Numerical Formulations for Approximating the Equations Governing Bed-Load Sediment Transport in Channels and Rivers*

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Abstract

This report looks at how to accurately numerically approximate the equations governing bed-load sediment transport in rivers and channels. We will discuss five different formulations that can be used to numerically approximate the equations. For each formulation, two different numerical schemes will be used to solve the equations. Two different test problems are discussed which are used to compare the performance of the different formulations.

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1 Introduction

In recent years, bed-load sediment transport in channels and rivers has become a major problem. The understanding as to how sediment interacts in certain environments is crucial for both the environment and businesses. For example, if a dam is constructed such that sediment is deposited at the entrance of the dam, a build up of sediment could occur resulting in the sediment needing to be drained frequently. The walls of the dam may get severely damaged if there is a massive build up of sediment next to the walls. In the environment, the understanding of how sediment interacts is essential when chemicals are deposited into rivers. We need to be able to make sure that these chemicals are being dispersed safely and how they interact with the river.

Throughout this report, we will be discussing how we can accurately numerically approximate the equations governing bed-load sediment transport in channels and rivers. These comprise the equation for conservation of mass,

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{1.1}$$

the equation for conservation of momentum,

$$\frac{\partial (uh)}{\partial t} + \frac{\partial \left[hu^2 + \frac{1}{2}gh^2\right]}{\partial x} = -ghB_x, \tag{1.2}$$

and the Bed-Updating Equation,

$$\frac{\partial B}{\partial t} + \xi \frac{\partial q}{\partial x} = 0, \tag{1.3}$$

where $\xi = \frac{1}{1-\epsilon}$ and ϵ is the porosity of the riverbed. Here $h(x,t)$ represents the height of the river, $B(x,t)$ is the height of the riverbed, $u(x,t)$ is the velocity in the x direction and $q(u,h)$ is the total (suspended and bedload) volumetric sediment transport rate in the x direction, see Figure 1.1.

The Sediment Transport Flux, q, is not a direct function of B, which can cause difