \in \mathbb{R}^{Q \times Q}$ with entries

$$\mathbf{G}_{kl}(x_*) := \sum_{i=1}^{N} w_i(x_*)\pi_k(x_i)\pi_l(x_i) = \sum_{i \in n(x_*)} w(\frac{x_* - x_i}{\rho_i})\widetilde{\pi}_k(\frac{x_* - x_i}{\rho_*})\widetilde{\pi}_l(\frac{x_* - x_i}{\rho_*}).$$

By Theorem 2.13 the function value $\varphi_i(x_*)$ is given by

$$\varphi_i(x_*) = w_i(x_*) \sum_{k=1}^{Q} \lambda_k(x_*)\pi_k(x_i), \tag{2.22}$$

where the vector $\lambda(x_*) = (\lambda_1(x_*), \dots, \lambda_Q(x_*))^\top \in \mathbb{R}^Q$ is the solution of the linear system

$$\mathbf{G}(x_*)\lambda(x_*) = \begin{pmatrix} \widetilde{\pi}_1(0) \\ \vdots \\ \widetilde{\pi}_Q(0) \end{pmatrix}. \tag{2.23}$$

In order to get bounds on the derivatives of φ_i, we need to get bounds on the derivatives of the function λ. In this direction, we first notice that the product rule together with (2.21) gives

$$|D^\alpha \mathbf{G}(x_*)| \leq C_\alpha {\rho_*}^{-|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^d, \quad |\alpha| \leq k, \tag{2.24}$$

where the constant C_α depends only on α, the function w, and the choice of basis $\{\widetilde{\pi}_l \,|\, l = 1, \ldots, Q\}$. The analogous bound

$$|D^\alpha \mathbf{G}^{-1}(x_*)| \le C_\alpha {\rho_*}^{-|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^d, \quad |\alpha| \le k, \tag{2.25}$$

holds by Cramer's rule, provided that we can show the existence of $\underline{C} > 0$ such that

$$\inf_{x_* \in \Omega} |\det \mathbf{G}(x_*)| \ge \underline{C} > 0. \tag{2.26}$$

From (2.25) follows a bound similar to (2.25) for the derivatives of the solution λ_l, $l = 1, \ldots, Q$ of (2.23); the product rule applied to (2.22) together with (2.21) then gives the desired bound (2.20) for the shape functions φ_i. We are thus left with establishing (2.26).
2. step: To see (2.26) we prove a lower bound on the smallest eigenvalue of the symmetric matrix $\mathbf{G}(x_*)$. To that end, let $\mathbf{a} \in \mathbb{R}^Q$ be arbitrary but fixed. We define the polynomial

$$\pi := \sum_{k=1}^{Q} \mathbf{a}_k \pi_k$$

and observe

$$\mathbf{a}^\top \mathbf{G}(x_*)\mathbf{a} = \sum_{i,k,l} w_i(x_*)\mathbf{a}_k \mathbf{a}_l \pi_k(x_i)\pi_l(x_i) = \sum_{i=1}^{N} w_i(x_*)|\pi(x_i)|^2. \tag{2.27}$$

We wish to exploit that Assumption 2.18 gives us the existence of $C_{min} > 0$ such that

$$\min\{w(x) \,|\, x \in B_\alpha(0)\} = C_{min} > 0. \tag{2.28}$$

To do so, we define $\eta < 1/2$ by

$$\eta := \frac{1}{2}\frac{\alpha}{C_{comp}} \le \frac{1}{2}\alpha < \frac{1}{2}, \tag{2.29}$$

where we used $C_{comp} \ge 1$. Next, we choose δ appearing in (2.19) according to the definition (2.33) below; in particular, therefore, $\delta < \eta$ so that there exists an index $i \in \mathbb{N}$ such that $x_* \in B_{\eta\hat{\rho}_i}(x_i)$. We fix this index and define

$$\widetilde{n}(x_*) := \{j \in \mathbb{N} \,|\, x_j \in X_N \cap B_{\eta\hat{\rho}_i}(x_*)\}. \tag{2.30}$$

Our goal in this 2. step is to show

$$\mathbf{a}^\top \mathbf{G}(x_*)\mathbf{a} \ge \sum_{j \in \widetilde{n}(x_*)} w_j(x_*)|\pi(x_j)|^2 \ge C_{min} \sum_{j \in \widetilde{n}(x_*)} |\pi(x_j)|^2. \tag{2.31}$$

The first bound in (2.31) is obvious since $w_j \ge 0$ for all j. To see the second estimate, in view of (2.28), it suffices to see $\|x_j - x_*\|_2 < \alpha\hat{\rho}_j$ for $j \in \widetilde{n}(x_*)$. Let therefore $j \in \widetilde{n}(x_*)$. Then

$$\|x_j - x_i\|_2 \le \|x_j - x_*\|_2 + \|x_i - x_*\|_2 < 2\eta\hat{\rho}_i \le \hat{\rho}_i,$$

where in the last step, we used $\eta \leq 1/2$. Hence, $x_j \in B_{\rho_i}(x_i)$, and thus $j \in n(i)$. We conclude with Assumption 2.4

$$\hat{\rho}_i \leq C_{comp}\hat{\rho}_j \qquad \forall j \in \widetilde{n}(x_*).$$

Together with the definition of η in (2.29), we arrive at the desired bound $\|x_j - x_*\|_2 < \eta\hat{\rho}_i \leq \eta C_{comp}\hat{\rho}_j \leq \frac{1}{2}\alpha\hat{\rho}_j \leq \alpha\hat{\rho}_j$.
3. step: To get further, we apply Lemma 2.24. Our choice of δ above is precisely the choice of Lemma 2.24 so that we can find $C > 0$ depending only on Ω, η, and p such that

$$\|\pi\|_{L^\infty(B_{\hat{\rho}_i}(x_*))} \leq C \max\{|\pi(x_j)| \,|\, j \in \widetilde{n}(x_*)\}.$$

Thus, we get from (2.31)

$$\mathbf{a}^\top \mathbf{G}(x_*)\mathbf{a} \geq C\|\pi\|^2_{L^\infty(B_{\hat{\rho}_i}(x_*))}.$$

In view of (2.21), we get from Bernstein's estimate Lemma B.4 the existence of $C > 0$ (depending only on p, C_{comp} and the parameter r of the cone condition) such that $\|\pi\|_{L^\infty(B_{\rho_*}(x_*))} \leq C\|\pi\|_{L^\infty(B_{\hat{\rho}_i}(x_*))}$. Thus, we get

$$\mathbf{a}^\top \mathbf{G}(x_*)\mathbf{a} \geq C\|\pi\|^2_{L^\infty(B_{\rho_*}(x_*))}. \tag{2.32}$$

To control the smallest eigenvalue of $\mathbf{G}(x_*)$, we are therefore left with estimating $\sum_{k=1}^{Q} |\mathbf{a}_k|^2$ by $\|\pi\|^2_{L^\infty(B_{\rho_*}(x_*))}$. We achieve this by a scaling argument: We define the function $\overline{\pi}(x) := \pi((x - x_*)/\rho_*)$ on $B_1(0)$ and note

$$\overline{\pi}(x) = \sum_{k=1}^{Q} \mathbf{a}_k \widetilde{\pi}_k(x).$$

We observe $\|\overline{\pi}\|_{L^\infty(B_1(0))} = \|\pi\|_{L^\infty(B_{\rho_*}(x_*))}$. By the equivalence of norms on finite dimensional space, we then get the existence of $C > 0$ (depending solely on p and the choice of the basis $\{\widetilde{\pi}_k \,|\, k = 1, \ldots, Q\}$) such that

$$C^{-1}\sum_{k=1}^{Q} |\mathbf{a}_k|^2 \leq \|\overline{\pi}\|^2_{L^\infty(B_1(0))} \leq C\sum_{k=1}^{Q} |\mathbf{a}_k|^2.$$

This establishes the desired lower bound on the eigenvalues of $\mathbf{G}(x_*)$. Finally, we note that this bound holds in fact uniformly in $x_* \in \Omega$, thus completing the proof of (2.26). □
The following lemma allows us to bound the L^∞-norm of a polynomial in terms of values in discrete points:

Lemma 2.24. *Let $X_N = \{x_i \,|\, i = 1, \ldots, N\} \subset \mathbb{R}^d$ and $\{\rho_i \,|\, i = 1, \ldots, N\} \subset \mathbb{R}^+$. Let $\Omega \subset \mathbb{R}^d$ satisfy an interior cone condition with angle θ and radius $r > 0$. Define*

$$\hat{\rho}_i := \min\{\rho_i, r\}, \qquad i = 1, \ldots, N.$$

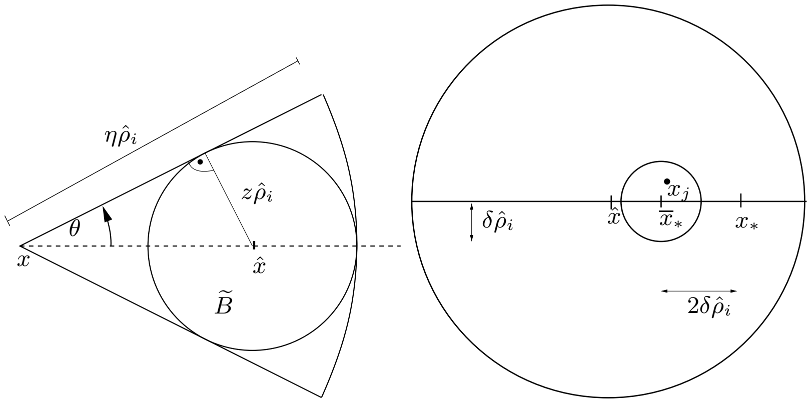

Fig. 2.2. Notation for Lemma 2.24. Left: Ball $\widetilde{B}$. Right: Location of $\hat{x}$, x_*, x_j.

Let $\eta \in (0,1]$ and $p \in \mathbb{N}_0$. Set

$$\delta := \eta \frac{\sin\theta}{1+\sin\theta} \min\left\{\frac{1}{3}, \frac{1}{36p^2}\right\}. \tag{2.33}$$

Then the following holds: If $\Omega \subset \cup_{i=1}^N B_{\hat\rho_i}(x_i)$ and if for all $i \in \{1,\dots,N\}$

$$\sup_{y \in B_{\hat\rho_i}(x_i)\cap\Omega} h(y) \le \delta\hat\rho_i, \tag{2.34}$$

then for each $x \in \Omega$ and any $x_i \in X_N \cap B_{\hat\rho_i}(x)$ and all $\pi \in \mathcal{P}_p$

$$\begin{aligned}\|\pi\|_{L^\infty(B_{\hat\rho_i}(x))} &\le \|\pi\|_{L^\infty(B_{2\hat\rho_i}(x_i))}\\ &\le 2\left(\frac{4(1+\sin\theta)}{\eta\sin\theta}\right)^p \max\{|\pi(x_j)|\,|\, x_j \in X_N \cap B_{\eta\hat\rho_i}(x)\}.\end{aligned}$$

Proof. The proof follows the arguments of [109] and proceeds in several steps. We fix $x \in \Omega \cap B_{\hat\rho_i}(x_i)$ and $\pi \in \mathcal{P}_p$. We also define

$$z := \eta\frac{\sin\theta}{1+\sin\theta}$$

and note that δ, z are chosen such that

$$3\delta \le z.$$

1. step: By the cone condition, there exists a cone $C_1 = C(x,\xi,\theta,\eta\hat\rho_i) \subset \Omega$. Elementary geometric considerations (see Fig. 2.2) then show the existence of a ball $\widetilde{B} = B_{z\hat\rho_i}(\hat{x})$, where $\hat{x} = x + \frac{\eta}{1+\sin\theta}\xi$ with the following properties:

$$\widetilde{B} \subset C_1 \subset \Omega \cap B_{\eta\hat\rho_i}(x) \cap B_{2\hat\rho_i}(x_i). \tag{2.35}$$

2. step: From Lemma B.4, we get

$$\|\pi\|_{L^\infty(B_{2\hat\rho_i}(x_i))} \le \left(\frac{4}{z}\right)^p \|\pi\|_{L^\infty(\widetilde{B})}. \tag{2.36}$$

It therefore suffices to bound $\|\pi\|_{L^\infty(\widetilde{B})}$ in terms of the values of π in the discrete set $X_N \cap B_{\eta\hat\rho_i}(x)$. Towards this goal, we construct in this 2. step an $x_j \in X_N \cap B_{\eta\hat\rho_i}(x)$ that will be seen in the 4. step to have the property that $|\pi(x_j)|$ is comparable to $\|\pi\|_{L^\infty(\widetilde{B})}$. Choose $x_* \in \overline{\widetilde{B}}$ such that

$$\|\pi\|_{L^\infty(\widetilde{B})} = |\pi(x_*)|.$$

We claim the existence of $x_j \in X_N \cap \widetilde{B} \cap \overline{B_{3\delta\hat\rho_i}(x_*)}$. To see this, we recall that $\hat{x}$ is the center of $\widetilde{B}$ and define the auxiliary point

$$\overline{x}_* := \begin{cases} x_* + 2\delta\hat\rho_i \frac{1}{\|\hat{x}-x_*\|_2}(\hat{x}-x_*) & \text{if } x_* \neq \hat{x}, \\ x_* & \text{if } x_* = \hat{x}. \end{cases}$$

Since $3\delta \le z$, elementary considerations show $\|\overline{x}_* - \hat{x}\|_2 < (z-\delta)\hat\rho_i$; hence $\overline{B_{\delta\hat\rho_i}(\overline{x}_*)} \subset \widetilde{B}$. The assumption (2.34) then implies the existence of an $x_j \in X_N \cap \overline{B_{\delta\hat\rho_i}(\overline{x}_*)} \subset X_N \cap \widetilde{B}$. By the triangle inequality we furthermore get $x_j \in \overline{B_{3\delta\hat\rho_i}(x_*)}$.

3. step: Let x_j be the point constructed in the 2. step and set

$$\zeta := \frac{1}{\|x_j - x_*\|_2}(x_j - x_*) \qquad \text{if } x_j \neq x_*.$$

If $x_j = x_*$, then choose an arbitrary $\zeta \in \mathbb{R}^d$ with $\|\zeta\|_2 = 1$. We claim:

$$\{x_* + t\zeta \,|\, t \in [0, \tfrac{1}{3}z\hat\rho_i]\} \subset \overline{\widetilde{B}}.$$

To see this, we first note that the case $x_* = \hat{x}$ is trivial. We therefore assume that $x_* \neq \hat{x}$. From the 2. step we recall

$$\|\overline{x}_* - x_j\|_2 \le \delta\hat\rho_i, \qquad \|\overline{x}_* - x_*\|_2 = 2\delta\hat\rho_i, \tag{2.37}$$

so that we can conclude

$$\|x_j - x_*\|_2 \ge \delta\hat\rho_i. \tag{2.38}$$

In order to see that $x_* + t\zeta \in \widetilde{B}$ for $t \in [0, \frac{1}{3}z\hat\rho_i]$ we write

$$x_j = \overline{x}_* + (x_j - \overline{x}_*) = x_* + \frac{2\delta\hat\rho_i}{\|\hat{x} - x_*\|_2}(\hat{x} - x_*) + (x_j - \overline{x}_*).$$

and compute

$$\|x_* + t\zeta - \hat{x}\|_2 \le \left| \|x_* - \hat{x}\|_2 - \frac{2\delta\hat\rho_i}{\|x_j - x_*\|_2} t \right| + \frac{\|x_j - \overline{x}_*\|_2}{\|x_j - x_*\|_2} t.$$

Requiring

$$\left| \|x_* - \hat{x}\|_2 - \frac{2\delta\hat{\rho}_i}{\|x_j - x_*\|_2} t \right| + \frac{\|x_j - \overline{x}_*\|_2}{\|x_j - x_*\|_2} t \leq z\hat{\rho}_i$$

is equivalent to the following two inequalities:

$$\|x_* - \hat{x}\|_2 - z\hat{\rho}_i \leq \frac{2\delta\hat{\rho}_i - \|x_j - \overline{x}_*\|_2}{\|x_j - x_*\|_2} t \qquad \text{and}$$

$$t \leq (\|x_* - \hat{x}\|_2 + z\hat{\rho}_i) \frac{\|x_j - x_*\|_2}{2\delta\hat{\rho}_i + \|x_j - \overline{x}_*\|_2},$$

which are indeed both satisfied for $t \in [0, \frac{1}{3} z\hat{\rho}_i]$ in view of $\|x_* - \hat{x}\|_2 \leq z\hat{\rho}_i$ and (2.37), (2.38).
4. step: We now turn to estimating $|\pi(x_*)|$ in terms of $|\pi(x_j)|$. To that end, we define with the vector ζ of the fourth step the polynomial

$$p(t) := \pi(x_* + t\zeta), \qquad t \in [0, \tfrac{1}{3} z\hat{\rho}_i],$$

and note that $x_j = x_* + \tau\zeta$ for some τ with $0 \leq \tau \leq 3\delta\hat{\rho}_i$ since $x_j \in \overline{B_{3\delta\hat{\rho}_i}(x_*)}$. Additionally, we have (for $p \geq 1$) in view of the definition of δ that $\tau \leq \frac{1}{3} z$. Using Markov's inequality (see, e.g., [33, Chap. 4, Thm. 1.4]), we can bound

$$\begin{aligned} |\pi(x_*) - \pi(x_j)| &= |p(\|x_* - x_j\|_2) - p(0)| = \left| \int_0^\tau p'(t)\, dt \right| \\ &\leq \tau \|p'\|_{L^\infty(0, \frac{1}{3} z\hat{\rho}_i)} \leq \frac{2\tau p^2}{\frac{1}{3} z\hat{\rho}_i} \|p\|_{L^\infty(0, \frac{1}{3} z\hat{\rho}_i)} \leq \frac{18\delta}{z} p^2 \|\pi\|_{L^\infty(\widetilde{B})}. \end{aligned}$$

Recalling now that $|\pi(x_*)| = \|\pi\|_{L^\infty(\widetilde{B})}$, we get

$$\|\pi\|_{L^\infty(\widetilde{B})} \leq \frac{1}{1 - 18p^2\delta/z} |\pi(x_j)|.$$

This estimate is also trivially true for $p = 0$. We therefore conclude, since $x_j \in X_N \cap \widetilde{B} \subset X_N \cap B_{\eta\hat{\rho}_i}(x)$

$$\|\pi\|_{L^\infty(B_{2\hat{\rho}_i}(x_i))} \leq \left(\frac{4}{z}\right)^p \frac{1}{1 - 18p^2\delta/z} \max\{x_j \mid x_j \in X_N \cap B_{\eta\hat{\rho}_i}(x)\}.$$

Using $\delta \leq \frac{1}{36p^2} z$ and the definition of z, we arrive at the desired bound. □

Exercise 2.25. Assumption 2.18 requires the function w to be k-times continuously differentiable. Consider what assumptions (e.g., on the definition of $n(x)$) need to be changed if w is in $C^{k-1,1}$. ■

2.4 Bibliographical Remarks

The construction of the Q_N in the proof of Theorem 2.6 that is based on point evaluations of locally approximating polynomials is just one possible technique; variations of such constructions can be found in [6,1]. The proof of the stability result Theorem 2.20 follows in essence [109]. Variants can be found, for example, in [55,41,1].
The moving least squares technique originates from scattered data approximation. Early references include [97,45]. It is, however, just one way of generating shape functions that reproduce polynomials. Alternatives include the reproducing kernel particle methods (RKPM), [72–75,70].
One reason for introducing meshless methods is to alleviate the costly meshing. Completely regular meshes on the other hand are very simple to generate and have many advantages. With this in mind, the web-splines (weighted extended B-splines) were introduced in [57]. The computational domain is covered with a regular mesh on which standard splines can be defined easily. Appropriate adjustments near the boundary are made to be able to handle essential boundary conditions.

3 Approximation properties of radial basis functions

A second class of shape functions that can be motivated from scattered data interpolation are radial basis functions (RBFs). In scattered data *interpolation* the basic problem is as follows: given a norm $\|\cdot\|$ on $\mathbb{R}^d$, a function $\Phi : \mathbb{R}_0^+ \to \mathbb{R}$, distinct points $X_N = \{x_i \,|\, i = 1, \dots, N\} \subset \mathbb{R}^d$ and function values f_i, $i = 1, \dots, N$, the goal is to find If of the form

$$If = \textstyle\sum_{j=1}^N u_j \Phi(\|\cdot - x_j\|) \quad \text{s.t.} \quad If(x_i) = f_i \qquad i = 1, \dots, N. \tag{3.1}$$

The problem (3.1) represents a linear system of equations. Clearly, existence and uniqueness of If depends on the function Φ. An important class for which this can be established is that of *positive definite* functions Φ:

Definition 3.1. A continuous function $\Phi : \mathbb{R}_0^+ \to \mathbb{R}$ is *positive definite*, if for any set $X = \{x_1, \dots, x_M\}$ of M distinct points the Gram matrix $\mathbf{G} \in \mathbb{R}^{M\times M}$ with entries $\mathbf{G}_{ij} = \Phi(\|x_i - x_j\|)$ is symmetric positive definite.

Proposition 3.2. *If Φ is positive definite, then the interpolation problem (3.1) is uniquely solvable.*

Proof. Exercise. □

Example 3.3. Classically, the norm $\|\cdot\|$ on $\mathbb{R}^d$ is taken to be the Euclidean norm $\|\cdot\|_2$. Popular examples of radial basis functions Φ are the Gaussians ($\Phi(r) = e^{-r^2}$), Hardy's multiquadrics $\Phi(r) = \sqrt{1 + r^2}$, and the inverse multiquadrics $\Phi(r) = (1 + r^2)^{-1/2}$. It is also a widely used practice to employ scaled versions, that is, to use the function $\widetilde{\Phi}(r) = \Phi(r/h)$ with a suitable scaling parameter $h > 0$. These RBFs can be used for scattered data interpolation in any dimension. Another class is obtained by taking the fundamental solution of the iterated Laplacian Δ^m. For $2m \geq d$, these RBFs are given by $\Phi(r) = r^{2m-d} \ln r$ if d is even and $\Phi(r) = r^{2m-d}$ if d is odd. The function Φ in the special case $m = d = 2$ is called the thin-plate spline since in the Kirchhoff plate model, which is a biharmonic equation, the deflection of an infinite plate under a point load coincides with Φ (up to scaling). ∎

The functions of Example 3.3 do not have bounded support. As was shown in [106,107] it is possible to construct RBFs that have compact support:

Example 3.4. A class of RBFs $\Phi_{d',k}$, $k \in \mathbb{N}_0$ for applications in spatial dimension $d \leq d'$ are the compactly supported RBFs of H. Wendland, [106,107]. A few examples of this class are:

function	smoothness	for problems in $\mathbb{R}^d$
$\Phi_{1,0}(r) = (1-r)_+$	C^0	$d = 1$
$\Phi_{1,1}(r) = (1-r)_+^3(3r+1)$	C^2	$d = 1$
$\Phi_{1,2}(r) = (1-r)_+^5(8r^2+5r+1)$	C^4	$d = 1$
$\Phi_{3,0}(r) = (1-r)_+^2$	C^0	$d \leq 3$
$\Phi_{3,1}(r) = (1-r)_+^4(4r+1)$	C^2	$d \leq 3$
$\Phi_{3,2}(r) = (1-r)_+^6(35r^2+18r+3)$	C^4	$d \leq 3$

With the exception of $\Phi_{1,0}$, $\Phi_{3,0}$, the functions $\Phi_{k,d'}$ satisfy Assumption 3.5 below (see [107] and Exercise 3.6) and hence are positive definite. As in Example 3.3 scaled version $\Phi_{k,d}(r/\rho)$ for a scaling parameter $\rho > 0$ are frequently employed as well. ∎

3.1 Analysis of a class of RBFs

We consider the following class of RBF functions $x \mapsto \Phi(\|x\|_2)$:

Assumption 3.5. The Fourier transform[2] ψ of the function $x \mapsto \Phi(\|x\|_2)$ satisfies for some $\tau > d/2$ and $C > 0$

$$C^{-1}(1 + \|\xi\|_2^2)^{-\tau} \leq \psi(\xi) \leq C(1 + \|\xi\|_2^2)^{-\tau} \qquad \forall \xi \in \mathbb{R}^d.$$

The set of RBFs that satisfy Assumption 3.5 is not empty:

Exercise 3.6. Check that the compactly supported RBF $\Phi_{1,1}$ of Example 3.4 for $d = 1$ satisfies Assumption 3.5 with $\tau = 2$. ∎

[2] $\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) e^{-\mathbf{i}x\cdot\xi}\, dx$ denotes the Fourier transform $\hat{f}$ of a function f. The inversion formula takes the form $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{\mathbf{i}x\cdot\xi}\, d\xi$.

The strict positivity of ψ stipulated in Assumption 3.5 allows us to define an inner product $\langle\cdot,\rangle_\Phi$ and the corresponding Hilbert space H_Φ, which is called the "native space":

$$\langle f,g\rangle_\Phi := \int_{\mathbb{R}^d} \frac{1}{\psi}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi, \qquad H_\Phi := \{f \mid \|f\|_\Phi^2 := \langle f,f\rangle_\Phi < \infty\}. \quad (3.2)$$

We have

Proposition 3.7. *Let Φ satisfy Assumption 3.5. Then*

1. *$H_\Phi \subset C(\mathbb{R}^d)$.*
2. *$H_\Phi = H^\tau(\mathbb{R}^d)$ with equivalent norms.*
3. *$\Phi \in H_\Phi$.*
4. *Φ is positive definite.*

Proof. The second assertion is just one of several equivalent definitions of the Sobolev spaces $H^\tau(\mathbb{R}^d)$. The other assertions are left as an exercise. □

Theorem 3.8. *Let Assumption 3.5 be valid. Then for distinct points $X_N = \{x_i \mid i = 1,\dots,N\}$ and $f \in H_\Phi$ the scattered interpolation problem:*

$$\begin{aligned} &\textit{Find } If \in V_N := \operatorname{span}\{\Phi(\|\cdot - x_i\|_2) \mid i = 1,\dots,N\} \\ &\textit{such that} \quad If(x_i) = f(x_i) \quad i = 1,\dots,N, \end{aligned}$$

has a unique solution, which satisfies

$$\langle f - If, v\rangle_\Phi = 0 \quad \forall v \in V_N \quad (3.3)$$

and

$$\|f - If\|_\Phi = \min_{v\in V_N} \|f - v\|_\Phi. \quad (3.4)$$

Proof. Existence and unique follows from the fact that $x \mapsto \Phi(\|x\|_2)$ is positive definite. The orthogonality relation can be seen as follows: The function $v_k = \Phi(\|\cdot - x_k\|_2)$ satisfies $v_k \in V_N$ and $\widehat{v}_k(\xi) = \psi(\xi)e^{\mathbf{i}x_k\xi}$. Next,

$$\langle f - If, v_k\rangle_\Phi = \int_{\mathbb{R}^d} \frac{1}{\psi}\left(\hat{f} - \widehat{If}\right)\psi e^{\mathbf{i}x_k\xi}\,d\xi = f(x_k) - If(x_k) = 0,$$

where the last step follows from the interpolation property. Hence, (3.3) is true. This orthogonality relation implies the best approximation result (3.4) in the $\|\cdot\|_\Phi$-norm in the standard way (see, e.g., the proof of Céa's Lemma in [23, Thm. 2.8.1]). □

Corollary 3.9 (stability of scattered data interpolation). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain (or $\Omega = \mathbb{R}^d$). Let $X_N = \{x_i \mid i = 1,\dots,N\} \subset \Omega$ and suppose Assumption 3.5. Then for all $f \in H^\tau(\Omega)$*

$$\|f - If\|_{H^\tau(\Omega)} \le C\|f\|_{H^\tau(\Omega)}.$$

Proof. We will only treat the case of Ω being a Lipschitz domain. Let $E : H^\tau(\Omega) \to H^\tau(\mathbb{R}^d)$ be the universal extension operator of Theorem A.1. Since $X_N \subset \Omega$, we have $Ef(x_i) = f(x_i)$, $i = 1, \ldots, N$. By Proposition 3.7, the interpolant If exists and is unique. Since $H^\tau(\mathbb{R}^d) = H_\Phi$, we have $Ef \in H_\Phi$. By Proposition 3.7 and Theorem 3.8 we arrive at

$$\begin{aligned}\|Ef - If\|^2_{H^\tau(\mathbb{R}^d)} &\leq C\langle Ef - If, Ef - If\rangle_\Phi = C\langle Ef - If, Ef\rangle_\Phi \\ &\leq C\|Ef - If\|_\Phi \|Ef\|_\Phi \leq C\|Ef - If\|_{H^\tau(\mathbb{R}^d)}\|Ef\|_{H^\tau(\mathbb{R}^d)} \\ &\leq C\|Ef - If\|_{H^\tau(\mathbb{R}^d)}\|f\|_{H^\tau(\Omega)}.\end{aligned}$$

We conclude $\|Ef-If\|_{H^\tau(\mathbb{R}^d)} \leq C\|f\|_{H^\tau(\Omega)}$. Since $Ef = f$ on Ω and trivially $\|Ef - If\|_{H^\tau(\Omega)} \leq C\|Ef - If\|_{H^\tau(\mathbb{R}^d)}$, the proof is complete. $\square$

This stability result is the key to approximation results for the scattered data interpolant If:

Corollary 3.10. *Let Assumption 3.5 be satisfied and let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Define the* fill distance

$$h := \sup_{x\in\Omega} \min_{i=1,\ldots,N} \|x - x_i\|_2. \tag{3.5}$$

Then there exists $C > 0$ such that for $f \in H^\tau(\Omega)$ there holds

$$\|f - If\|_{H^s(\Omega)} \leq Ch^{\tau-s}\|f\|_{H^\tau(\Omega)}, \qquad 0 \leq s \leq \tau.$$

Proof. We proceed in two steps.
1. step: By Theorem 3.8, the linear operator $\operatorname{Id} - I : H^\tau(\Omega) \to V_N \subset H^\tau(\Omega)$ satisfies $\|\operatorname{Id} - I\|_{H^\tau(\Omega)\to H^\tau(\Omega)} \leq C$. If we can show the claim for $s = 0$, i.e., $\|\operatorname{Id} - I\|_{H^\tau(\Omega)\to L^2(\Omega)} \leq Ch^\tau$, then the desired bound $\|\operatorname{Id} - I\|_{H^\tau(\Omega)\to H^s(\Omega)} \leq Ch^{\tau-s}$ for any $s \in [0,\tau]$ follows by interpolation. We are thus left with showing the special case $s = 0$.
2.step: Choose $p \in \mathbb{N}_0$ such that $\tau \leq p$. By Lemma 2.24 there exist C, $\hat{C} > 0$ depending only on Ω such that for $\rho = Ch$ we have for all balls $B_\rho(x)$, $x \in \Omega$:

$$\|\pi\|_{L^\infty(B_\rho(x))} \leq \hat{C} \max_{x_i\in B_\rho(x)} |\pi(x_i)| \qquad \forall \pi \in \mathcal{P}_p. \tag{3.6}$$

We cover $\Omega \subset \bigcup_{x\in\Omega} \overline{B_\rho(x)}$. By the Besicovitch covering theorem, Theorem A.4, we can extract from the cover $\mathcal{B} = \{\overline{B_\rho(x)}\,|\, x \in \Omega\}$ a subcover $\mathcal{B}_j$, $i = j, \ldots, M$, with the following properties: $\Omega \subset \cup_{j=1}^M \cup_{\overline{B}\in\mathcal{B}_j} \overline{B}$ and each collection $\mathcal{B}_j$ consists of countably many disjoint balls.
We set $z := f - If$ and assume for notational convenience, as we may using the extension operator of Theorem A.1, that z is defined on $\mathbb{R}^d$ with $\|z\|_{H^\tau(\mathbb{R}^d)} \leq C\|z\|_{H^\tau(\Omega)}$. For each ball $\overline{B}$ of $\cup_{j=1}^M \mathcal{B}_j$ we select $Q \in \mathcal{P}_p$ as given by the polynomial approximation result Theorem B.1. We can then bound with the triangle inequality and the polynomial inverse estimate of Theorem B.3

$$\|z\|_{L^2(B)} \leq \|z - Q\|_{L^2(B)} + \|Q\|_{L^2(B)} \leq C\left\{\rho^\tau\|z\|_{H^\tau(B)} + \rho^{d/2}\|Q\|_{L^\infty(B)}\right\}.$$

Our choice of the balls B in $\mathcal{B}$ guarantees (3.6). Hence, we can estimate

$$\|Q\|_{L^\infty(B)} \leq \hat{C} \max\{|Q(x_i)| \,|\, x_i \in \overline{B}\} = C \sup\{|Q(x_i)| \,|\, x_i \in B\}.$$

Since z vanishes in the interpolation points x_i, we get

$$\begin{aligned}\|Q\|_{L^\infty(B)} &\leq \hat{C} \sup\{|Q(x_i) - z(z_i)| \,|\, x_i \in B\} \\ &\leq \hat{C}\|z - Q\|_{L^\infty(B)} \leq C\rho^{\tau - d/2}\|z\|_{H^\tau(B)},\end{aligned}$$

where we used again the approximation properties in L^∞ ascertained in Theorem B.1. Using the fact that $\Omega \subset \cup_{j=1}^M \cup_{\overline{B}\in\mathcal{B}_j} \overline{B}$ and that for each $j \in \{1, \ldots, M\}$ the balls of the collection $\mathcal{B}_j$ are pairwise disjoint, we get

$$\|z\|^2_{L^2(\Omega)} \leq \sum_{j=1}^M \sum_{\overline{B}\in\mathcal{B}_j} \|z\|^2_{L^2(B)} \leq C\rho^{2\tau} \sum_{j=1}^M \sum_{\overline{B}\in\mathcal{B}_j} \|z\|^2_{H^\tau(B)} \leq C\rho^{2\tau} \sum_{j=1}^M \|z\|^2_{H^\tau(\Omega)}.$$

This concludes the proof in view of the stability result Corollary 3.9. □

It is of interest to consider functions $f \in H^k(\Omega)$ with $k < \tau$. Since in this case the function f may not be continuous, we cannot define the scattered data interpolant; nevertheless, the space $V_N = \operatorname{span}\{\Phi(\|\cdot - x_i\|_2) \,|\, i = 1, \ldots, N\}$ can still have good approximation properties. Indeed, we have the following:

Proposition 3.11. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Assume that Φ satisfies Assumption 3.5. Let X_N be a particle distribution with fill distance h given by (3.5). Set $V_N := \operatorname{span}\{\Phi(\|\cdot - x_i\|_2) \,|\, x_i \in X_N\}$. Then for $0 \leq k \leq \tau$ and real numbers $0 \leq s_1 \leq \cdots \leq s_m = k$, we have for some $C > 0$ independent of h and f:*

$$\inf_{v\in V_N} \sum_{j=1}^m h^{s_j} \|f - v\|_{H^{s_j}(\Omega)} \leq Ch^k \|f\|_{H^k(\Omega)}.$$

Proof. We will prove the following, weaker statement:

$$\inf_{v\in V_N} \|f - v\|_{H^s(\Omega)} \leq Ch^{k-s}\|f\|_{H^k(\Omega)}, \qquad 0 \leq s \leq k. \tag{3.7}$$

The statement of the proposition then follows from (3.7) and a result on simultaneous approximation in Sobolev space, [22]. To see (3.7), fix s and let $\Pi : H^s(\Omega) \to V_N$ be the $H^s(\Omega)$-orthogonal projection. Then by Corollary 3.10

$$\|\operatorname{Id} - \Pi\|_{H^s(\Omega)\to H^s(\Omega)} = 1, \qquad \|\operatorname{Id} - \Pi\|_{H^\tau(\Omega)\to H^s(\Omega)} \leq Ch^{\tau - s}.$$

Since the space $H^k(\Omega)$ can be obtained by interpolation between $H^s(\Omega)$ and $H^\tau(\Omega)$ we arrive at the desired bound.

3.2 Bibliographical Remarks

The presentation here follows [86]. The presentation is restricted to positive definite RBFs for simplicity. A very important, more general class of functions is that of conditionally positive RBFs: For given $p \in \mathbb{N}_0$, norm $\|\cdot\|$ on $\mathbb{R}^d$, a function $\Phi(\|\cdot\|)$ is called conditionally positive definite if for any set $X_M = \{x_1, \ldots, x_M\}$ of distinct points, the matrix $\mathbf{G} \in \mathbb{R}^{M\times M}$ defined by $\mathbf{G}_{ij} = \Phi(\|x_i - x_j\|)$ is positive definite on the set subspace $\{\mathbf{a} \in \mathbb{R}^M \mid \sum_{k=1}^{M} \mathbf{a}_k \pi(x_k) = 0 \quad \forall \pi \in \mathcal{P}_p\}$. The interpolation problem (3.1) is then replaced with the problem of finding If of the form $\sum_{j=1}^{N} u_j \Phi(\|\cdot - x_j\|) + \pi$ for a $\pi \in \mathcal{P}_p$ such that $If(x_i) = f_i$ for $i = 1, \ldots, N$. For a detailed survey of RBF functions we refer to [25,26,61,110].

The approximation theory for RBFs can be traced back to the work of Duchon, [35,36], where in particular the RBFs Φ that are fundamental solutions of the iterated Laplacian are analyzed.

The approximation result Proposition 3.11 is just one example of a setting where the function f to be approximated is not in the native space H_Φ. We refer to [24] and the reference there for a more detailed discussion.

It should be noted that even for the compactly supported radial basis functions of Example 3.4 the Gram matrix $\mathbf{G}$ of the interpolation problem or the stiffness matrix, if they are used as shape functions in Galerkin methods, is not sparse. Multiresolution analysis ideas have been proposed and employed in the context of radial basis functions. For example, if for each level $l \in \{0, \ldots, L\}$ a collection of points $x_{i,l}$, $i = 1, \ldots, N_l$, is given or constructed, one can approximate from the space $\operatorname{span}\{\Phi(\|(\cdot - x_{i,l})/h_l\|_2) \,|\, i = 1, \ldots, N_l, l = 0, \ldots, L\}$, where the scaling parameters h_l are additional, suitably chosen parameters. We refer [61] and the references there for more details.

4 Partition of Unity Method and Generalized FEM

The approximation properties of the spaces discussed in Sections 2, 3 ultimately rely on the local approximation properties of polynomials. The Partition of Unity Method/generalized FEM [7,78,79,82,9,101–103] is a generalization of the classical FEM and the above approaches in that it allows the creation of special approximation spaces that are tailored to a particular problem. As we will see in Theorem 4.1, one can construct, starting from local approximation spaces V_i, a global approximation space V by means of a partition of unity, where the global space V inherits the approximation properties from the local spaces V_i. As we will illustrate in Section 5, the approximation properties of the local spaces V_i need not rely on those of polynomials.

4.1 Approximation Theory

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $\{\psi_i \,|\, i = 1, \ldots, N\}$ be a collection of $W^{1,\infty}(\Omega)$ functions. Set $\Omega_i := (\operatorname{supp}\psi_i)^\circ \subset \Omega$, $h_i := \operatorname{diam}\Omega_i$, and assume*

$$\|\psi_i\|_{L^\infty(\Omega)} \le C_\infty, \qquad \|\nabla\psi_i\|_{L^\infty(\Omega)} \le \frac{C_G}{h_i} \qquad i = 1, \ldots, N,$$

$$\sum_{i=1}^N \psi_i \equiv 1 \quad \text{on } \Omega, \qquad \sup_{x\in\Omega} \operatorname{card}\{i \in \mathbb{N} \,|\, x \in \Omega_i\} \le M.$$

Assume that each Ω_i, $i = 1, \ldots, N$, is a Lipschitz domain as well.
For each $i \in \{1, \ldots, N\}$ let $V_i \subset H^1(\Omega_i)$ be given and set

$$V := \sum_{i=1}^N \psi_i V_i = \left\{ \sum_{i=1}^N \psi_i v_i \,|\, v_i \in V_i \right\}. \tag{4.1}$$

Then $V \subset H^1(\Omega)$.
Assume that for a given $u \in H^1(\Omega)$ the spaces V_i have a local approximation property*, i.e., there exist $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i)} =: \varepsilon_1(i), \qquad \|\nabla(u - v_i)\|_{L^2(\Omega_i)} =: \varepsilon_2(i). \tag{4.2}$$

Then the approximant $v := \sum_{i=1}^N \psi_i v_i \in V$ satisfies

$$\|u - v\|^2_{L^2(\Omega)} \le M C_\infty^2 \sum_{i=1}^N |\varepsilon_1(i)|^2, \tag{4.3}$$

$$\|\nabla(u - v)\|^2_{L^2(\Omega)} \le 2M \sum_{i=1}^N \left[\left(\frac{C_G}{h_i}\right)^2 |\varepsilon_1(i)|^2 + C_\infty^2 |\varepsilon_2(i)|^2 \right]. \tag{4.4}$$

Proof. The assumption that the patches Ω_i be Lipschitz domain is required to ensure that $V \subset H^1(\Omega)$ as we now show: By the extension result Theorem A.1, there exist extension operators $E_i : H^1(\Omega_i) \to H^1(\mathbb{R}^d)$. For each $i \in \{1, \ldots, N\}$ we choose $v_i \in V_i$. We then check that $\psi_i(E_i v_i) \in H^1(\Omega)$ as the product of a Lipschitz continuous function and an $H^1(\Omega)$-function. Hence, $\sum_{i=1}^N \psi_i E_i v_i \in H^1(\Omega)$. By the support properties of the functions ψ_i we get $\sum_{i=1}^N \psi_i v_i = \sum_{i=1}^N \psi_i E_i v_i$. In this way, we see that $V = \sum_{i=1}^N \psi_i V_i \subset H^1(\Omega)$. We will now prove (4.4) and leave (4.3) as an exercise. Using $\sum_{i=1}^N \psi_i \equiv 1$ on Ω we can write with the product rule

$$\nabla(u - \sum_{i=1}^N \psi_i v_i) = \nabla \sum_{i=1}^N \psi_i (u - v_i) = \sum_{i=1}^N (u - v_i)\nabla\psi_i + \psi_i \nabla(u - v_i).$$

This allows us to bound the error $e := u - \sum_{i=1}^N \psi_i v_i$ by

$$\int_\Omega |e|^2\,dx \le 2\int_\Omega \left|\sum_{i=1}^N (u-v_i)\nabla\psi_i\right|^2 + \left|\sum_{i=1}^N \psi_i \nabla(u-v_i)\right|^2 dx. \tag{4.5}$$

The assumption $\sup_{x\in\Omega} \operatorname{card}\{i \,|\, x \in \Omega_i\} \le M$ implies that for each fixed $x \in \Omega$ each of the sums consists of at most M terms. Hence, exploiting the bound $(\sum_{j=1}^M |a_j|)^2 \le M \sum_{j=1}^M |a_j|^2$, which is valid for any finite sequence $(a_j)_{j=1}^M$, and using the bounds on the functions ψ_i, $\nabla\psi_i$, we arrive at

$$\begin{aligned}
\left|\sum_{i=1}^N (u-v_i)(x)\nabla\psi_i(x)\right|^2 &\le M\sum_{i=1}^N |\nabla\psi_i(x)|^2\,|(u-v_i)(x)|^2 \\
&\le MC_G^2 \sum_{i=1}^N \frac{1}{h_i^2}|(u-v_i)(x)|^2, \\
\left|\sum_{i=1}^N \psi_i(x)\nabla(u-v_i)(x)\right|^2 &\le M\sum_{i=1}^N |\psi_i(x)|^2\,|\nabla(u-v_i)(x)|^2 \\
&\le MC_\infty^2 \sum_{i=1}^N |\nabla(u-v_i)(x)|^2.
\end{aligned}$$

Inserting these bounds in (4.5) then gives the desired estimate. □

Remark 4.2. Theorem 4.1 is formulated for L^2-based spaces—an extension to spaces $W^{k,q}$, $1 \le q < \infty$ is possible. If the partition of unity is smoother, i.e., $\psi_i \in W^{k,\infty}(\Omega)$ and the local spaces V_i satisfy $V_i \subset H^k(\Omega_i)$, then again $V \subset H^k(\Omega)$ and analogous approximation results in H^k can be obtained. Thus, applications requiring subspaces of $H^k(\Omega)$ instead of $H^1(\Omega)$ as approximation spaces can easily be constructed. ■

A prominent example of a partition of unity satisfying the assumptions of Theorem 4.1 consists of the standard basis of a FEM space:

Example 4.3. Let $\mathcal{T}$ be a shape-regular mesh on a domain $\Omega \subset \mathbb{R}^d$. Let $\{x_i \,|\, i = 1,\dots,N\}$ be the vertices of $\mathcal{T}$ and let $\{\psi_i \,|\, i = 1,\dots,N\}$ be the standard piecewise linear basis of $S^{1,1}(\mathcal{T})$. Then $\{\psi_i \,|\, i = 1,\dots,N\}$ is a partition of unity satisfying the assumptions of Theorem 4.1. ■

Remark 4.4. Partitions of unity are systems of functions that reproduce polynomials of degree $p = 0$. Hence, one can obtain a partition of unity with the Shephard construction of Exercise 2.14 from a collection of particles $X_N = \{x_i \,|\, i = 1,\dots,N\}$ and corresponding weight functions w_i, $i = 1,\dots,N$. As discussed in Section 2.3, the regularity of the shape functions obtained in this way is determined by the regularity of the weight functions w_i.

Of particular note in the Shephard construction is the case when each patch Ω_i contains an open subset Ω_i' such that $\Omega_i' \cap \Omega_j = \emptyset$ for $j \neq i$. Then $\psi_i \equiv 1$ on Ω_i'. Such a partition of unity is employed in the particle partition of unity method of [96]. ∎

For practical implementations, it is important to identify a basis of the space V. It appears natural to base it on bases $\mathcal{B}_i = \{b_{i,j} \,|\, j = 1, \dots, \dim V_i\}$, $i = 1, \dots, N$, and consider the set $\mathcal{B} = \{\psi_i b_{i,j} \,|\, i = 1, \dots, N, j = 1, \dots, \dim V_i\}$. In general $\mathcal{B}$ is *not* a basis of V as the following exercise shows:

Exercise 4.5. Let $\Omega = (0, 1)$ and $0 = x_0 < x_1 < \cdots x_N = 1$ be a partition of Ω. let ψ_i, $i = 0, \dots, N$, be the standard piecewise linear hat function associated with node x_i. Let $V_i = \mathcal{P}_p = \operatorname{span}\{b_j \,|\, j = 0, \dots, p\}$ for each $i = 0, \dots, p$. Show by a dimension argument that $\{\psi_i b_j \,|\, i = 0, \dots, N, j = 0, \dots, p\}$ is not a basis of $V = \sum_{i=0}^{N} \psi_i V_i$. ∎

If the partition of unity is suitably chosen, then the set $\mathcal{B}$ is a basis of V:

Exercise 4.6. Let the partition of unity $\{\psi_i \,|\, i = 1, \dots, N\}$ be such that for each i there exists an open set Ω_i' with $\Omega_i' \cap \operatorname{supp} \psi_j = \emptyset$ for all $j \neq i$. Show: The set $\mathcal{B}$ is a basis of V. This fact is exploited in the particle partition of unity of [96]. ∎

4.2 Example: polynomial local approximation spaces

There are several ways to employ the approximation result Theorem 4.1 in a numerical scheme. One way is to use polynomials as local approximation spaces V_i; the partition of unity method could, for example, be obtained from a collection of particles and the partition of unity is based on the Shepard function of Exercise 2.14. This is approach is pursued in a series of papers by Griebel and Schweitzer [47–51] and collected in the monograph [96]. The approximation properties of this method are comparable to the classical FEM as is shown in the following Exercises 4.7, 4.8.

Exercise 4.7. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. For each patch Ω_i choose a polynomial degree $p_i \in \mathbb{N}_0$ and set $V_i := \mathcal{P}_{p_i}$. For each $i \in \{1, \dots, N\}$ let $\widetilde{B}_i$ be a ball of diameter $\operatorname{diam} \widetilde{B}_i \leq C h_i$ such that $\Omega_i \subset \widetilde{B}_i$. Assume additionally that the balls $\widetilde{B}_i$ satisfy an overlap condition, i.e.,

$$\sup_{x \in \mathbb{R}^d} \{i \,|\, x \in \widetilde{B}_i\} \leq M. \tag{4.6}$$

Show: Under the hypotheses of Theorem 4.1 on the functions ψ_i there holds

$$\inf_{v \in V_N} \|u - v\|^2_{H^1(\Omega)} \leq C \sum_{i=1}^{N} h_i^{2(\min\{p_i+1,k\}-1)} \|u\|^2_{H^k(\widetilde{B}_i)}.$$

In particular, if $p_i = p$ for all i and if we set $h := \max h_i$, then

$$\inf_{v \in V_N} \|u - v\|^2_{H^1(\Omega)} \leq C h^{2\min\{p,k-1\}} \|u\|^2_{H^k(\Omega)}.$$

■

The size diam $\widetilde{B}_i$ of the ball $\widetilde{B}_i$ in Exercise 4.7 plays the role of the local mesh size in the classical FEM. Graded meshes can also be simulated as illustrated in the following exercise.

Exercise 4.8. Continue Exercise 4.7 for the approximation of singularity functions of the form $u(r,\varphi) = r^\alpha \Theta(\varphi)$ as discussed in Exercise 2.11. Let $\Omega = (0,1/2)^2$, let X_N be the particle distribution given in Exercise 2.11 with $\beta > p/\alpha$. Let the patches Ω_i be such that $x_i \in \Omega_i \subset \widetilde{B}_i$, where $\widetilde{B}_i = B_{\rho_i}(x_i)$ with ρ_i given in Exercise 2.11. Let $V_i = \mathcal{P}_p$ as in the preceding exercise. Show: (4.6) holds, and the approximation space V satisfies

$$\inf_{v \in V} \|u - v\|_{H^1(\Omega)} \leq C N^{-p},$$

i.e., the optimal rate of convergence is achieved. ■

5 Examples of operator adapted approximation spaces

Theorem 4.1 allows us to construct approximation spaces V where the global space V inherits the approximation properties of the local spaces V_i. These spaces can be custom tailored to the approximation of a function u. We illustrate this with a few examples.

5.1 A one-dimensional example

We consider the following one-dimensional model problem:

$$Lu := -(a(Mx)u')' + b(x)u = f \quad \text{on } \Omega = (0,1), \quad u(0) = u(1) = 0, \tag{5.1}$$

where $M \in \mathbb{N}$ and $a \in L^\infty(\mathbb{R})$ is 1-periodic. Additionally, we assume ellipticity, i.e., $0 < \underline{a} \leq a(x)$ for all $x \in \mathbb{R}$ and $0 \leq b(x) \leq \|b\|_{L^\infty(\Omega)}$ for all $x \in \Omega$. If M is large, then the coefficient $a(M\cdot)$ is highly oscillatory and so is the solution u. The standard FEM performs poorly in the situation, namely, convergence is only observed under the assumption of scale resolution, i.e., if the mesh size h is sufficiently small to resolve all scales. The following example illustrates this.

Example 5.1. We consider the case $a = \frac{1}{2+\cos(2\pi x)}$, $b \equiv 0$, and $f \equiv 1$. In the left graph in Fig. 5.3 we show the convergence behavior of the classical FEM

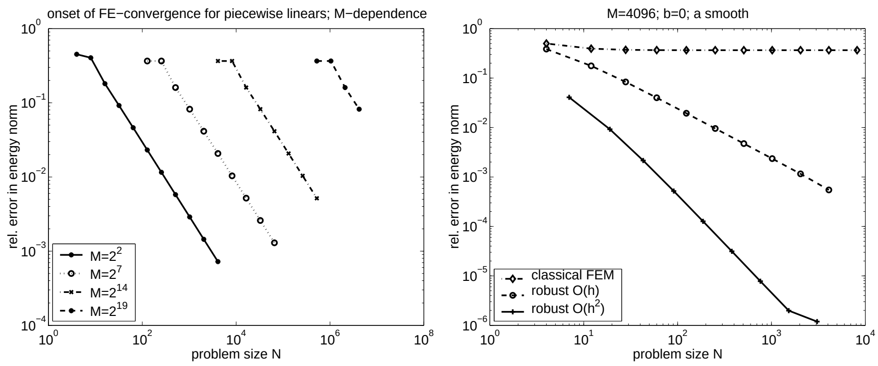

Fig. 5.3. Left: Convergence of the classical FEM. Right: Convergence of the PUM.

based on the space $S_0^{1,1}(\mathcal{T})$ on uniform meshes. The error measure is relative error in the energy norm, i.e.,

$$\frac{\|u-u_N\|_E}{\|u\|_E} = \sqrt{\frac{\int_\Omega a(Mx)|(u-u_N)'|^2\,dx}{\int_\Omega a(Mx)|u'|^2\,dx}}. \tag{5.2}$$

The solution u can be computed analytically and it can be checked that $\|u'\|_{L^2(\Omega)} \sim M$ and $\|u''\|_{L^2(\Omega)} = O(M^2)$. The classical FEM convergence analysis then gives

$$\frac{\|(u-u_N)'\|_{L^2(\Omega)}}{\|u'\|_{L^2(\Omega)}} \le C \min\left\{1, \frac{h\|u''\|_{L^2(\Omega)}}{\|u'\|_{L^2(\Omega)}}\right\} \le C\min\{1, hM\}. \tag{5.3}$$

We clearly observe in Fig. 5.3 the expected asymptotic first order convergence; nevertheless, the asymptotic convergence behavior is not observed until $h \approx 1/M$, that is, until scale resolution is reached. Note that this is in agreement with (5.3). ∎

It is possible to design local approximation spaces that have good approximation properties for the solution of (5.1).

Lemma 5.2. *Let $I = (x_0, x_0+h)$ and $\gamma < 1$. Let $h^2\frac{\|b\|_{L^\infty(I)}}{\underline{a}} \le \gamma < 1$. Let $\mathcal{B} = \{u_0, u_1\}$ be a* fundamental system *for L, i.e., $Lu_0 = Lu_1 = 0$ on I and u_0, u_1 are linearly independent. Then there exists a $C > 0$ depending only on $\underline{a}$, $\|a\|_{L^\infty(I)}$, $\|b\|_{L^\infty(I)}$, γ, such that for a solution $u \in H^1(I)$ of $Lu = f$ there holds*

$$\inf_{v\in V} \|u-v\|_{L^\infty(I)} + h\|(u-v)'\|_{L^\infty(I)} \le Ch^2\|f\|_{L^\infty(I)},$$

where $V := \operatorname{span}\mathcal{B}$.

Proof. Since $f \in L^\infty(I)$ and $u \in H^1(I)$ we get that u and au' are continuous. We then choose $v \in V$ such that $v(x_0) = u(x_0)$ and $(av')(x_0) = (au')(x_0)$.

The error $e := u - v$ then satisfies $e(x_0) = 0$ and $(ae')(x_0)$ together with $Le = f$. The differential equation $Le = f$ gives us $-(ae')' = f - be$ so that

$$
\begin{aligned}
|e(x)| &\le \left| \int_{x_0}^{x} e'(t)\,dt \right| \le h\|e'\|_{L^\infty(I)}, \\
|e'(x)| &\le \frac{1}{\underline{a}} |(ae')(x)| \le \frac{1}{\underline{a}} \left| \int_{x_0}^{x} f - be\,dt \right| \le \frac{1}{\underline{a}} h\|f\|_{L^\infty(I)} + \frac{\|b\|_{L^\infty(I)}}{\underline{a}} h\|e\|_{L^\infty(I)}.
\end{aligned}
$$

Combining these two estimates, we arrive at

$$
\|e'\|_{L^\infty(I)} \le \frac{\|f\|_{L^\infty(I)}}{\underline{a}} h + \underbrace{h^2 \frac{\|b\|_{L^\infty(I)}}{\underline{a}}}_{\le \gamma < 1} \|e'\|_{L^\infty(I)},
$$

which allows us to conclude $\|e'\|_{L^\infty(I)} \le h \frac{1}{\underline{a}(1-\gamma)} \|f\|_{L^\infty(I)}$. $\square$

Remark 5.3. It should be noted that the approximation spaces constructed in Lemma 5.2 merely require a and b to be L^∞—no further regularity is required. ■

Extensions of the approximation result Lemma 5.2 are obtained in the following exercise.

Exercise 5.4. (a) Construct a one-dimensional space $V_0 = \operatorname{span}\{u_0\}$ such that $u_0(x_0) = 0$ and V_0 satisfies, for $u(x_0) = 0$ and $Lu = f$,

$$
\inf_{v \in V_0} \|u - v\|_{L^\infty(I)} + h\|(u - v)'\|_{L^\infty(I)} \le Ch^2 \|f\|_{L^\infty(I)}.
$$

(b) Let u_2 be such that $Lu_2 = 1$. Let u_0, u_1 be defined in Lemma 5.2. Set $V_2 := \operatorname{span}\{u_0, u_1, u_2\}$. Show:

$$
\inf_{v \in V_2} \|u - v\|_{L^\infty(I)} + h\|(u - v)'\|_{L^\infty(I)} \le Ch^3 \|f'\|_{L^\infty(I)}.
$$

(c) Construct a two-dimensional space $V_{0,2} = \operatorname{span}\{u_0, u_1\}$ such that $v(x_0) = 0$ for $v \in V_{0,2}$ and $V_{0,2}$ satisfies, for $u(x_0) = 0$ and $Lu = f$,

$$
\inf_{v \in V_{0,2}} \|u - v\|_{L^\infty(I)} + h\|(u - v)'\|_{L^\infty(I)} \le Ch^3 \|f'\|_{L^\infty(I)}.
$$

■

Example 5.5. We use the partition of unity method (PUM) with a partition of unity given by the piecewise linear functions on a uniform mesh with mesh size h for the approximation of the solution of (5.1) where $a = \frac{1}{2+\cos(2\pi x)}$, $b \equiv 0$, and $f(x) = x$. We choose $M = 4096$. In the first experiment the local approximation spaces are taken as the spaces V constructed in Lemma 5.2

for the internal nodes and the space V_0 constructed in Exercise 5.4 for the two nodes at the boundary of Ω. In view of Lemma 5.2 and Theorem 4.1 we expect convergence $O(h)$ in the energy norm (cf. (5.2)), where the constant in the $O(h)$ convergence is *independent* M. The convergence behavior of this projection method is depicted in the graph labelled "robust $O(h)$" in the right picture of Fig. 5.3. Since the problem size $N \sim 1/h$, the expected convergence $O(h)$ is indeed confirmed numerically.
In the second experiment, the local spaces for the internal nodes are taken as the spaces V_2 of Exercise 5.4 and the spaces $V_{0,2}$ of Exercise 5.4 for the boundary nodes. In view of Exercise 5.4 and Theorem 4.1 we expect a convergence $O(h^2)$ in the energy norm. This expectation is confirmed by the graph labelled "robust $O(h^2)$" in the right picture of Fig. 5.3. Again, the constant hidden in the $O(h^2)$ convergence result is independent of M. For more details on this one-dimensional problem, we refer to [82]. ∎

Exercise 5.6. The approximation properties of the space V constructed in Lemma 5.2 can also be understood by transforming the problem. Consider the case $b \equiv 0$. Then

$$V = \operatorname{span}\left\{1, \int_{x_0}^{x} \frac{1}{a(t)}\,dt\right\}.$$

Let $f \in L^2(I)$ and define the change of variable $\widetilde{x} := \int_{x_0}^{x} \frac{1}{a(t)}\,dt$. Show: The function $\widetilde{u}(\widetilde{x}) := u(x)$ is in H^2 (*hint:* write down a differential equation satisfied by $\widetilde{u}$). Hence it can be approximated well from $\mathcal{P}_1$. Infer from that approximation results for u for the approximation from V. ∎

Remark 5.7. The construction in Lemma 5.2 exploits in a crucial way the fact that a one-dimensional problem is considered: the solution space of homogeneous linear second order differential equations is two-dimensional. Nevertheless analogous approximation results can be shown for quasi one-dimensional cases. Exercise 5.6 illustrates an old, but powerful tool of numerical mathematics, namely, the use of suitable transformations. This device is also the reasons for the results of [7]. In [7] problems of the form

$$-\partial_x\left(a(x)\partial_x u\right) - \partial_y\left(a(x)\partial_y u\right)) = f \qquad \text{on } (0,1)^2$$

are considered; the coefficient $a \geq \underline{a} > 0$ depends on the single variable x but may be merely bounded and measurable. For such problems, it is shown that local approximation space of the form

$$V := \operatorname{span}\left\{1, \int_{x_0}^{x} \frac{1}{a(t)}\,dt, y\right\}$$

can lead to the optimal rate $O(h)$. ∎

5.2 Laplace's Equation

We consider the two-dimensional case $\Omega \subset \mathbb{R}^2$ and solutions to Laplace's equation

$$-\Delta u = 0 \qquad \text{on } \Omega. \tag{5.4}$$

It seems reasonable to try to approximate the solutions to a differential equation with systems of functions that likewise solve the differential equation. For the Laplace equation one such system is that of harmonic polynomials:

$$\mathcal{HP}_p := \operatorname{span}\{\operatorname{Re} z^n, \operatorname{Im} z^n \mid n = 0, \dots, p\}, \tag{5.5}$$

where $z = x + \mathbf{i}y \in \mathbb{C}$. Note that $\dim \mathcal{HP}_p = 2p + 1$. We have exponential convergence if the function u to be approximated is harmonic on set that strictly contains the domain of interest:

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let $\Omega' \subset\subset \Omega$ be a compact subset. Let $k \in \mathbb{N}_0$. Let u satisfy $-\Delta u = 0$ on Ω. Then there exist C, $b > 0$ such that for all $p \in \mathbb{N}_0$*

$$\inf_{v \in \mathcal{HP}_p} \|u - v\|_{W^{k,\infty}(\Omega')} \le Ce^{-bp}.$$

Proof. This result is due to Szegö. We refer to [80] for a proof. □

Example 5.9. We consider the approximation of the solution u of (5.4), where $\Omega = (0, 1)^2$. The exact solution is given by

$$u(x, y) = \operatorname{Re}\left(\frac{1}{a^2 + z^2} + \frac{1}{a^2 - z^2}\right), \qquad a = 1.05.$$

Ω is partitioned into n^2 square of equal size, and the partition of unity is taken as the standard bilinear hat functions associated with this mesh. This partition of unity is fixed and the local approximation spaces V_i are taken as $\mathcal{HP}_p$ for different values of $p \in \mathbb{N}_0$. The numerical results in the left graph in Fig. 5.4 present the result of the minimization problem

$$\min\Big\{\frac{\|\nabla(u - v)\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}} \,\Big|\, v \in V := \sum_{i=1}^{(n+1)^2} \varphi_i V_i\Big\}$$

in dependence on the polynomial degree p. ■

Algebraic convergence results are also available:

Theorem 5.10. *Let $\Omega \subset \mathbb{R}^2$ be star-shaped with respect to a ball and let Ω satisfy an exterior cone condition with angle $\lambda\pi$. Let $k \ge 1$ and let $u \in H^k(\Omega)$ satisfy (5.4). Then there exists $C > 0$ and harmonic polynomials $u_p \in \mathcal{HP}_p$ such that*

$$\|u - u_p\|_{H^j(\Omega)} \le C\left(\frac{\ln(p + 2)}{p + 2}\right)^{\lambda(k-j)}, \qquad j = 0, 1.$$

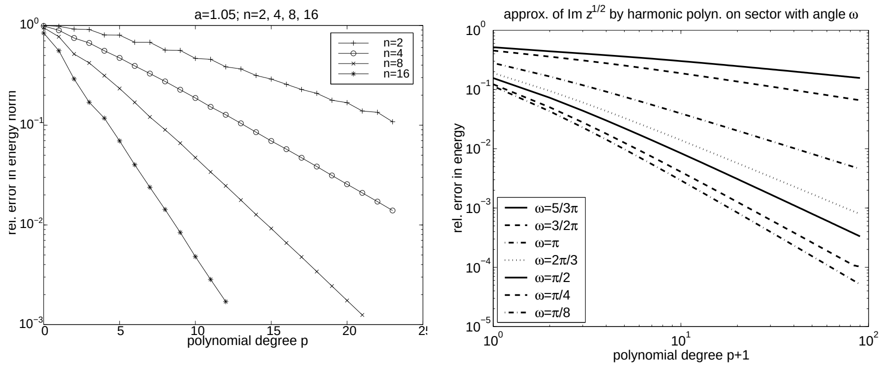

Fig. 5.4. Left: Exponential convergence of Example 5.9. Right: Algebraic convergence of Example 5.11.

Proof. See [80]. □

Example 5.11. Theorem 5.10 can be sharpened in the following situation (see [80] for a more detailed discussion of this effect): Define the sector $S_\omega = \{(r\cos\varphi, r\sin\varphi) \,|\, 0 < r < 1, 0 < \varphi < \omega\}$ and let $u(x,y) = \operatorname{Re} z^\alpha$ or $u(x,y) = \operatorname{Im} z^\alpha$ for some $\alpha > 0$. Then we have with $\lambda = 2 - \frac{\omega}{\pi}$ and any $\varepsilon > 0$

$$\inf_{v \in \mathcal{HP}_p} \|u - v\|_{H^1(S_\omega)} \leq C_\varepsilon p^{-\lambda\alpha+\varepsilon},$$

where C_ε depends on α, ω, and ε. Fig. 5.4 illustrates this convergence behavior by plotting for different values of ω the result of the minimization problem

$$\min\left\{ \frac{\|\nabla(u_{1/2} - v)\|^2_{L^2(S_\omega)}}{\|\nabla u_{1/2}\|^2_{L^2(S_\omega)}} \,\Big|\, v \in \mathcal{HP}_p \right\}, \qquad u_{1/2} = \operatorname{Im} z^{1/2},$$

in dependence on the polynomial degree p. It is noteworthy that in this particular example λ may be bigger than 1—this cannot be expected in the situation of Theorem 5.10. ∎

Remark 5.12. The system of harmonic polynomials is just one possible choice. Near corners, the solution of (5.4) has singularities, which are known. The corresponding singularity functions could be used as approximation systems. We will describe the idea of augmenting a standard FEM space with such singularity function in more detail in Section 6. ∎

5.3 Helmholtz equation

We consider for two-dimensional problems the Helmholtz equation

$$-\Delta u - k^2 u = 0 \qquad \text{on } \Omega \subset \mathbb{R}^2, \tag{5.6}$$

and we discuss the following two choices of local approximation systems:

1. Systems of *plane waves*, $W(p)$, given by

$$W(p) := \operatorname{span}\left\{e^{\mathbf{i}k\omega_n\cdot(x,y)} \mid n = 0,\dots,p-1\right\}, \tag{5.7}$$

   where the vectors ω_n are given by $\omega_n := (\cos\frac{2\pi n}{p}, \sin\frac{2\pi n}{p})^\top$.
2. *Generalized harmonic polynomials* given by

$$V(p) := \operatorname{span}\left\{J_n(kr)\sin(n\varphi), J_n(kr)\cos(n\varphi) \mid n = 0,\dots,p\right\}, \tag{5.8}$$

   where we employed polar coordinates (r,φ) in the definition of $V(p)$; the functions J_n are the first kind Bessel function.

We note that $\dim V(p) = O(p)$, $\dim W(p) = O(p)$. These spaces have the following approximation properties:

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega' \subset\subset \Omega$ be a compact subset. Let u solve (5.6). Then there exist C, $b > 0$ such that for all $p \in \mathbb{N}$, $p \geq 2$:*

$$\inf_{v\in V(p)} \|u-v\|_{H^1(\Omega')} \leq Ce^{-bp}, \qquad \inf_{v\in W(p)} \|u-v\|_{H^1(\Omega')} \leq Ce^{-bp/\log p}. \tag{5.9}$$

Proof. The first estimate is proved in [80]. The second one can be proved using the arguments detailed in Section C.2. □

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^2$ be star-shaped with respect to a ball. Let Ω satisfy an exterior cone condition with angle $\lambda\pi$. Let $u \in H^k(\Omega)$, $k \geq 1$, solve (5.6). Then there exists $C > 0$ such that*

$$\inf_{v\in V(p)} \|u-v\|_{H^1(\Omega)} \leq C\left(\frac{\log(p+2)}{p+2}\right)^{\lambda(k-1)}, \tag{5.10}$$

$$\inf_{v\in W(p)} \|u-v\|_{H^1(\Omega)} \leq C\left(\frac{\log^2(p+2)}{p+2}\right)^{\lambda(k-1)}. \tag{5.11}$$

Proof. (5.10) is proved in [80]. See Section C.2 for the proof of (5.11). □

Example 5.15. The function

$$u(x,y) = e^{\mathbf{i}k(\cos\theta,\sin\theta)}, \qquad \theta = \frac{\pi}{16},$$

is a solution of (5.6). Let $\Omega = (0,1)$, and let g be defined on $\partial\Omega$ by $g := \partial_n u + \mathbf{i}ku$. Then u solves

$$-\Delta u - k^2 u = 0 \quad \text{on } \Omega, \qquad \partial_n u + \mathbf{i}ku = g \quad \text{on } \partial\Omega. \tag{5.12}$$

Let Ω be partitioned into $n\times n$ squares of equal size. We take as the partition of unity ψ_i, $i = 1,\dots,(n+1)^2$, the standard bilinear hat functions associated

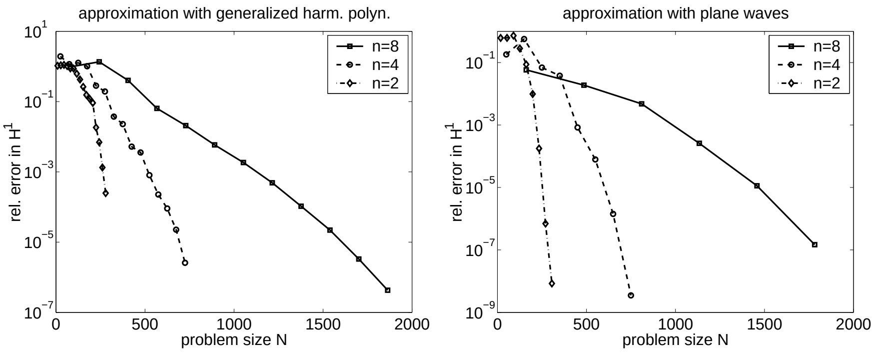

Fig. 5.5. Operator adapted methods for Helmholtz equation; see Example 5.15. Local approximation space $V(p)$ (left) and $W(p)$ (right).

with the $(n+1)^2$ nodes. The approximation space V is then constructed as in Theorem 4.1 with local spaces taken either as $V(p)$ (with p ranging from 1 to 15) or as $W(p)$ (with $p \in \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38\}$). Contrary to our exposition so far, all spaces are taken as spaces over the field $\mathbb{C}$ instead of $\mathbb{R}$. The numerical approximation u_N is obtained as the standard Galerkin approximation for problem (5.12), viz.,

$$\text{Find } u_N \in V \text{ s.t. } \int_\Omega (\nabla u_N \cdot \nabla \overline{v} - k^2 u \overline{v}) + \mathbf{i}k \int_{\partial\Omega} u\overline{v} = \int_\Omega f\overline{v} + \int_{\partial\Omega} g\overline{v} \quad \forall v \in V.$$

Theorem 5.13 suggests that an exponential rate of convergence could be achieved. The numerical results for $k = 32$ are displayed in Fig. 5.5. Indeed, we observe for fixed n an exponential convergence in $p \sim N$ for the relative error $\|u - u_N\|_{H^1(\Omega)}/|u\|_{H^1(\Omega)}$. We refer to [79] for more details. ■

5.4 Linear Elasticity

In two-dimensional linear elasticity and in the absense of body forces, the displacement field (u, v) satisfies the following system of equations:

$$\partial_x \sigma_x + \partial_y \tau_{xy} = 0, \qquad \partial_x \tau_{xy} + \partial_y \sigma_y = 0; \tag{5.13}$$

here, the stresses σ_x, σ_y, and τ_{xy} are defined by

$$\sigma_x = \lambda(\partial_x u + \partial_y v) + 2\mu\partial_x u, \quad \sigma_y = \lambda(\partial_x u + \partial_y v) + 2\mu\partial_y v, \quad \tau_{xy} = \mu(\partial_y u + \partial_x v).$$

The material constants λ, μ are called the Lamé constants.

Remark 5.16. The above system is written for the so-called plane strain case. For plane stress, λ should be replaced with $\lambda^* = 2\lambda\mu/(\lambda + 2\mu)$. ■

Let Ω be simply connected. By [85], the displacement field (u, v) can then be expressed in terms of two holomorphic functions φ, ψ, namely,

$$2\mu\,[u(x,y) + \mathbf{i}v(x,y)] = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi}(z); \tag{5.14}$$

here, we set $\kappa = (\lambda + 3\mu)/(\lambda + \mu)$. This representation is unique if we require additionally $\varphi(z_0) = 0$ for an arbitrarily chosen point z_0. This representation suggests to use as an approximation space for the approximation of the complex function $u + \mathbf{i}v$ the space

$$V_p^{elast} := \operatorname{span}\{\kappa\pi(z) - z\overline{\pi'(z)} - \overline{\rho(z)} \,|\, \pi, \rho \in \mathcal{H}_p\}, \tag{5.15}$$

where $\mathcal{H}_p$ denote the space of (complex) polynomials of degree p. An approximation result analogous to Theorem 5.10 can indeed be obtained:

Theorem 5.17. *Let $\Omega \subset \mathbb{R}^2$ be star-shaped with respect to a ball. Let Ω satisfy an exterior cone condition with angle $\hat\lambda\pi$. Let $m \in \mathbb{N}$, $s \in [0,1)$ and assume that the displacement field $(u,v) \in H^{m+s}(\Omega)$ satisfies the homogeneous elasticity equations (5.13). Then the function $\mathbf{u} := u + \mathbf{i}v$ can be approximated from V_p^{elast} such that*

$$\inf_{\mathbf{u}_{ap} \in V_p^{elast}} \|\mathbf{u} - \mathbf{u}_{ap}\|_{H^1(\Omega)} \le C \left(\frac{\log(p+2)}{p+2}\right)^{\hat\lambda(m+s-1)} \|\mathbf{u}\|_{H^{m+s}(\Omega)}.$$

Proof. See Section C.3. □

Remark 5.18. The proof of Theorem 5.17 shows that the improved rate of convergence for the typical singularity functions that we observed in Example 5.11 are also obtained for the elasticity equations. ∎

5.5 Further examples

The Laplace equation and the Helmholtz equation are merely two examples of elliptic equations for which special approximation systems can be constructed. A more general theory by S. Bergman [16–18] and I.N. Vekua [105] is in fact available: For two-dimensional elliptic equations of the form

$$-\Delta u + a(x,y)\partial_x u + b(x,y)\partial_y u + c(x,y)u = 0, \tag{5.16}$$

where the functions a, b, c are real analytic on Ω, there exists a linear operator ReV that maps functions holomorphic on Ω onto solutions of solution of (5.16). Essentially, this operator is a bijection and bicontinuous in Sobolev norms. That is: regularity assertions for u can be translated into regularity assertions for the corresponding holomorphic functions; this function may then be approximation by (complex) polynomials; the image of (complex) polynomials under ReV then yields a good approximation space. In some cases, the operator ReV can be computed explicitly (e.g., in the case of the

Helmholtz equations, where the space $V(p)$ is precisely the image of complex polynomials under the map ReV); we refer to Appendix C and [80] for more details on this. The representation theory of Bergman and Vekua is, due to its close link with complex analysis, largely a two-dimensional theory. Some extensions to three dimensions have been done in [28].

5.6 Local approximation spaces obtained numerically

In the above examples the local approximation spaces were given in closed form. They can, however, be obtained numerically as well. For example, while the form of the singularity functions of linear elasticity is known, the precise exponents have to be determined as solutions of small auxiliary problems. More in the spirit of domain decomposition is the following approach for problems of the form $Lu = 0$: For each patch Ω_i, one chooses a finite dimensional space $V_{i,\partial\Omega_i} = \operatorname{span}\{\tilde{b}_{i,j} \,|\, j = 1, \ldots, N_i\}$ of functions that are defined on $\partial\Omega_i$. The space V_i is then obtained by (numerically) solving boundary value problems

$$Lb_{i,j} = 0 \quad \text{on } \Omega_i, \qquad b_{i,j}|_{\partial\Omega_i} = \tilde{b}_{i,j}.$$

The total computation is therefore done in two steps: first, many local problems are solved (which can be done completely in parallel), and in a second step a global problems is solved. Conceptually, this is the approach taken for example in [5] and [58,59,37] for calculations of very heterogeneous media.

Remark 5.19. The functions $b_{i,j}$ were computed above as solutions of Dirichlet problems. The approximation space V_i could be determined by solving other boundary value problems, e.g., by solving Neumann problems. It has also been observed that it is advantageous to define them as solutions of boundary value problems defined on Ω_i', where $\Omega_i \subset\subset \Omega_i'$. We refer, for example, to [5] for more details on this. ∎

Another example of a method where the approximation spaces are determined numerically in a preprocessing step is the generalized FEM of [77,95] for problems with periodic microstructures.

5.7 Bibliographical Remarks

Approximation systems that are tailored to the differential operator are used by engineers, where such methods are known, among others, under the name of Trefftz methods, see, e.g., [63,64,56]. In the context of the partition of unity method/generalized FEM special approximation systems have been used in [69] for Helmholtz problems and in [90,34] for elasticity and crack problems. The "method of particular solutions" [43], [19] (see, in particular, the references in [19]) is closely related to the ideas presented here.

We have seen the poor performance of the classical FEM in Section 5.1. Indeed, it was already shown in [10] that the classical FEM can perform arbitrarily poorly. On the other hand, the constructions in [81] show that for reasonable classes of right-hand sides, it is in principle possible to construct good approximation spaces. Such approximation spaces have to be adapted to a particular problem at hand.

6 Augmenting classical FEM spaces

The partition of unity method/generalized FEM can be viewed as a framework for incorporating information about the problem into the approximation space. The simplest such technique is to augment a standard finite element space with special functions.

6.1 Singular functions

The power of augmenting a classical FEM space with special functions can be seen in the following model problem: Let $\Omega \subset \mathbb{R}^2$ be a polygon and consider

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{6.1}$$

If we denote by A_j, $j = 1, \dots, J$, the vertices of Ω and by $\omega_j \in (0, 2\pi)$ the internal angle of Ω at A_j, then it is well-known that the classical FEM-space $S_0^{1,1}(\mathcal{T})$ that is based on a quasi-uniform mesh $\mathcal{T}$ of mesh size h performs poorly if $\max_{j=1,\dots,J} \omega_j > \pi$; namely, the rate of convergence is

$$\inf_{v \in S_0^{1,1}(\mathcal{T})} \|u - v\|_{H^1(\Omega)} \le Ch^\alpha, \qquad \alpha = \min_{j=1,\dots,J} \frac{\pi}{\omega_j} < 1.$$

This is indeed observed in practice. By augmenting this FEM space by a few suitably chosen singularity functions, however, we recover the optimal rate of convergence. To this end, it is important to note the following regularity assertion for the solution u of (6.1):

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a polygon with vertices A_j, $j = 1, \dots, J$, and internal angles ω_j, $j = 1, \dots, J$. Define for each vertex A_j the singularity functions $S_{j,i}$, $i = 1, 2, \dots,$ by*

$$S_{j,i}(r_j, \varphi_j) := \begin{cases} r_j^{i\pi/\omega_j} \sin(i\frac{\pi}{\omega_j}\varphi_j) & \text{if } i\pi/\omega_j \notin \mathbb{N} \\ r_j^{i\pi/\omega_j} \left[\ln r_j \sin(i\frac{\pi}{\omega_j}\varphi_j) + \varphi_j \cos(i\frac{\pi}{\omega_j}\varphi_j)\right] & \text{if } i\pi/\omega_j \in \mathbb{N} \end{cases} \tag{6.2}$$

where (r_j, φ_j) represent polar coordinates with origin A_j such that the two edges of Ω meeting at A_j fall on the lines $\varphi_j = 0$ and $\varphi_j = \omega_j$.
Let $f \in H^{-1+k}(\Omega)$, $k > 0$ and $k \notin \mathbb{N}$. Then the solution u of (6.1) can be written in the form

$$u = \sum_{j=1}^{J} \sum_{\substack{i \in \mathbb{N} \\ i\frac{\pi}{\omega_j} < k}} a_{ij} S_{ij} + u_0, \tag{6.3}$$

for some numbers $a_{ij} \in \mathbb{R}$ *and* $u_0 \in H^{1+k}(\Omega)$.

Proof. Such decompositions can be found, for example, in [52,53]. □

This regularity assertion allows us to design approximation spaces that recover the optimal rate of convergence (in terms of "error vs. problem size"):

Exercise 6.2. Fix a cut-off function $\chi_j \in C_0^\infty(\mathbb{R}^2)$ for each corner A_j such that $\chi_j \equiv 1$ in a neighborhood of A_j and such that $\chi_j \equiv 0$ in a neighborhood of the vertices A_i, $i \neq j$.

(a) Show: The decomposition (6.3) can take the form

$$u = \sum_{j=1}^{J} \sum_{\substack{i \in \mathbb{N} \\ i\frac{\pi}{\omega_j} < k}} a_{ij} \chi_j S_{ij} + \tilde{u}_0,$$

where $\tilde{u}_0 \in H^{1+k}(\Omega) \cap H_0^1(\Omega)$. Additionally, $\chi_j S_{i,j} \in H_0^1(\Omega)$.

(b) Show: The space

$$V_N := S_0^{p,1}(\mathcal{T}) \oplus \operatorname{span}\{\chi_j S_{j,i} \,|\, j = 1, \ldots, J, i\frac{\pi}{\omega_j} < k\} \subset H_0^1(\Omega)$$

satisfies

$$\inf_{\in V_N} \|u - v\|_{H^1(\Omega)} \leq C h^{\min\{p,k\}}. \tag{6.4}$$

Note that $\dim V_N \sim \dim S_0^{p,1}(\mathcal{T})$. ■

The purpose of the cut-off functions χ_j is to localize the singularity functions. This could also be achieved with the aid classical FEM functions:

Exercise 6.3. Let $\mathcal{T}$ be a quasi-uniform mesh on the polygon $\Omega \subset \mathbb{R}^2$. Let $\{\psi_i \,|\, i = 1, \ldots, N_1\}$ be set of the classical piecewise linear hat functions associated with $\mathcal{T}$ and $S^{1,1}(\mathcal{T}) = \operatorname{span}\{\psi_i \,|\, i = 1, \ldots, N_1\}$. Fix $\rho > 0$ and define, for each $j \in \{1, \ldots, J\}$, the set $I_j := \{i \,|\, \operatorname{supp} \psi_i \subset B_\rho(A_j)\}$. Define

$$V_N := S_0^{p,1}(\mathcal{T}) \oplus \operatorname{span}\{\psi_i S_{j,m} \,|\, m\frac{\pi}{\omega_j} < k, \quad i \in I_j, \quad j = 1, \ldots, J\}.$$

Show: Also for this choice of approximation space the approximation property (6.4) holds. Note: $V_N \subset H_0^1(\Omega)$ and $\dim V_N \sim \dim S_0^{p,1}(\mathcal{T})$. ■

The above construction involves only classical FEM functions and the singularity functions $S_{j,i}$. Of course, since $\rho > 0$ is fixed, a rather large number of nodes is affected (see the left picture in Fig. 6.6, where the nodes that require multiplication with singularity functions are denoted •), namely, $O(h^{-2})$ nodes. A variety of practitioners have therefore looked at further simplifications:

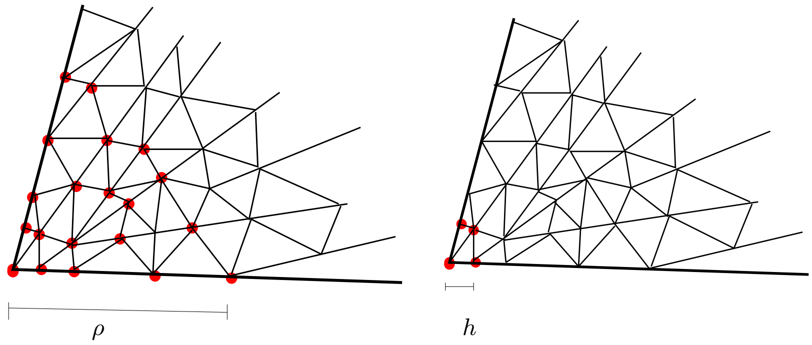

Fig. 6.6. Nodes marked • are augmented with singularity function. Left: $O(h^{-2})$ nodes are augmented to ensure optimal rate of convergence. Right: augmenting very few nodes often suffices in practice.

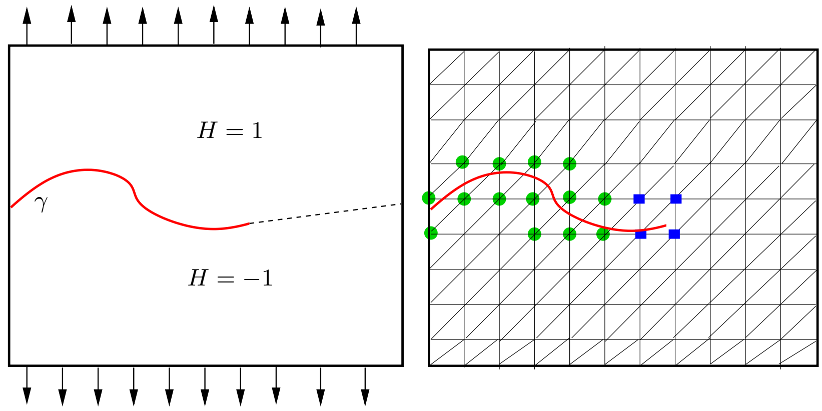

Fig. 6.7. Left: Crack problem. Right: Classical FEM mesh. Nodes • are enriched with discontinuity functions; nodes marked ▪ are enriched with singularity function.

Example 6.4. In practice, a) only the strongest singularity functions are added (typically only $S_{j,1}$), b) only those singularity functions at re-entrant corners (i.e., for corners A_j where $\pi/\omega_j < 1$) and c) $\rho \sim h$ is chosen (see the right picture in Fig. 6.6). While the choice $\rho \sim h$ does not improve the rate of convergence, the constant is greatly improved so that in many cases good engineering accuracy is reached. ▪

6.2 Crack propagation problems

Crack propagation problems have been put forward as an example where augmenting a standard FEM space with special functions is advantageous. In many 2D crack problems, the crack is modelled as a curve γ (see Fig. 6.7). A linear elasticity problem is solved on $\Omega \setminus \gamma$; then the so-called stress intensity factors are extracted from the FEM solution; from these stress intensity factors the crack propagation is determined according to some engineering model; finally, the crack is extended, and the next iteration of this loop is performed. Performing such a crack propagation analysis is costly since the domain $\Omega \setminus \gamma$ on which the elasticity equations have to be solved, changes in each iteration step thus requiring (at least local) remeshing. Additionally, since the solution exhibits a strong singularity at the crack tip, a strongly refined mesh is required near the crack tip to resolve this singularity and guarantee reliable results. The technique of augmenting a standard FEM space by a few special functions to overcome these two difficulties seems very attractive and has been proposed, for example, under the name X-FEM (extended FEM) in [84,29,98] and in the context of the generalized FEM. We will only sketch the key ideas of the X-FEM applied to crack propagation problems. For that, we will not consider the elasticity equation but the simpler scalar case of

$$-\Delta u = 0 \quad \text{on } \Omega \setminus \gamma, \qquad \partial_n u = 0 \quad \text{on } \gamma^+ \text{ and on } \gamma^- \tag{6.5}$$

together with further boundary conditions on $\partial\Omega$. Here, γ^+ and γ^- denote the upper and lower part of the curve γ (see Fig. 6.7). If γ is sufficiently smooth, then an expansion analogous to that of Lemma 6.1 can be obtained, namely, near the crack tip (located at the origin), the solution u of (6.5) takes the form

$$u = \sum_{n=0}^{\infty} S_n r^{n/2} \cos\left(\frac{n}{2}\varphi\right);$$

here, the coefficient $S_1 \in \mathbb{R}$ of the first singularity function is called, in analogy to the elasticity case, the stress intensity factor. The solution u need not be continuous across the curve γ. It is, away from the crack tip, only piecewise smooth. The idea of the X-FEM is to employ a standard FEM space V_{FE} on Ω. This space ignores the crack γ but takes care of the geometry of Ω and the boundary conditions on $\partial\Omega$. The crack γ is then accounted for as follows: nodes near γ but far from the crack tip are collected in the set I_H, nodes near the crack tip are collected in the set I_{CT} (see Fig. 6.7 where these sets are denoted $\bullet$ and $\blacksquare$). One defines the discontinuity function

$$H(x) := \begin{cases} 1 & \text{if } x \text{ is above } \gamma \\ -1 & \text{if } x \text{ is below } \gamma \end{cases}$$

and takes as approximation space

$$V_N := V_{FE} \oplus \operatorname{span}\{H\psi \,|\, i \in I_H\} \oplus \operatorname{span}\{\psi_i r^{1/2} \cos \frac{1}{2}\varphi \,|\, i \in I_{CT}\}.$$

This approximation space is chosen so as to account for the expected solution behavior near the crack tip. Near the crack but away from the crack tip, the space V_N contains discontinuous functions, reflecting the fact that the sought solution may jump across the crack γ.

Remark 6.5. Some extensions of this choice would be: a) add more singularity functions, b) use higher order discontinuity functions, e.g., $H(x)\pi(x)$, where $\pi \in \mathcal{P}_p$ (the above construction corresponds to $p = 0$). ∎

We will not analyze the approximation properties of the space V_N defined above. The following exercise, however, gives an indication of what can be expected away from the crack tip.

Exercise 6.6. Let $\Omega = (-1, 1)$ and consider a uniform mesh $\mathcal{T}$ of mesh size $h = 2/(2N + 1)$ with nodes $x_i = -1 + ih$, $i = 0, \ldots, 2N + 1$. Let $S^{p,1}(\mathcal{T}) \subset C(\Omega)$ be a standard FEM space on the mesh $\mathcal{T}$ and consider the approximation of a function u that is smooth on $[-1, 0) \cup (0, 1]$ but has a jump discontinuity at 0. What convergence rate (in L^2) can be expected? Augment the nodes x_N, x_{N+1} with the Heaviside function $H(x) = \operatorname{sign} x$, i.e., consider $S^{p,1}(\mathcal{T}) \oplus \operatorname{span}\{\psi_N(x)H(x), \psi_{N+1}(x)H(x)\}$, where ψ_i is the standard hat function associated with nodes x_i. What convergence rate can be expected? Consider $S^{p,1}(\mathcal{T}) \oplus \operatorname{span}\{\psi_N(x)H(x)x^j, \psi_{N+1}H(x)x^j \,|\, j = 0, \ldots, p-1\}$. ∎

Remark 6.7. If only very few close neighbors of the crack tip are enriched with the singularity function, then the rate of convergence cannot be expected to be good. Nevertheless, as already pointed out in Example 6.4, good engineering accuracy can be reached. ∎

6.3 Further examples: the generalized FEM

The generalized FEM in the form [101–103] is very similar to the X-FEM. The versatility of the generalized FEM is demonstrated in [101–103] by calculations on complicated domains, for example, domains with many holes or cracks. A classical FEM is augmented by special functions the reflect the proper behavior of the solution near these features. Related earlier work on the generalized FEM for elasticity and crack problems can be found in [90,34].

6.4 Bibliographical Remarks

The idea of augmenting classical FEM spaces with special functions adapted to a problem has a long history. For problems with singularities (e.g., corner singularities) it can be found in [42,20].

The bilinear form a in all the above examples involves an integration over Ω. In practice, this integration is replaced by numerical quadrature. Based on modern adaptive quadrature techniques (possibly including adaptive order control for higher efficiency) it is possible to perform the integration in a completely black box fashion where the user merely needs to provide information whether a point $x \in \mathbb{R}^d$ is in Ω. The "pixelation" technique of [102] can be viewed as an example of such an approach. For geometries whose boundary is piecewise smooth or piecewise affine, it can be much more efficient to deviate from the black box approach by employing local meshing near the boundary, [101,103,48,96]. Note that this local mesh near the boundary need not be regular since it is only used for quadrature purposes. The structure of the shape functions also greatly affects the cost of the quadrature. Consider as an example the particle partition of unity method of [96]. There, the shape functions whose support is contained in Ω are constructed such that they are piecewise smooth, where the regions of smoothness are axis parallel boxes. Clearly, this choice greatly simplifies the design of appropriate quadrature rules. We finally mention that the use of numerical quadrature entails errors; some ideas for their control are discussed in [101].

7 Enforcement of essential boundary conditions

In many applications, essential boundary conditions have to be enforced. As a model problem we consider the classical Poisson problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx = F(v) := \int_\Omega f v, \; dx \quad \forall v \in H_0^1(\Omega). \tag{7.1}$$

The ideas how to enforce essential boundary conditions in meshless mehods are essentially the same ones as in the classical FEM. They can be split into two categories:

- *Conforming methods:* The approximation space V_N is chosen as a subspace of $H_0^1(\Omega)$, i.e., $V_N \subset H_0^1(\Omega)$. This can be achieved by
  - using cut-off functions;
  - combining the classical FEM near the boundary with particle methods in the interior;
  - creating $H_0^1(\Omega)$-conforming spaces in the framework of the partition of unity method by properly selecting the local approximation spaces V_i near the boundary.
- *Non-conforming methods:* In these methods, the variational formulation is changed. These methods include
  - Lagrange multiplier methods,
  - collocation of boundary conditions,
  - penalty methods,
  - Nitsche's method.

7.1 Conforming methods

For the model case (7.1) the approximation space V_N has to be chosen to satisfy $V_N \subset H_0^1(\Omega)$.

A simple approach The simplest approach is to select from a given set $\mathcal{B} = \{\varphi_i \,|\, i = 1, \dots, N\}$ of shape functions only those that satisfy $\varphi_i \in H_0^1(\Omega)$, i.e., to take

$$V_{N,0} := \operatorname{span}\{\varphi_i \,|\, (\operatorname{supp} \varphi_i)^\circ \subset \Omega\}. \tag{7.2}$$

Good approximation properties cannot be expected of $V_{N,0}$, however, even if the function to be approximated is smooth:

Exercise 7.1. Let $V_N := \operatorname{span}\{\varphi_i \,|\, i = 0, \dots, N\}$ be the space of piecewise linear functions associated with the mesh given by the points $x_i = -\frac{h}{2} + ih$, $i = 0, \dots, N$, $h = 1/(N-1)$. Consider for $\Omega = (0,1)$ the subspace $V_{N,0} \subset V_N$ given by $V_{N,0} = \operatorname{span}\{\varphi_i \,|\, (\operatorname{supp} \varphi_i)^\circ \subset \Omega\}$. Show that for the smooth function $u(x) = x(1-x) \in H_0^1(\Omega)$ we have

$$\inf_{v \in V_{N,0}} \|u - v\|_{H^1(\Omega)} \geq C\sqrt{h}.$$

■

Cut-off function methods In cut-off function methods, the essential boundary conditions are enforced by multiplying an approximation space V_N by a weight function w, where w vanishes on $\partial\Omega$ and satisfies $w \sim \operatorname{dist}(\cdot, \partial\Omega)$. If w is sufficiently smooth and $V_N \subset H^1(\Omega)$, then we obtain an $H_0^1(\Omega)$-conforming subspace $V_{w,N}$ by setting $V_{w,N} := wV_N \subset H_0^1(\Omega)$. These ideas can be traced back to [67,83] and were revived in [57]. Concerning the approximation properties of the space $V_{w,N}$ we follow [57].

Lemma 7.2. *Let $k \geq 2$, and let $w \in W^{k,\infty}(\Omega)$ be such that $w \sim \operatorname{dist}(\cdot, \partial\Omega)$. Then there exists $C > 0$ such that for any compact subset $\Omega' \subset\subset \Omega$ we have for functions u, v satisfying $u = vw$*

$$\|v\|_{H^k(\Omega)} \leq C\delta^{-1}\left[\|u\|_{H^k(\Omega)} + \|v\|_{H^{k-1}(\Omega')}\right], \qquad \|v\|_{H^{k-1}(\Omega)} \leq C\|u\|_{H^k(\Omega)},$$

where $\delta = \operatorname{dist}(\Omega', \partial\Omega)$.

Proof. The proof follows from Hardy's inequality. The details can be found in [57, Thm. 6.1]. □

Lemma 7.2 can be employed to recover the optimal rate of convergence if $u \in H^k(\Omega) \cap H_0^1(\Omega)$:

Exercise 7.3. Let $\Omega \subset \mathbb{R}^d$ have a smooth boundary. Assume the setting of Exercise 4.7. Suppose that $p_i = p \geq k - 1 \geq 1$ for all i and that $h_i \sim h$ for all i. Show, using Lemma 7.2, that the space $V_{w,N} = wV_N$ satisfies

$$\inf_{v \in V_{w,N}} \|u - v\|_{H^1(\Omega)} \leq Ch^{k-1}\|u\|_{H^k(\Omega)} \qquad \forall u \in H^k(\Omega) \cap H_0^1(\Omega);$$

here V_N is chosen as in Exercise 4.7. ■

Remark 7.4. The existence of a weight function w with the above regularity properties is closely related to the smoothness of $\partial\Omega$: the "natural" choice $w(x) := \operatorname{dist}(x, \partial\Omega)$ is only smooth if $\partial\Omega$ is. ■

Combination with the classical FEM A technique proposed, e.g., in [68], is to combine shape functions of the classical FEM with general particle methods. In the vicinity of the boundary $\partial\Omega$, a standard mesh is defined and a standard FE space is employed. This space guarantees optimal approximation properties and gives the flexibility of the classical FEM to handle boundary conditions. For the approximation in the interior of Ω, any system can be used, e.g., systems $V_{N,0}$ of the form (7.2). These ideas can be shaped into several forms. In order to illustrate what can be expected, we present the following example:

Example 7.5. Let $\Omega \subset \mathbb{R}^2$ be a polygon, and let $2 \leq k \leq p$. Let $V_N \subset H^1(\Omega)$ be an approximation space with the property

$$\inf_{v \in V_N} \|u - v\|_{L^2(\Omega)} + h\|u - v\|_{H^1(\Omega)} \leq Ch^k \|u\|_{H^k(\Omega)}. \tag{7.3}$$

Let $S_h := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) < h\}$ be a tubular neighborhood of $\partial\Omega$. Let $\mathcal{T}$ be an affine, quasi-uniform triangulation of mesh size $O(h)$ of a set $\Omega' \subset \Omega$ that satisfies $S_h \subset \Omega'$. Let $S^{p,1}(\mathcal{T})$ be the standard finite element space of piecewise polynomials of degree p on the mesh $\mathcal{T}$ and set $S_0^{p,1}(\mathcal{T}) = S^{p,1}(\mathcal{T}) \cap H_0^1(\Omega')$. Note that by extending functions of $S_0^{p,1}(\mathcal{T})$ by zero outside of Ω', we may think of $S_0^{p,1}(\mathcal{T})$ as a subset of $H_0^1(\Omega)$. Let $\{\psi_i \,|\, i \in I_{\partial\Omega}\} \subset S^{1,1}(\mathcal{T})$ be the standard piecewise linear hat functions associated with the nodes on $\partial\Omega$ and set

$$\omega := \sum_{i \in I_{\partial\Omega}} \psi_i.$$

Again, by the support properties of the piecewise linear hat functions ψ_i, we may think of ω as being defined on Ω. We observe:

$$\begin{aligned} &\omega \equiv 1 \quad \text{on } \partial\Omega, && \omega \equiv 0 \quad \text{on } \Omega \setminus \Omega', \\ &\omega \in W^{1,\infty}(\Omega), && \|\nabla\omega\|_{L^\infty(\Omega)} \leq Ch^{-1}. \end{aligned}$$

We select as the approximation space

$$V_{p,N} := (1 - \omega)V_N \oplus S_0^{p,1}(\mathcal{T}) \subset H_0^1(\Omega).$$

We claim that for $u \in H^k(\Omega) \cap H_0^1(\Omega)$

$$\inf_{v \in V_{p,N}} \|u - v\|_{H^1(\Omega)} \leq Ch^{k-1} \|u\|_{H^k(\Omega)}. \tag{7.4}$$

(7.4) is shown using the same ideas as in the proof of Theorem 4.1. Let $u_N \in V_N$ be an approximation of u from V_N such that

$$\|u - u_N\|_{L^2(\Omega)} + h\|u - u_N\|_{H^1(\Omega)} \leq Ch^k \|u\|_{H^k(\Omega)}.$$

We will take the approximant to u from $V_{p,N}$ of the form $(1 - \omega)u_N + v$, where $v \in S_0^{p,1}(\mathcal{T})$ will be determined below. The error can be written as

$u-(1-\omega)u_N - v = (1-\omega)(u-u_N) + (\omega u - v)$. For the first term, we calculate

$$\|(1-\omega)(u-u_N)\|_{L^2(\Omega)} + h\|(1-\omega)(u-u_N)\|_{H^1(\Omega)} \le Ch^k\|u\|_{H^k(\Omega)},$$

which has the desired form (7.4). We now turn to the definition of $v \in S_0^{p,1}(\mathcal{T})$, which approximates ωu. We select $I_{p-1}u \in S^{p-1,1}(\mathcal{T})$ by a standard FEM interpolation procedure. Then, $(I_{p-1}u)|_{\partial\Omega} = 0$ and

$$\|u - I_{p-1}u\|_{L^2(K)} + h\|\nabla(u - I_{p-1}u)\|_{L^2(K)} \le Ch^k|u|_{H^k(K)} \qquad \forall K \in \mathcal{T}.$$

Here, we exploited the assumption $p \ge k$. As the product of a piecewise linear function and a piecewise polynomial of degree $p-1$, the function $\omega I_{p-1}u$ satisfies $\omega I_{p-1}u \in S_0^{p,1}(\mathcal{T})$. We conclude using the support properties of ω and $\|\nabla\omega\|_{L^\infty(\Omega)} \le Ch^{-1}$

$$\|\omega u - \omega I_{p-1}u\|_{L^2(\Omega)} + h|\omega u - \omega I_{p-1}u|_{H^1(\Omega)} \le Ch^k.$$

Thus taking $v := \omega I_{p-1}u$ gives an approximation $(1-\omega)u_N + \omega I_{p-1}u \in V_{p,N}$ that realizes the desired bound (7.4). ∎

Local approximation spaces satisfying essential boundary conditions The previous idea of combining the classical FEM in a strip near the boundary with general approximation spaces V_N in the interior of Ω can be viewed as a variant of the partition of unity method where the local approximation spaces V_i for the patches Ω_i near the boundary are chosen such that they conform to the boundary conditions. A more general approach is the outlined in the following exercise.

Exercise 7.6. Assume the hypotheses of Theorem 4.1. Suppose additionally: if $\Gamma_{i,D} := \partial\Omega_i \cap \partial\Omega \ne \emptyset$, then $V_i \subset H_D^1(\Omega_i) := \{u \in H^1(\Omega_i)\,|\,u|_{\Gamma_{i,D}} = 0\}$. Show: The space V of Theorem 4.1 satisfies $V \subset H_0^1(\Omega)$, and the approximation result of Theorem 4.1 is still valid. ∎

Local approximation spaces V_i that satisfy the correct boundary conditions can be derived in different ways. They can be determined analytically or numerically.

Example 7.7. Let u solve Laplace's equation and assume that u vanishes on a straight line. Extending u by reflection across this line yields a function (again denoted u) that is anti-symmetric with respect to this line and again solves Laplace's equation. It is shown in [78] that harmonic polynomials that are anti-symmetric with respect to this line (and hence vanish on it), can approximate the function u at the same rate as the full space $\mathcal{HP}_p$ of harmonic polynomials. ∎

As discussed in Section 5.6, local approximation spaces V_i can also be computed numerically. If these spaces are computed using the standard FEM, then it is easy to enforce essential boundary conditions.

7.2 Non-conforming methods: Lagrange Multiplier methods and collocation techniques

The essential boundary condition could also be enforced in a weak sense. The simplest such approach is to collocate the boundary condition in a (finite) set of points $Y \subset \partial\Omega$ as was proposed, for example, in [2,54,111]. Such methods are, however, difficult to analyze even in the setting of the classical FEM. Early references to the Lagrange Multiplier Method are [3,4]. One introduces a bilinear form $b : H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ by

$$b(v,\mu) := \langle \gamma_0 v, \mu \rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)},$$

where $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator $\gamma_0 v = v|_{\partial\Omega}$. One then considers the problem: Find $(u,\lambda) \in H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= F(v) \qquad && \forall v \in H^1(\Omega), \\ b(u,\mu) &= 0 && \forall \mu \in H^{-1/2}(\partial\Omega). \end{aligned} \tag{7.5}$$

The function u of the pair (u,λ) solving (7.5) is in fact an element of $H_0^1(\Omega)$ and also a solution of the original problem (7.1). A natural discretization of (7.5) is to take subspaces $V_N \subset H^1(\Omega)$, $M_N \subset H^{-1/2}(\partial\Omega)$ and then consider the problem: Find $(u_N,\lambda_N) \in V_N \times M_N$ such that

$$\begin{aligned} a(u_N,v) + b(v,\lambda_N) &= F(v) \qquad && \forall v \in V_N, \\ b(u_N,\mu) &= 0 && \forall \mu \in M_N. \end{aligned} \tag{7.6}$$

We mention in passing that $\langle v,\mu \rangle_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} = \int_{\partial\Omega} v\mu\, ds$ if $\mu \in L^2(\partial\Omega)$ so that the discrete problem (7.6) represents a linear system of equations that can be set up for any reasonable choice of space M_N (e.g., a space of piecewise constant functions). One challenge in the Lagrange multiplier method is that the spaces V_N and M_N cannot be chosen independently. As is well-known the so-called "inf-sup" condition, or Babuška-Brezzi condition, needs to be satisfied: If

$$\inf_{\mu \in M_N} \sup_{v \in V_N} \frac{b(v,\mu)}{\|v\|_{H^1(\Omega}\|\mu\|_{H^{-1/2}(\partial\Omega)}} \geq \gamma_N > 0, \tag{7.7}$$

then the error $u - u_N$ satisfies (see, e.g., [94, Thm. 5.13])

$$\|u-u_N\|_{H^1(\Omega)} \leq C \left(1 + \frac{1}{\gamma_N}\right) \inf_{(v,\mu)\in V_N \times M_N} \|u-v\|_{H^1(\Omega)} + \|\lambda - \mu\|_{H^{-1/2}(\partial\Omega)}.$$

This bound suggests that the inf-sup constant γ_N should be bounded away from zero uniformly in the discretization parameter N to guarantee good performance. The condition $\gamma_N > 0$ is indeed necessary as the following exercise shows.

Exercise 7.8. Show: $\gamma_N = 0$ implies that the matrix representing the linear system (7.6) is not invertible. ■

In the classical FEM, various combinations of spaces V_N and M_N are known to be "stable" in the sense that (7.6) holds for a constant independent of the mesh size; we refer to [100] for a more detailed discussion and appropriate references. In the context of the classical FEM, a key ingredient in the stability proofs for pairs V_N, M_N are inverse estimates. To the knowledge of the author, such estimates are not available for meshless methods, and an analysis is therefore hard. We will encounter a similar difficulty in our analysis of Nitsche's method below; the appropriate inverse estimate is therefore stipulated as Assumption 7.13.

7.3 Non-conforming methods: penalty method

In the conforming FEM, one would have to choose $V_N \subset H_0^1(\Omega)$. In the penalty method, the essential boundary conditions are weakened by changing the problem: Taking $V_N \subset H^1(\Omega)$ and $\psi \geq 1$ the problem is to find $u_N \in V_N$ such that

$$a_\psi(u_N, v) := a(u_N, v) + \int_{\partial\Omega} \psi u_N v \, ds = F(v) \qquad \forall v \in V_N. \tag{7.8}$$

We recognize this as the Galerkin approximation to the following problem:

$$\text{Find } u_\psi \in H^1(\Omega) \text{ s.t. } a_\psi(u_\psi, v) = F(v) \qquad \forall v \in H^1(\Omega). \tag{7.9}$$

The strong form of this problem is:

$$-\Delta u_\psi = f \qquad \text{on } \Omega, \qquad \partial_n u_\psi + \psi u = 0 \qquad \text{on } \partial\Omega. \tag{7.10}$$

One sees that, if $\psi \to \infty$, then $u_\psi \to u$, where u is the solution of (7.1). We will make this more precise below.

Theorem 7.9 (penalty method). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Let $k \geq 2$. Assume $u \in H^k(\Omega)$ is the solution of (7.1). Let $\xi \in H^{k-1}(\Omega)$ solve*

$$-\Delta\xi + \xi = 0 \quad \text{on } \Omega, \qquad \xi|_{\partial\Omega} = \partial_n u \quad \text{on } \partial\Omega. \tag{7.11}$$

Assume that the approximation space $V_N \subset H^1(\Omega)$ satisfies:

$$\inf_{v \in V_N} \|u - v\|_{L^2(\Omega)} + h\|\nabla(u - v)\|_{L^2(\Omega)} \leq Ch^k, \tag{7.12}$$

$$\inf_{v \in V_N} \|\xi - v\|_{L^2(\Omega)} + h\|\nabla(\xi - v)\|_{L^2(\Omega)} \leq Ch^{k-1}. \tag{7.13}$$

Then there holds for a $C > 0$ independent of ψ and h

$$\|u - u_N\|_{H^1(\Omega)} \leq C\left\{\psi^{-1} + \psi^{-1/2}h^{k-3/2} + \psi^{1/2}h^{k-1/2} + h^{k-1}\right\}.$$

Setting $\psi = h^\sigma$ *with the optimal value* $\sigma = \frac{2k-1}{3}$ *gives*

$$\|u - u_N\|_{H^1(\Omega)} \leq h^\sigma, \qquad \sigma = \frac{2k-1}{3}.$$

Remark 7.10. The regularity assumption $\xi \in H^{k-1}(\Omega)$ is satisfied, for example, if $\partial\Omega$ is smooth. ■

Proof of Theorem 7.9. The proof follows the exposition of [3, Thm. 7.2.2]. From the Lax-Milgram Lemma (see, e.g., [23, Thm. 2.7.7]) we have upon equipping the space $H^1(\Omega)$ with the norm $\|\cdot\|_\psi := \sqrt{a_\psi(\cdot,\cdot)}$, which is equivalent to the standard $\|\cdot\|_{H^1(\Omega)}$ norm,

$$\|u_\psi - u_N\|_\psi = \inf_{v\in V_N} \|u_\psi - v\|_\psi.$$

We now write

$$u = u_\psi + \frac{1}{\psi}\xi + \zeta.$$

The function ζ satisfies

$$\begin{aligned} a_\psi(\zeta, v) &= \underbrace{a(u,v)}_{=\int_\Omega fv\,dx + \int_{\partial\Omega} \partial_n uv\,ds} + \psi \underbrace{\int_{\partial\Omega} uv\,ds}_{=0} - \underbrace{a_\psi(u_\psi, v)}_{=\int_\Omega fv\,dx} - \frac{1}{\psi} a_\psi(\xi, v) \\ &= \int_{\partial\Omega} \partial_n uv\,ds - \frac{1}{\psi} a(\xi, v) - \underbrace{\int_{\partial\Omega} \xi v\,ds}_{=\int_{\partial\Omega} \partial_n uv\,ds} = -\frac{1}{\psi} \int_\Omega \nabla\xi \cdot \nabla v\,dx. \end{aligned}$$

Hence, the Lax-Milgram Lemma gives us

$$\|\zeta\|_\psi \leq \frac{1}{\psi} \|\xi\|_{H^1(\Omega)}. \tag{7.14}$$

The function u_N is the Galerkin approximation to u_ψ, so we get $\|u_\psi - u_N\|_\psi = \inf_{v\in V_N} \|u_\psi - v\|_\psi$. Thus:

$$\|u_\psi - u_N\|_\psi = \inf_{v\in V_N} \|u_\psi - v\|_\psi \leq \inf_{v\in V_N} \|u - v\|_\psi + \frac{1}{\psi} \inf_{v\in V_N} \|\xi - v\|_\psi + \|\zeta\|_\psi.$$

Using the bound $\|z\|^2_{L^2(\partial\Omega)} \leq C\|z\|_{L^2(\Omega)} \|z\|_{H^1(\Omega)}$ (see, e.g., Theorem A.2), we can bound with our assumptions on the approximation properties of V_N

$$\|u_\psi - u_N\|_\psi \leq C\bigl\{h^{k-1} + \psi^{1/2} h^{k-1/2} + \psi^{-1/2} h^{k-3/2} + \psi^{-1}\bigr\}.$$

Choosing $\psi = h^{-\sigma}$ gives

$$\|u_\psi - u_N\|_\psi \leq C h^{\min\{\sigma, \sigma/2 + k - 3/2, -\sigma/2 + k - 1/2, k-1\}}.$$

The optimal rate of convergence is obtained for $\sigma = \frac{2k-1}{3}$. We get

$$\|u - u_N\|_{H^1(\Omega)} \leq \|u_\psi - u_N\|_{H^1(\Omega)} + \frac{1}{\psi}\|\xi\|_{H^1(\Omega)} + \|\zeta\|_{H^1(\Omega)} \leq \|u_\psi - u_N\|_\psi + C\psi^{-1},$$

which gives the desired bound. □

Remark 7.11. In the case $k = 2$, we see that the choice $\sigma = (2k - 1)/3$ leads to the optimal rate of convergence. For $k > 2$, the penalty method leads to suboptimal rates. ■

7.4 Non-conforming methods: Nitsche's method

Nitsche's method was introduced in [88]; a good accound that relates it to various forms of Lagrange Multiplier Methods can be found in [100]. Like the penalty method, Nitsche's method alters the variational formulation albeit in a more subtle way. For definiteness' sake, we consider again the model problem (7.1).

For simplicity, we will assume that the approximation space V_N satisfies $V_N \subset H^2(\Omega)$, although weaker assumptions suffice[3]. We need to identify the shape functions φ_i that are near the boundary. Hence, upon recalling the definition of patches, $\Omega_i = (\operatorname{supp} \varphi_i)^\circ$, we define

$$I_{\partial\Omega} := \{i \in \mathbb{N} \mid \Omega_i \cap \partial\Omega \neq \emptyset\}. \tag{7.15}$$

For $i \in I_{\partial\Omega}$ we set

$$\Gamma_i := \Omega_i \cap \partial\Omega, \qquad \widetilde{h}_i := \operatorname{diam} \Gamma_i. \tag{7.16}$$

For a penalty parameter $\gamma > 0$ define

$$a_N(u, v) := a(u, v) - \int_{\partial\Omega} \partial_n u v \, ds - \int_{\partial\Omega} u \partial_n v \, ds + \gamma \sum_{i \in I_{\partial\Omega}} \widetilde{h}_i^{-1} \int_{\Gamma_i} uv \, ds. \tag{7.17}$$

One variant of Nitsche's method can then be formulated as:

$$\text{Find} \quad u_N \in V_N \ \text{ s.t. } \ a_N(u_N, v) = F(v) \qquad \forall v \in V_N. \tag{7.18}$$

In contrast to the penalty method, Nitsche's method is consistent if the exact solution is sufficiently regular:

Lemma 7.12 (consistency of Nitsche's method). *Let Ω be a Lipschitz domain. If for some $\varepsilon > 0$ the solution u of (7.1) satisfies $u \in H^{3/2+\varepsilon}(\Omega)$, then $a_N(u, v) = F(v)$ for all $v \in V_N$.*

Proof. By the trace theorem, the assumption $u \in H^{3/2+\varepsilon}(\Omega)$ guarantees that $\partial_n u$ is well-defined and $\partial_n u \in L^2(\partial\Omega)$. Since also the Gauß-Green theorem holds, the result now follows by inspection. □

[3] One has to be able to define the conormal derivative $\partial_n u$ for $u \in V_N$ as an element of $H^{-1/2}(\partial\Omega)$ in a meaningful way. In view of practical computations, one would like $\partial_n u \in L^2(\partial\Omega)$. For example, $V_N \subset H^s(\Omega)$ for some $s > 3/2$ suffices.

The consistency result Lemma 7.12 will allow us to obtain quasi-optimality results in appropriate norms. In order to perform this analysis, we introduce a few discrete norms on the space $H^{3/2+\varepsilon}(\Omega)$:

$$\|u\|_{1/2,h}^2 := \sum_{i\in I_{\partial\Omega}} \widetilde{h}_i^{-1}\|u\|_{L^2(\Gamma_i)}^2, \tag{7.19}$$

$$\|\partial_n u\|_{-1/2,h}^2 := \sum_{i\in I_{\partial\Omega}} \widetilde{h}_i\|\partial_n u\|_{L^2(\Gamma_i)}^2, \tag{7.20}$$

$$\|u\|_{1,h}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{1/2,h}^2 + \|\partial_n u\|_{-1/2,h}^2. \tag{7.21}$$

Central to the analysis of Nitsche's method is an inverse assumption:

Assumption 7.13 (inverse assumption). There exists $C_{inv} > 0$ such that

$$\|\partial_n u\|_{-1/2,h} \le C_{inv}\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in V_N.$$

In the case of the classical FEM, this inverse assumption can be proved:

Exercise 7.14. Let $\mathcal{T}$ be a shape-regular triangulation of a polygon in $\mathbb{R}^2$. For the space of piecewise linears $S^{1,1}(\mathcal{T})$, let $\mathcal{E}_{\partial\Omega}$ be the set of edges that lie on $\partial\Omega$ and let h_e be the length of edge $e \in \mathcal{E}_{\partial\Omega}$. Show: There exists $C > 0$ depending solely on the shape-regularity constant of $\mathcal{T}$ such that upon setting

$$\|\partial_n u\|_{-1/2,h}^2 := \sum_{e\in\mathcal{E}_{\partial\Omega}} h_e\|\partial_n u\|_{L^2(e)}^2$$

we have $\|\partial_n u\|_{-1/2,h} \le C_{inv}\|\nabla u\|_{L^2(\Omega)}$ for all $u \in S^{1,1}(\mathcal{T})$ for some suitable $C_{inv} > 0$. ■

If the inverse Assumption 7.13 is satisfied, then the bilinear form a_N is coercive on V_N provided that the parameter γ is chosen suffiently large:

Lemma 7.15. *If Assumption 7.13 is satisfied, then we have for $\gamma > 2C_{inv}^2$*

$$\min\left\{\frac{1}{4}, \frac{1}{4C_{inv}}, \gamma - 2C_{inv}^2\right\}\|u\|_{1,h}^2 \le a_N(u,u) \qquad \forall u \in V_N, \tag{7.22}$$

$$|a_N(u,v)| \le (1+\gamma)\|u\|_{1,h}\|v\|_{1,h} \qquad \forall u, v \in H^{3/2+\varepsilon}(\Omega). \tag{7.23}$$

Proof. Using the fact that $\partial\Omega \subset \cup_{i\in I_{\partial\Omega}}\Gamma_i$, we can estimate with the Cauchy-Schwarz inequality

$$\left|\int_{\partial\Omega} \partial_n u u\, ds\right| \le \|\partial_n u\|_{-1/2,h}\|u\|_{1/2,h}.$$

Using next the bound $2|ab| \le \epsilon a^2 + \frac{1}{\epsilon}b^2$, which is valid for all $\epsilon > 0$, we get

$$\begin{aligned}
a_N(u,u) &\ge \|\nabla u\|_{L^2(\Omega)}^2 - 2\|\partial_n u\|_{-1/2,h}\|u\|_{1/2,h} + \gamma\|u\|_{1/2,h}^2 \\
&\ge \|\nabla u\|_{L^2(\Omega)}^2 - \epsilon\|\partial_n u\|_{-1/2,h}^2 - \epsilon^{-1}\|u\|_{1/2,h}^2 + \gamma\|u\|_{1/2,h}^2 \\
&\ge (1-\epsilon C_{inv}^2)\|\nabla u\|_{L^2(\Omega)}^2 + (\gamma - \epsilon^{-1})\|u\|_{1/2,h}^2,
\end{aligned}$$

where we appealed to the inverse assumption. Choosing now $\epsilon = (2C_{inv}^2)^{-1}$ gives the desired bound (7.22).
The bound (7.23) follows from the trace theorem. $\square$

Remark 7.16. Lemma 7.15 shows that the problem (7.18) is well-defined and leads to a symmetric positive definite stiffness matrix, provided that the parameter γ is choosen sufficiently large. A good estimate on C_{inv} is required for that. Determining C_{inv} can be formulated as an eigenvalue problem, and a numerical scheme that works well has been proposed in [51,96]. ∎

The consistency result Lemma 7.12 allows us to get quasi-optimality of the Nitsche method:

Lemma 7.17. *Set $\underline{a} := \min\{\frac{1}{4}, \frac{1}{4C_{inv}}, \gamma - 2C_{inv}^2\}$. Assume that the solution u of (7.1) satisfies $u \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then*

$$\|u - u_N\|_{1,h} \leq \left(1 + \frac{1+\gamma}{\underline{a}}\right) \inf_{v \in V_N} \|u - v\|_{1,h}.$$

Proof. The proof is the same as the proof of Céa's lemma, for which we refer, for example, to [23, Thm. 2.8.1]. $\square$

Theorem 7.18 (Convergence of Nitsche's method). *Let the solution u of (7.1) satisfy $u \in H^k(\Omega)$ for some $k \geq 2$. Assume:*

(a) the constant $\underline{a}$ of Lemma 7.17 is positive;
(b) the sets Γ_i, $i \in I_{\partial\Omega}$ satisfy an overlap condition;
(c) $h_i \sim h$ for all $i \in I_{\partial\Omega}$,
(d) $\inf_{v \in V_N} \|u - v\|_{L^2(\Omega)} + h\|u - v\|_{H^1(\Omega)} + h^2\|u - v\|_{H^2(\Omega)} \leq Ch^k\|u\|_{H^k(\Omega)}$.

Then

$$\|u - u_N\|_{H^1(\Omega)} \leq Ch^{k-1}.$$

Proof. By the quasi-optimality result Lemma 7.17 it suffices to bound the expression $\inf_{v \in V_N} \|u - v\|_{1,h}$. Using $h_i \sim h$ for all $i \in I_{\partial\Omega}$ and the overlap condition on the sets Γ_i gives us for arbitrary $v \in V_N$

$$\|u - v\|_{1,h}^2 \leq \|u - v\|_{H^1(\Omega)}^2 + Ch\|\partial_n(u - v)\|_{L^2(\partial\Omega)}^2 + h^{-1}\|\partial_n(u - v)\|_{L^2(\partial\Omega)}^2.$$

The trace Theorem A.2 applied to $z \in H^2(\Omega)$ gives in view of $\nabla z \in H^1(\Omega)$

$$\|u - v\|_{1,h}^2 \leq C\big\{\|u - v\|_{H^1(\Omega)}^2 + h\|u - v\|_{H^1(\Omega)}\|u - v\|_{H^2(\Omega)} + \frac{1}{h}\|u - v\|_{L^2(\Omega)}\|u - v\|_{H^1(\Omega)}\big\}.$$

The assumptions on the approximation properties of V_N allow us to conclude the argument. $\square$

We required $k \geq 2$ in the proof of Theorem 7.18 for convenience only. The follow exercise shows that $k > 3/2$ is in fact sufficient:

Exercise 7.19. Use Theorem A.2 to show that the approximation result of Theorem 7.18 is true for $k \in (3/2, 2)$ provided

$$\inf_{v \in V_N} \|u - v\|_{L^2(\Omega)} + h\|u - v\|_{H^1(\Omega)} + h^k \|u - v\|_{H^k(\Omega)} \leq Ch^k \|u\|_{H^k(\Omega)}.$$

■

Remark 7.20. The approximation properties of V_N stipulated in Theorem 7.18 required simultaneous approximation properties of V_N in three norms. Such results were established in Theorem 2.6 and Proposition 3.11. ■

A Results from Analysis

Theorem A.1 (universal extension operator). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then there exists a linear operator $E : L^1(\Omega) \to L^1(\mathbb{R}^d)$ with the following properties:*

(i) $(Eu)|_\Omega = u$ *for all* $u \in L^1(\Omega)$.
(ii) *For each $k \in \mathbb{N}_0$, $p \in [1, \infty]$, there exists $C > 0$ such that $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(\Omega)}$ for all $u \in W^{k,p}(\Omega)$.*

Proof. See [99, Chap. VI.3]. □

Theorem A.2 (multiplicative trace theorem). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $s \in (1/2, 1]$. Then there exists a constant $C > 0$ such that for all $u \in H^s(\Omega)$ the trace $\gamma_0 u = u|_{\partial\Omega}$ satisfies*

$$\|\gamma_0 u\|_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)}^{1-1/(2s)} \|u\|_{H^s(\Omega)}^{1/(2s)}.$$

Proof. The case $s = 1$ is well-known (see, e.g., [23, Prop. 1.6.3]). For the case $s \in (1/2, 1)$, a proof that is based on elementary techniques can be found in Exercise A.3. A short proof resting on the theory of interpolation spaces is as follows. From [104, Thm. 2.9.3], we can infer the trace theorem

$$\|\gamma_0 u\|_{L^2(\partial\Omega)} \leq C\|u\|_{B^{1/2}_{2,1}(\Omega)}, \tag{A.1}$$

where the Besov space $B^{1/2}_{2,1}(\Omega) = (L^2(\Omega), H^1(\Omega))_{1/2,1}$; here, the K-method of interpolation, [15,104] is employed. For $s \in (1/2, 1]$, the reiteration theorem then allows us to recognize $B^{1/2}_{2,1}$ as an interpolation space between $L^2(\Omega)$ and $H^s(\Omega)$, namely, $B^{1/2}_{2,1}(\Omega) = (L^2(\Omega), H^s(\Omega))_{\theta,1}$, where $\theta = 1/(2s)$. Inserting into (A.1) the interpolation inequality $\|u\|_{B^{1/2}_{2,1}(\Omega)} \leq C_\theta \|u\|^{1-\theta}_{L^2(\Omega)} \|u\|^\theta_{H^s(\Omega)}$ then gives the desired result. □

Exercise A.3 (alternative proof of Theorem A.2). The present exercise illustrates a very useful device of analysis, namely, how scaling arguments can lead to multiplicative bounds.
For simplicity, consider the case $\Omega = (0,1)^d$. Write $\Gamma := \mathbb{R}^{d-1} \times \{0\}$. Using the extension operator of Theorem A.1, we may assume $u \in H^s(\mathbb{R}^d)$. Proceed in several steps:

(a) Starting from the estimate $\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^s(\Omega)}$ for all $v \in H^s(\Omega)$, show that

$$\|v\|_{L^2(\Gamma)} \leq C\left[\|v\|_{L^2(\mathbb{R}^d)} + |v|_{H^s(\mathbb{R}^d)}\right] \qquad \forall v \in C_0^\infty(\mathbb{R}^d) \qquad \text{(A.2)}$$

where we recall that $|\cdot|_{H^s(\mathbb{R}^d)}$ is defined as the Slobodeckij norm (1.1).
(b) By scaling (i.e., considering the function $\tilde{u}(x) := u(Rx)$) show that (A.2) has actually the form

$$\|v\|^2_{L^2(\Gamma)} \leq C\left[R\|v\|^2_{L^2(\mathbb{R}^d)} + R^{1-2s}|v|^2_{H^s(\mathbb{R}^d)}\right] \qquad \forall v \in C_0^\infty(\mathbb{R}^d) \quad \text{(A.3)}$$

for *arbitrary* $R > 0$.
(c) Choose R in (A.3) suitably to obtain $\|v\|^2_{L^2(\Gamma)} \leq C\|v\|^{2-1/s}_{L^2(\mathbb{R}^d)}|v|^{1/s}_{H^s(\mathbb{R}^d)}$. ■

The following theorem shows that it is possible to cover arbitrary bounded sets by balls that satisfy a finite overlap property:

Theorem A.4 (Besicovitch covering theorem). *Let $d \in \mathbb{N}$. Then there exists a constant $M_d > 0$ (depending solely on d) with the following property: Let $\mathcal{B}$ be a collection of nondegenerate closed balls in $\mathbb{R}^d$ with* $\sup\{\operatorname{diam} B \mid B \in \mathcal{B}\} < \infty$. *Let A be the set of centers of the balls of $\mathcal{B}$. Then there exist countable collections $\mathcal{B}_1, \ldots, \mathcal{B}_{M_d} \subset \mathcal{B}$ such that each $\mathcal{B}_i$, $i = 1, \ldots, M_d$, is a collection of* disjoint *balls and*

$$A \subset \bigcup_{i=1}^{M_d} \bigcup_{B \in \mathcal{B}_i} B.$$

Proof. See, for example, [112, Thm. 1.3.5] or [40, Sec. 1.5.2]. □

B Properties of Polynomials

Theorem B.1 (polynomial approximation). *Let $B \subset \mathbb{R}^d$ be a ball of diameter $h \leq 1$. Then for each polynomial degree $p \in \mathbb{N}_0$ there exists a linear operator $Q_p : L^1(B) \to \mathcal{P}_p$ with the following properties:*

$$Q_p u = u \qquad \forall u \in \mathcal{P}_p, \qquad \text{(B.1)}$$

$$\|u - Q_p u\|_{W^{s,q}(B)} \leq C_{p,q,k} h^{(\min\{p+1,k\}-s)_+}\|u\|_{W^{k,q}(B)}, \quad 0 \leq s \leq k. \quad \text{(B.2)}$$

Here, the notation $(\cdot)_+$ represents the function $x \mapsto (x)_+ = \max\{x, 0\}$. The constant $C_{p,q,k}$ depends only on $p \in \mathbb{N}_0$, $q \in [1, \infty)$, d, and $k \geq 0$. The bound (B.2) also holds for $q = \infty$ if k and s are restricted to integer values s, $k \in \mathbb{N}_0$.
If $q \in (1, \infty)$ and $k > d/q$ or if $q = 1$ and $k \geq d$, then additionally

$$\|u - Q_p u\|_{L^\infty(B)} \leq \tilde{C}_{p,q,k} h^{\min\{p+1,k\}-d/q} \|u\|_{W^{k,q}(B)}, \tag{B.3}$$

where $\tilde{C}_{p,q,k}$ depends only on p, q, d, and k.

Proof. The L^∞-bound (B.3) will be treated in the following Exercise B.2. We elaborate the arguments of [23, Chap. 4] in order to show the statements (B.1), (B.2). We proceed in several steps.
1. step: Let $F : B_1(0) \to B$ be an affine bijection. We define $u \mapsto Q_p u$ by $(Q_p u) \circ F := \widehat{Q}_p(u \circ F)$, where $\widehat{Q}_p : L^1(B_1(0)) \to \mathcal{P}_p$ is defined as in [23, Chap. 4]. From [23, Prop. 4.3.8 and Cor. 4.1.15] we have

$$\widehat{Q}_p u = u \qquad \forall u \in \mathcal{P}_p, \tag{B.4}$$

$$\|\widehat{Q}_p u\|_{W^{m,\infty}(B_1(0))} \leq C_m \|u\|_{L^1(B_1(0))} \qquad \text{for any } m \in \mathbb{N}_0. \tag{B.5}$$

(B.4) implies (B.1). We therefore turn to the proof of (B.2). We set $\mu := \min\{p+1, k\}$, let $v \in \mathcal{P}_p$ be arbitrary, and calculate for $s \in [0, \mu]$ using (B.4) and the stability result (B.5)

$$\begin{aligned} &\|u - \widehat{Q}_p u\|_{W^{s,q}(B_1(0))} \leq \|u - \widehat{Q}_p u\|_{W^{\mu,q}(B_1(0))} \\ &\leq \|u - v\|_{W^{\mu,q}(B_1(0))} + \|\widehat{Q}_p(u - v)\|_{W^{\mu,q}(B_1(0))} \\ &\leq \|u - v\|_{W^{\mu,q}(B_1(0))} + C\|(u - v)\|_{L^1(B_1(0))} \leq C\|u - v\|_{W^{\mu,q}(B_1(0))}. \end{aligned} \tag{B.6}$$

2. step: In order to employ scaling arguments, we have to replace the full norm on the right-hand side of (B.6) by a semi-norm. The technique for doing this can be traced back to [21,31] and is based on a compactness argument: From Rellich's theorem, [39, Chap. 5.7], we have that the embedding $W^{k,q}(B_1(0)) \subset\subset W^{k-1,q}(B_1(0))$ is compact for $k \in \mathbb{N}$; for $k = \tilde{k} + s$ with $\tilde{k} \in \mathbb{N}_0$ and $s \in (0, 1)$ we have $W^{k,q}(B_1(0)) \subset\subset W^{\tilde{k},q}(B_1(0))$, [104, Sec. 1.16.4, Thm. 2]. Reasoning in the same way by contradiction as in the classical proof of the Poincaré inequality (see, e.g., [39, Sec. 5.8.1]), we can infer for $p \in \mathbb{N}_0$ with $p \geq k - 1$

$$\inf_{v \in \mathcal{P}_p} \|u - v\|_{W^{k,q}(B_1(0))} \leq C|u|_{W^{k,q}(B_1(0))} \qquad \forall u \in W^{k,q}(B_1(0)). \tag{B.7}$$

3. step: Since $v \in \mathcal{P}_p$ in (B.6) is arbitrary and $\mu \leq p + 1$, we get for $s \in [0, \mu]$

$$\|u - \widehat{Q}_p u\|_{W^{s,q}(B_1(0))} \leq C \inf_{v \in \mathcal{P}_p} \|u - v\|_{W^{\mu,q}(B_1(0))} \leq C|u|_{W^{\mu,q}(B_1(0))}.$$

By transforming to B and observing how the semi-norms $|\cdot|_{W^{s,q}}$, $|\cdot|_{W^{\mu,q}}$ scale (cf. (1.1)) we obtain the desired bound (B.2) for $s \in [0, \mu]$.
4. step: It remains to see the bound for $\min\{p+1, k\} < s \leq k$. This can only happen for $p+1 < k$. But then $p+1 < s$ and an easy calculation shows that $|Q_p|_{W^{s,q}(B)} = 0$. We conclude for the semi norm

$$|u - Q_p u|_{W^{s,q}(B)} \leq |u|_{W^{s,q}(B)} + |Q_p u|_{W^{s,q}(B)} = |u|_{W^{s,q}(B)} \leq C\|u\|_{W^{k,q}(B)}.$$

This allows us to obtain the desired bound (B.2) for the case $\min\{p+1, k\} < s \leq k$. $\square$

Exercise B.2. Show (B.3) by proving the following two results.

(a) Show the following generalization of (B.7) for $p+1 < k$ and $\Omega := B_1(0)$:

$$\inf_{v \in \mathcal{P}_p} \|u - v\|_{W^{k,q}(\Omega)} \leq C|u|_{W^{p+1,q}(\Omega)} + \sum_{\substack{j \in \mathbb{N} \\ p+2 \leq j < k}} |u|_{W^{j,q}(\Omega)} + |u|_{W^{k,q}(\Omega)}.$$

(b) The parameter k in the statement of Theorem B.1 is such that the Sobolev embedding theorem $W^{k,q}(B_1(0)) \subset L^\infty(B_1(0))$ holds. By proceeding as in the proof of Theorem B.1 show the estimate (B.3). ■

Theorem B.3 (polynomial inverse estimates). *Let $p \in \mathbb{N}_0$, $d \in \mathbb{N}$, $k \in \mathbb{N}$. Then there exists a constant $C > 0$ depending only on p, d, and there exists a constant C_k depending only on d, p, k such that for any ball $B \subset \mathbb{R}^d$ of radius $h \leq 1$ there holds for all $\pi \in \mathcal{P}_p$:*

$$\|\pi\|_{L^\infty(B)} \leq Ch^{-d/2}\|\pi\|_{L^2(B)},$$
$$\|\pi\|_{H^k(B)} \leq C_k h^{-k}\|\pi\|_{L^2(B)}.$$

Proof. For $h = 1$ this estimate follows from the equivalence of norm of the finite dimensional space $\mathcal{P}_p$. The general case $h \neq 1$ follows by a scaling argument (see also [23, Lemma 4.5.3]).

Lemma B.4. *Let $B_1 \subset B_2 \subset \mathbb{R}^d$ be two balls of radius r_1, r_2, respectively. Then*

$$\|\pi\|_{L^\infty(B_2)} \leq \left(\frac{2r_2}{r_1}\right)^p \|\pi\|_{L^\infty(B_1)} \qquad \forall \pi \in \mathcal{P}_p. \tag{B.8}$$

Proof. To show this, we employ the following one-dimensional Bernstein estimate for $r \geq 1$, [33, Chap. 4, Thm. 2.2]:

$$\|\pi\|_{L^\infty(-r,r)} \leq r^p\|\pi\|_{L^\infty(-1,1)} \qquad \forall \pi \in \mathcal{P}_p. \tag{B.9}$$

Let $B_1 = B_{r_1}(x_1)$, $B_2 = B_{r_2}(x_2)$. Let $y \in B_{r_2}(x_2) \setminus \{x_1\}$ be arbitrary; let l be the line passing through the points y and x_1. Then the length of $l \cap B_1$ is $2r_1$ and the length of $l \cap B_{r_2}(x_2)$ is bounded by $\operatorname{diam} B_{r_2}(x_2)$. Since the restriction of π to l can be viewed as a univariate polynomial, the one-dimensional result (B.9) implies

$$|\pi(y)| \leq \|\pi\|_{L^\infty(l)} \leq \left(\frac{\operatorname{diam} B_{r_2}(x_2)}{r_1}\right)^p \|\pi\|_{L^\infty(l\cap B_1)} \leq \left(\frac{2r_2}{r_1}\right)^p \|\pi\|_{L^\infty(B_1)}.$$

Since $y \in B_{r_2}(x_2)$ was arbitrary, the desired bound (B.8) follows. □

C Approximation with adapted function systems

In this appendix, we prove Theorems 5.13, 5.14, and 5.17. These results are restricted to two-dimensional problems and make use of complex variables. We will identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ where appropriate without explicit mention.

C.1 The theory of Bergman and Vekua

We consider equations of the form

$$-\Delta u + a\partial_x u + b\partial_y u + cu = 0 \qquad \text{on } \Omega \subset \mathbb{R}^2, \tag{C.1}$$

where the constants a, b, c are real. The theory of S. Bergman [16] and I.N. Vekua [105] asserts the existence of a bijection between (suitably normalized) holomorphic functions and the solutions of (C.1). This bijection is even bicontinuous in Sobolev norms:

Lemma C.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected Lipschitz domain. Fix $z_0 \in \Omega$. Let $\mathcal{H} := \{\varphi \,|\, \varphi$ holomorphic on Ω and $\varphi(z_0)$ real$\}$. Then there exists a linear map* ReV *with the following properties:*

1. *$\operatorname{ReV}(\varphi)$ solves (C.1) for every $\varphi \in \mathcal{H}$.*
2. *For every solution u of (C.1) there exists a unique $\varphi \in \mathcal{H}$ such that $\operatorname{ReV}(\varphi) = u$.*
3. *$\|\operatorname{ReV}(\varphi)\|_{H^k(\Omega)} \leq C\|\varphi\|_{H^k(\Omega)}$ for all $\varphi \in \mathcal{H}$ and $k \geq 0$.*
4. *If $u \in H^k(\Omega)$, $k \geq 1$, solves (C.1), then the corresponding $\varphi = \operatorname{ReV}^{-1}(u) \in \mathcal{H}$ is likewise in $H^k(\Omega)$ and $\|\varphi\|_{H^k(\Omega)} \leq C\|u\|_{H^k(\Omega)}$.*

In the last two estimates, the constant C depends on k, Ω, and the differential operator.

Proof. See [80]. Corresponding bicontinuity results in Hölder spaces have been obtained in [38]. □

Remark C.2. The case of Laplace's equation is particularly simple. Then ReV reduces to the operator Re, i.e., taking the real part of a holomorphic function. Lemma C.1 can be generalized to the case of real analytic coefficients a, b, c; we refer to [80] and [16,105] for the precise statements. ∎

An important observation is that the operator ReV can also be computed for Helmholtz's equation. For $z_0 = 0$ and writing (x, y) in polar coordinates, it is shown in [80] that

$$\mathrm{ReV}[z^n] = n! \left(\frac{2}{k}\right)^n \cos(n\varphi) J_n(kr), \tag{C.2a}$$

$$\mathrm{ReV}[\mathbf{i}z^n] = -n! \left(\frac{2}{k}\right)^n \sin(n\varphi) J_n(kr); \tag{C.2b}$$

here and in the remainder of this section (r, φ) denotes polar coordinates, i.e., $x = r\cos\varphi$, $y = r\sin\varphi$; the functions J_n are the first kind Bessel functions.

C.2 Proof of Theorems 5.13, 5.14

The approximation properties of the spaces $V(p)$ of (5.8) are proved in [80]. The purpose of the present section is to show how the approximation properties of $W(p)$ (see (5.7)) can be inferred from those of $V(p)$. To that end, we need to approximate the functions $e^{\mathbf{i}n\varphi} J_n(kr)$ from $W(p)$:

Lemma C.3. *Let the spaces $W(p)$ be defined by (5.7). Then there exists $C > 0$ independent of $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$ and there exists, for each $n \in \mathbb{N}_0$, a function $v \in W(p)$ such that for all $R \geq 1$, $(x, y) \in \mathbb{R}^2$, $k \geq 0$ we have*

$$|e^{\mathbf{i}n\varphi} J_n(kr) - v(x, y)| \leq C e^{nR} e^{ke^R(|x|+|y|)} e^{-pR/e},$$
$$|\nabla(e^{\mathbf{i}n\varphi} J_n(kr) - v(x, y))| \leq C e^{nR}(1 + ke^R) e^{ke^R(|x|+|y|)} e^{-pR/e}.$$

Proof. Given n and p, we will construct the function $v \in W(p)$ explicitly.
1. step: We start by deriving an integral representation for $e^{\mathbf{i}n\varphi} J_n(kr)$. From [46, 8.411] we have for $z \in \mathbb{C}$ the integral representation

$$J_n(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-n\mathbf{i}\theta + \mathbf{i}z\sin\theta} \, d\theta. \tag{C.3}$$

Next, we recall $x = r\cos\varphi$, $y = r\sin\varphi$, and we get using the periodicity of the integrand in (C.3)

$$\begin{aligned}
\pi e^{\mathbf{i}n\pi/2} e^{\mathbf{i}n\varphi} J_n(kr) &= e^{\mathbf{i}n\pi/2} e^{\mathbf{i}n\varphi} \int_{-\pi}^{\pi} e^{-\mathbf{i}n(\theta+\varphi+\pi/2) + \mathbf{i}kr\sin(\theta+\varphi+\pi/2)} \, d\theta \\
&= \int_{-\pi}^{\pi} e^{-\mathbf{i}n\theta + \mathbf{i}kr\{\cos\theta\cos\varphi - \sin\theta\sin\varphi\}} \, d\theta = \int_{-\pi}^{\pi} e^{-\mathbf{i}n\theta + \mathbf{i}k\{x\cos\theta - y\sin\theta\}} \, d\theta \\
&= \int_{-\pi}^{\pi} e^{-\mathbf{i}n\theta + \mathbf{i}k\{x\cos\theta + y\sin\theta\}} \, d\theta. \qquad \text{(C.4)}
\end{aligned}$$

By differentiating under the integral sign with respect to x and y, we obtain a similar expression for the gradient of $e^{\mathbf{i}n\varphi} J_n(kr)$.

2. step: For $\rho > 0$ we define the strip $S_\rho := \{z \in \mathbb{C} \,|\, |\operatorname{Im} z| < \rho\}$. We claim that the Fourier coefficients g_ν of periodic functions g that are holomorphic on a strip S_R decay exponentially. For $\rho < R$ the expression $g_\rho := \sup_{z \in S_\rho} |g(z)|$ is finite and an m-fold integration by parts gives for $\nu \neq 0$

$$g_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathbf{i}\nu\theta} g(\theta)\, d\theta = \frac{1}{2\pi} \left(\frac{1}{\mathbf{i}\nu}\right)^m \int_{-\pi}^{\pi} e^{-\mathbf{i}\nu\theta} g^{(m)}(\theta)\, d\theta.$$

Using the Cauchy integral representation formula we get for $\nu \neq 0$

$$|g_\nu| = \left| \frac{1}{2\pi} \frac{m!}{2\pi\mathbf{i}} \left(\frac{1}{\mathbf{i}\nu}\right)^m \oint_{|t|=\rho} \int_{-\pi}^{\pi} e^{\mathbf{i}\nu\theta} \frac{g(\theta+t)}{(-t)^{m+1}}\, d\theta\, dt \right| \leq C \frac{m!}{(\rho|\nu|)^m} g_\rho.$$

The parameter $m \in \mathbb{N}_0$ is at our disposal. We choose it as $\lfloor |\nu|\rho/e \rfloor$ and get, using the generous bound $m! \leq m^m$,

$$\frac{m!}{(\rho|\nu|)^m} \leq \left(\frac{m}{\rho|\nu|}\right)^m \leq \left(\frac{|\nu|\rho/e}{|\nu|\rho}\right)^{|\nu|\rho/e-1} = e e^{-|\nu|\rho/e}.$$

Thus, we arrive at

$$|g_\nu| \leq e e^{-\rho|\nu|/e} g_\rho \qquad \forall \nu \in \mathbb{Z}$$

and conclude

$$\sum_{|\nu| \geq p} |g_\nu| = \frac{2e}{1 - e^{-\rho/e}} e^{-p\rho/e} g_\rho. \tag{C.5}$$

3. step: For $p \in \mathbb{N}$ and $\theta_j := -\pi + \frac{2\pi}{p} j$, $j = 0, \ldots, p-1$, we denote by T_p the trapezoidal rule for integration on the interval $(-\pi, \pi)$, i.e.,

$$T_p f := \frac{2\pi}{p} \sum_{j=0}^{p-1} f(\theta_j).$$

The rule T_p is exact for trigonometric polynomials of degree $p-1$, i.e.,

$$T_p f = \int_{-\pi}^{\pi} f(\theta)\, d\theta \qquad \forall f \in \mathcal{T}_p := \operatorname{span}\{e^{j\theta}, e^{-j\theta} \,|\, j = 0, \ldots, p-1\}.$$

Hence, if the periodic function g has the Fourier representation $g(\theta) = \sum_{\nu \in \mathbb{Z}} g_\nu e^{\mathbf{i}\nu\theta}$, we can bound

$$\left| \int_{-\pi}^{\pi} g(\theta)\, d\theta - T_p g \right| \leq 4\pi \inf_{v \in \mathcal{T}_p} \|g - v\|_{L^\infty(-\pi,\pi)} \leq 4\pi \sum_{|\nu| \geq p} |g_\nu|. \tag{C.6}$$

4. step: We observe that an approximation of $e^{\mathbf{i}n\varphi} J_n(kr)$ from $W(p)$ can be obtained by applying the trapezoidal rule to the integral (C.4). We set

$$g(\theta) := \frac{1}{\pi} e^{-\mathbf{i}n\pi/2} e^{-\mathbf{i}n\theta + \mathbf{i}k\{x\cos\theta + y\sin\theta\}}$$

and note $v := T_p g \in W(p)$. It therefore remains to get bounds on the error $e^{\mathbf{i}n\varphi} J_n(kr) - v$. The function g is entire, and we can bound for any $R > 0$

$$\sup_{z \in S_R} |g(z)| \leq e^{nR} e^{ke^R(|x|+|y|)}. \tag{C.7}$$

Hence, we get by combining (C.5), (C.6), (C.7)

$$|e^{\mathbf{i}n\varphi} J_n(kr) - v| \leq 4\pi \sum_{|\nu| \geq p} |g_\nu| \leq C e^{nR} e^{ke^R(|x|+|y|)} e^{-pR/e},$$

where the constant $C > 0$ is independent of n, p, $R \geq 1$, and x, y.
5. step: The bound for the gradient $\nabla(e^{\mathbf{i}n\varphi} J_n(kr) - v)$ is obtained similarly: By differentiating under the integral sign, we have the representation formula $\partial_x e^{\mathbf{i}n\varphi} J_n(kr) = \int_{-\pi}^{\pi} \partial_x g(\theta)\, d\theta$; by linearity of the operator T_p we have $\partial_x v = T_p(\partial_x g)$. Reasoning as above then gives the desired bound. □

Proof of Theorems 5.13 and 5.14. It only remains to prove the approximation properties of the space $W(p)$. We will only show Theorem 5.14 and leave the proof of Theorem 5.13 to the reader. Let Ω be star-shaped with respect to the ball $B_\rho(0)$. The real and imaginary parts $u_1 := \operatorname{Re} u$ and $u_2 := \operatorname{Im} u$ of the complex-valued solution u of the Helmholtz equation also solve the Helmholtz equation. Additionally, $\|u_1\|_{H^k(\Omega)} + \|u_2\|_{H^k(\Omega)} \leq C\|u\|_{H^k(\Omega)}$. From the approximation properties of $V(p)$ detailed in (5.10) and the observation (C.2) we have the existence of holomorphic polynomials $P_j \in \mathcal{H}_N$ of degree N such that

$$\|u_j - \mathrm{ReV}\, P_j\|_{H^1(\Omega)} \leq C \left(\frac{\ln N}{N}\right)^{\lambda(k-1)}. \tag{C.8}$$

Lemma C.1 asserts that ReV is bicontinuous in Sobolev spaces, so we get

$$\|P_j\|_{H^1(\Omega)} \leq C\| \mathrm{ReV}\, P_j\|_{H^1(\Omega)} \leq C \tag{C.9}$$

for some $C > 0$ that is independent of N. We now approximate ReV P_j from $W(p)$. To that end, we write the polynomial P_j as $P_j(z) = \sum_{n=0}^{N} a_{n,j} z^n$. Cauchy's integral representation then gives

$$a_{n,j} = \frac{1}{2\pi\mathbf{i}} \oint_{|t|=\rho/2} \frac{P_j(t)}{(-t)^{n+1}}\, dt.$$

The bound (C.9) and Lemma C.8 then imply

$$\|P_j\|_{L^\infty(B_{\rho/2}(0))} \leq \frac{1}{\sqrt{\pi}\, \operatorname{dist}(\partial B_{\rho/2}(0), \partial B_\rho(0))} \|P_j\|_{L^2(B_\rho(0))} \leq C$$

for some $C > 0$ independent of N. From this, we infer for the coefficients $a_{n,j}$ of the polynomial P_j

$$|a_{n,j}| \leq C \frac{1}{(\rho/2)^n} \|P_j\|_{L^2(B_\rho(0))} \leq C \frac{1}{(\rho/2)^n}.$$

In view of (C.2) and Lemma C.3, we can approximate for $p \geq N$

$$\inf_{v \in W(p)} \|\operatorname{Re} V P_j - v\|_{H^1(\Omega)} \leq C \sum_{n=0}^{N} |a_{n,j}| n! \left(\frac{2}{k}\right)^n e^{nR} (1 + k e^R) e^{k e^R \operatorname{diam}\Omega} e^{-pR/e}.$$

Here, the constant $C > 0$ is independent of the parameters R and N, both of which we will now choose. We estimate

$$\sum_{n=0}^{N} |a_{n,j}| n! e^{nR} \left(\frac{2}{k}\right)^n \leq C N! e^{(\gamma+R)N}$$

for suitable C, $\gamma > 0$ independent of N, R. Choosing now (ignoring the complications do to rounding $p/\log p$ to the nearest integer)

$$N = \frac{p}{\log p} \tag{C.10}$$

we can bound $\log N! \leq N \log N = \frac{p}{\log p} \log(p/\log p) \leq p$ to arrive at

$$\sum_{n=0}^{N} |a_{n,j}| n! e^{nR} \left(\frac{2}{n}\right)^n \leq C e^{\gamma' p}$$

for some C, $\gamma' > 0$ independent of p and R. Hence, choosing $R > 0$ sufficiently large allows us to estimate

$$\inf_{v \in W(p)} \|\operatorname{Re} V P_j - v\|_{H^1(\Omega)} \leq C e^{-bp}, \tag{C.11}$$

for some appropriate $b > 0$ independent of p. The triangle inequality $\|u_j - v\|_{H^1(\Omega)} \leq \|u_j - \operatorname{Re} V P_j\|_{H^1(\Omega)} + \|\operatorname{Re} V P_j - v\|_{H^1(\Omega)}$ and making use of (C.8), (C.10), (C.11) allows us to conclude the proof. $\square$

C.3 Two-dimensional elasticity

For complex-valued functions, we use the standard abbreviations $\partial_z = \frac{1}{2}(\partial_x - \mathbf{i}\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + \mathbf{i}\partial_y)$. As discussed in (5.14), the displacement field (u, v) can be expressed on simply connected domains in terms of two holomorphic function φ, ψ. We can then check that

$$2\mu \partial_{\overline{z}}^m (u + \mathbf{i}v) = -z\overline{\varphi^{(m+1)}} - \overline{\psi^{(m)}}, \tag{C.12a}$$

$$\sigma_x + \sigma_y = 2 \operatorname{Re} \varphi', \tag{C.12b}$$

$$2\mu \partial_z (u + \mathbf{i}v) = (\kappa + 1) \operatorname{Re} \varphi' + \mathbf{i}(\kappa - 1) \operatorname{Im} \varphi', \tag{C.12c}$$

where the stresses σ_x, σ_y are defined in Section 5.4. It will be convenient to combine the components of the displacement field (u, v) into the complex-valued function

$$\mathbf{u}(x, y) := u(x, y) + \mathbf{i}v(x, y).$$

The next lemma shows that the functions φ, ψ appearing in the representation formula (5.14) inherit regularity from the displacement field $\mathbf{u}$:

Lemma C.4. *Let $\Omega \subset \mathbb{R}^2$ be star-shaped with respect to a ball $B_\rho(z_0)$. Let the displacement field $\mathbf{u} = u + \mathbf{i}v \in H^k(\Omega)$ for some $k \in \mathbb{N}$. Let $z_0 \in \Omega$. Let φ, ψ be the holomorphic functions appearing in the representation formula (5.14), which are uniquely determined by stipulating $\varphi(z_0) = 0$. Then*

$$\|\varphi\|_{H^k(\Omega)} + \|\psi\|_{H^{k-1}(\Omega)} \leq C\|\mathbf{u}\|_{H^k(\Omega)},$$

where $C > 0$ depends only on the Lamé constants, upper bounds on $\operatorname{diam}\Omega$, *and lower bounds on ρ.*

Proof. We will only show the case $k = 1$ and leave the case $k > 1$ to the reader. Equation (C.12b) implies that $\operatorname{Re}\varphi' \in L^2(\Omega)$ with $\|\operatorname{Re}\varphi'\|_{L^2(\Omega)} \leq C\|\mathbf{u}\|_{H^1(\Omega)}$. Equation (C.12c) then shows that also $\operatorname{Im}\varphi' \in L^2(\Omega)$ with $\|\operatorname{Im}\varphi'\|_{L^2(\Omega)} \leq C\|\mathbf{u}\|_{H^1(\Omega)}$. The condition $\varphi(z_0) = 0$ then allows us to infer from Lemma C.9 that $\|\varphi\|_{L^2(\Omega)} \leq C\|\varphi'\|_{L^2(\Omega)}$ for a constant $C > 0$ that depends only on upper bounds on $\operatorname{diam}\Omega$ and lower bounds on ρ. Finally, we use once more the representation formula (5.14) to get the desired L^2 estimate for ψ. □

Lemma C.5. *Let $\Omega \subset \mathbb{C}$ be a domain and define for $\varepsilon > 0$ the set $\Omega_\varepsilon = \{z \in \Omega \,|\, B_\varepsilon(z) \subset \Omega\}$. If f, g are holomorphic on Ω and satisfy $f \in H^s(\Omega)$, $z\overline{f'} + \overline{g} \in H^s(\Omega)$ for some $s \in [0,1]$, then*

$$\|z\overline{f'} + \overline{g}\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{s-1}\left\{\|f\|_{H^s(\Omega)} + \|z\overline{f'} + \overline{g}\|_{H^s(\Omega)}\right\}.$$

Proof. The case $s = 1$ is trivial and the case $s = 0$ is very similar to the case $s \in (0,1)$. We have to bound the $L^2(\Omega_\varepsilon)$-norms of

$$\partial_z(z\overline{f'} + \overline{g}) = \overline{f'}, \qquad \partial_{\overline{z}}(z\overline{f'} + \overline{g}) = z\overline{f''} + \overline{g'}.$$

By an interior estimate for holomorphic functions, [80, Lemma 2.4], we have for each $s' \in [0,1]$ a constant $C_{s'} > 0$ such that for all $f \in H^{s'}(\Omega)$ that are holomorphic on Ω

$$\|f'\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{s'-1}|f|_{H^{s'}(\Omega)}. \tag{C.13}$$

For the bound on $z\overline{f''} + \overline{g'}$ we use Cauchy's integral representation formula to get for $z \in \Omega_\varepsilon$

$$\begin{aligned}\overline{z\overline{f''} + \overline{g'}} &= \frac{1}{2\pi\mathbf{i}}\oint_{|t-z|=\varepsilon} \frac{(\overline{z} - \overline{t})f'(t)}{(z-t)^2}\,dt \\ &\quad + \frac{1}{2\pi\mathbf{i}}\oint_{|t-z|=\varepsilon} \frac{\overline{t}f'(t) + g(t) - (\overline{z}f'(z) + g(z))}{(z-t)^2}\,dt. \end{aligned}\tag{C.14}$$

For the second term, we used additionally $\oint_{|z-t|=\varepsilon} \frac{1}{(z-t)^2}\,dt = 0$. For the first integral in (C.14), we observe that $|t-z| = \varepsilon$ implies $\overline{z} - \overline{t} = \frac{\varepsilon^2}{z-t}$ and recognize the first integral to be

$$\frac{1}{2\pi\mathbf{i}}\oint_{|t-z|=\varepsilon} \frac{(\overline{z} - \overline{t})f'(t)}{(z-t)^2}\,dt = \frac{\varepsilon^2}{2!}\frac{2!}{2\pi\mathbf{i}}\oint_{|t-z|=\varepsilon} \frac{f'(t)}{(z-t)^3}\,dt = \frac{\varepsilon^2}{2!}f'''(z).$$

Together with bounds on the second integral, we arrive at

$$\left|\overline{z\overline{f''}+\overline{g'}}\right|^2 \leq C\varepsilon^4|f'''(z)|^2 + C\varepsilon^{+2s}\sup_{t\in\partial B_\varepsilon(z)}\frac{|\overline{t}f'(t)+g(t)-(\overline{z}f'(z)+g(z)|^2}{|z-t|^{2+2s}}.$$

Upon integrating in $z \in \Omega_\varepsilon$, we can bound $\varepsilon^2\|f'''\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{s-1}|f|_{H^s(\Omega)}$ if we note $\Omega_\varepsilon \subset \Omega_{\varepsilon/3} \subset \Omega_{2\varepsilon/3} \subset \Omega$ and use (C.13) repeatedly, namely, twice with $s' = 0$ and once with $s' = s$. For the second term involving the supremum, we use the interior estimate (C.22) to bound the supremum and then integrate in the z-variable to obtain the desired result. □

Lemma C.6. *Let Ω be star-shaped with respect to the ball $B_\rho(0)$. Let $m \in \mathbb{N}$, $s \in [0,1)$. Let the displacement field (u,v) be in $H^{m+s}(\Omega)$. Define the function*

$$g(t) := 2\mu\left(u((1-t)z) + \mathbf{i}v((1-t)z)\right).$$

Then for $t \in (0,1/2)$

$$\|g^{(m+1)}(t)\|_{L^2(\Omega)} + \|g^{(m)}(t)\|_{H^1(\Omega)} \leq Ct^{-(1-s)}\|\mathbf{u}\|_{H^{m+s}(\Omega)}. \tag{C.15}$$

Proof. We will only show the bound on $g^{(m)}$, the other one being handled similarly. Using the representation formula (5.14) for $\mathbf{u} = u + \mathbf{i}v$, we write

$$\begin{aligned}
g(t) &= -(1-t)z\overline{\varphi'((1-t)z)} - \overline{\psi((1-t)z))} + \kappa\varphi((1-t)z),\\
g^{(m)}(t) &= \left[-(1-t)z\overline{\varphi^{(m+1)}((1-t)z)} - \overline{\psi^{(m)}((1-t)z))}\right]\overline{(-z)^m}\\
&\quad + mz\overline{(-z)^{m-1}\varphi^{(m)}((1-t)z)} + \kappa(-z)^m\varphi^{(m)}((1-t)z),\\
\partial_z g^{(m)}(t) &= -(1-t)\overline{(-z)^m\varphi^{(m+1)}((1-t)z)} + m\overline{(-z)^{m-1}\varphi^{(m)}((1-t)z)}\\
&\quad + \kappa\frac{d}{dz}\left[(-z)^m\varphi^{(m)}((1-t)z)\right],\\
\partial_{\overline{z}} g^{(m)}(t) &= (1-t)\left[-(1-t)z\overline{\varphi^{(m+2)}((1-t)z)} - \overline{\psi^{(m+1)}((1-t)z)}\right]\overline{(-z)^m}\\
&\quad - m\left[-(1-t)z\overline{\varphi^{(m+1)}((1-t)z)} - \overline{\psi^{(m)}((1-t)z)}\right]\overline{(-z)^{m-1}}\\
&\quad + mz\overline{\frac{d}{dz}\left[(-z)^{m-1}\varphi^{(m)}((1-t)z)\right]}.
\end{aligned}$$

The estimate (C.15) follows from the change of variables $\zeta = (1-t)z$, the observations (C.12), and Lemma C.4. An additional ingredient to the proof is the fact that there exists $C > 0$ such that $B_{Ct}(z) \subset \Omega$ for all $z \in (1-t)\Omega$ so that Lemma C.5 can be employed. □

Lemma C.7. *Assume the hypotheses of Lemma C.6. Let T_m be the Taylor polynomial of g about the point $t_0 = \varepsilon$ that is evaluated at $t = 0$, i.e.,*

$$T_m = \sum_{\nu=0}^{m} g^{(\nu)}(\varepsilon)\frac{(-\varepsilon)^\nu}{\nu!}.$$

Then T_m *is defined on* $\frac{1}{1-\varepsilon}\Omega$ *and*

$$\|T_m\|_{L^2(\frac{1}{1-\varepsilon}\Omega)} \leq C\|\mathbf{u}\|_{H^m(\Omega)}, \qquad \text{(C.16)}$$

$$\|T_m\|_{H^1(\frac{1}{1-\varepsilon/2}\Omega)} \leq C\varepsilon^{-1}\|\mathbf{u}\|_{H^m(\Omega)}, \qquad \text{(C.17)}$$

$$\|g(0) - T_m\|_{L^2(\Omega)} + \varepsilon\|g(0) - T_m\|_{H^1(\Omega)} \leq C\varepsilon^{m+s}\|\mathbf{u}\|_{H^{m+s}(\Omega)}. \qquad \text{(C.18)}$$

Proof. The bound (C.16) follows from the change of variables $\zeta = (1-\varepsilon)z$, an inspection of the definition of the terms $g^{(j)}$, $j = 0, \ldots, m$, equation (C.12), and Lemma C.4. The proof of (C.17) follows along the same lines. Estimating $\partial_z g^{(m)}(t)$, however, requires additionally to use Lemma C.5 and the observation that $\frac{1}{1-\varepsilon/2}\Omega \subset \{z \in \frac{1}{1-\varepsilon}\Omega \,|\, B_{\varepsilon'}(z) \subset \frac{1}{1-\varepsilon}\Omega\}$ for some $\varepsilon' \sim \varepsilon$. In the bound (C.18), we will only show the $H^1(\Omega)$-estimate. We will also exclude the case $m = 1$, $s = 0$, which we leave to the reader. We choose $\delta \in (0, 1/2)$ such that $2(m-1) - 2(1-s) + 2\delta > 0$ and recall the Taylor formula

$$g(0) - T_m = -\frac{1}{m!}(-\varepsilon)^m g^{(m)}(\varepsilon) - \frac{1}{(m-1)!}\int_\varepsilon^0 g^{(m)}(t)(-t)^{m-1}\,dt.$$

The first term can be bounded by $\varepsilon^{m+s-1}\left[\|u\|_{H^{m+s}(\Omega)} + \|v\|_{H^{m+s}(\Omega)}\right]$ by Lemma C.6. For the integral, we estimate

$$\left\|\int_\varepsilon^0 g^{(m)}(t)t^{m-1}\,dt\right\|^2_{H^1(\Omega)} \leq \int_0^\varepsilon \|g^{(m)}(t)\|^2_{H^1(\Omega)} t^{2(1-s-\delta)}\,dt \int_0^\varepsilon |t^{-(1-s)+\delta+m-1}|^2\,dt,$$

which can again be estimated in the desired fashion using Lemma C.6. $\square$

Proof of Theorem 5.17. Without loss of generality, we assume that Ω is star-shaped with respect to the ball $B_\rho(0)$. For a parameter $\varepsilon > 0$ sufficiently small, which will be chosen below in dependence on the polynomial degree p, we define g and T_m as in Lemmas C.6, C.7. Then T_m is defined on $\frac{1}{1-\varepsilon}\Omega$ and, since $g(0) = \mathbf{u}$, we get from Lemma C.7

$$\|\mathbf{u} - T_m\|_{H^j(\Omega)} \leq C\varepsilon^{m+s-j}\|\mathbf{u}\|_{H^{m+s}(\Omega)}, \qquad j = 0, 1. \qquad \text{(C.19)}$$

From the representation formulas for the $g^{(j)}$, $j = 0, \ldots, m$, in the proof of Lemma C.6, we observe that T_m has the form $T_m = \kappa\varphi_1 - z\overline{\varphi_1'} - \overline{\psi_1}$, where φ_1, ψ_1 are functions holomorphic on $\frac{1}{1-\varepsilon}\Omega$ and $\varphi_1(0) = \varphi(0) = 0$. Lemma C.4 (together with the observation that the constant appearing in Lemma C.4 can be made independent of $\varepsilon \in (0, 1/2)$) and Lemma C.7 then imply

$$\begin{aligned}\|\varphi_1\|_{H^1(\frac{1}{1-\varepsilon/2}\Omega)} + \|\psi_1\|_{L^2(\frac{1}{1-\varepsilon/2}\Omega)} &\leq C\|T_m\|_{H^1(\frac{1}{1-\varepsilon/2}\Omega)} \\ &\leq C\varepsilon^{-1}\|\mathbf{u}\|_{H^m(\Omega)}. \qquad \text{(C.20)}\end{aligned}$$

Since φ_1, ψ_1 are holomorphic on $\frac{1}{1-\varepsilon}\Omega$, they can be approximated on Ω by (complex) polynomials at an exponential rate. Namely, by Szegö's approximation result (see [80, Thm. 2.6]) there exist complex polynomials φ_{ap},

$\psi_{ap} \in \mathcal{H}_p$ of degree p such that

$$\|\varphi_1 - \varphi_{ap}\|_{W^{j,\infty}(\Omega)} \le Ch^{-\alpha}(1+h)^{-p}\|\varphi_1\|_{L^2(\mathrm{Int}(L_4h))}, \quad j = 0,1,2, \quad \text{(C.21a)}$$
$$\|\psi_1 - \psi_{ap}\|_{W^{j,\infty}(\Omega)} \le Ch^{-\alpha}(1+h)^{-p}\|\varphi_1\|_{L^2(\mathrm{Int}(L_4h))}, \quad j = 0,1,2; \quad \text{(C.21b)}$$

here, $L_h = \{\varphi_\Omega(z) \,|\, |z| = 1+h\}$, where $\varphi_\Omega : \mathbb{C} \setminus B_1(0) \to \mathbb{C} \setminus \Omega$ is the unique conformal map with $\varphi_\Omega(\infty) = \infty$ and $\varphi'_\Omega(\infty) > 0$. The constants C, $\alpha > 0$ are independent of h and p. By geometric considerations (see [80, Lemma 2.3]), we can ascertain the existence of $D > 0$ such that for $h^{\hat{\lambda}} = D\varepsilon$ we have $\mathrm{Int}\, L_{4h} \subset \frac{1}{1-\varepsilon/2}\Omega$. Hence, combining (C.19), (C.20), (C.21), we can conclude for $j \in \{0,1\}$

$$\begin{aligned}&\|\mathbf{u} - (-z\overline{\varphi'_{ap}} - \overline{\psi_{ap}} + \kappa\varphi_{ap})\|_{H^j(\Omega)} \\ &\qquad \le C\varepsilon^{m+s-j}\|\mathbf{u}\|_{H^{m+s}(\Omega)} + \varepsilon^{-\hat{\lambda}\alpha}(1+(D\varepsilon)^{1/\hat{\lambda}})^{-p}\varepsilon^{-1}\|\mathbf{u}\|_{H^m(\Omega)}.\end{aligned}$$

Choosing

$$\varepsilon = K\left(\frac{\log(p+2)}{p+2}\right)^{\hat{\lambda}}$$

for sufficiently large K gives the desired bound stated in Theorem 5.17. □

Lemma C.8 (interior estimates for holomorphic functions). *Let $\Omega \subset \mathbb{C}$ be a domain. Define for $\varepsilon > 0$ the set $\Omega_\varepsilon := \{z \in \Omega \,|\, B_\varepsilon(z) \subset \Omega\}$. Then for any function f that is holomorphic on Ω*

$$\|f\|_{L^\infty(\Omega_\varepsilon)} \le \frac{1}{\sqrt{\pi}\varepsilon}\|f\|_{L^2(\Omega)}. \quad \text{(C.22)}$$

Proof. The proof can be found, for example, in [76]. For the reader's convenience, we reproduce it here: For fixed $z \in \Omega_\varepsilon$ we use Cauchy's integral representation theorem to write for any $r \in (0,\varepsilon)$

$$|f(z)| = \left|\frac{1}{2\pi\mathbf{i}}\oint_{|t|=r}\frac{f(z+t)}{-t}\,dt\right| = \frac{1}{2\pi}\left|\int_{\partial B_1(0)} f(z+rt)\,|dt|\right|.$$

Multiplying this equality by r and integrating over r from 0 to ε gives, if we note that the right-hand side integral is then an area integral in polar coordinates,

$$\begin{aligned}\frac{1}{2}\varepsilon^2|f(z)| &= \int_0^\varepsilon r|f(z)|\,dr = \frac{1}{2\pi}\int_0^\varepsilon\left|\int_{\partial B_1(0)} f(z+rt)\,|dt|\right| r\,dr \\ &\le \frac{\varepsilon}{2\sqrt{\pi}}\left(\int_0^\varepsilon\int_{\partial B_1(0)}|f(z+rt)|^2\,|dt| r\,dr\right)^{1/2} = \frac{\varepsilon}{2\sqrt{\pi}}\|f\|_{L^2(B_\varepsilon(z))}.\end{aligned}$$

Since $z \in \Omega_\varepsilon$ was arbitrary, the proof is complete. □

Lemma C.9. *Let $\Omega \subset \mathbb{C}^2$ be star-shaped with respect to* 0 *and assume that $B_\rho(0) \subset \Omega$. Then for $f \in H^1(\Omega)$ holomorphic on Ω we have*

$$\|f - f(0)\|_{L^2(\Omega)} \leq \sqrt{2}\operatorname{diam}\Omega \left[\frac{1}{\pi} + \left(\frac{2\operatorname{diam}\Omega}{\rho}\right)^2\right]^{1/2} \|f'\|_{L^2(\Omega)}. \qquad \text{(C.23)}$$

Proof. We define $\delta := \rho/(2\operatorname{diam}\Omega) < 1$. Since Ω is star-shaped with respect to 0, we can write for $z \in \Omega$ by integrating on the line connecting 0 and z

$$f(z) - f(0) = \int_{t=0}^1 z f'(tz)\,dt = \int_{t=0}^\delta z f'(tz)\,dt + \int_{t=\delta}^1 z f'(tz)\,dt.$$

For the first integral, we note that $t \in (0,\delta)$ and $z \in \Omega$ implies $|tz| \leq \rho/2$. Hence, Lemma C.8 implies

$$\left|\int_{t=0}^\delta z f'(tz)\,dt\right| \leq \frac{\delta \operatorname{diam}\Omega}{\sqrt{\pi}\rho/2}\|f'\|_{L^2(B_\rho(0))} \leq \frac{1}{\sqrt{\pi}}\|f'\|_{L^2(\Omega)}.$$

Thus,

$$\|f - f(0)\|^2_{L^2(\Omega)} \leq 2\frac{\operatorname{area}(\Omega)}{\pi}\|f'\|^2_{L^2(\Omega)} + 2\int_\Omega \left|\int_{t=\delta}^1 z f'(tz)\,dt\right|^2.$$

The second term is treated as follows: First, the Cauchy-Schwarz inequality is applied to the inner integral; then the order of integration is switched, and finally a change of variables $\zeta := tz$ is performed. This leads to

$$\int_\Omega \left|\int_{t=\delta}^1 z f'(tz)\,dt\right|^2 \leq \left(\frac{\operatorname{diam}\Omega}{\delta}\right)^2 \|f'\|^2_{L^2(\Omega)}.$$

Combining the above estimates leads to (C.23). □

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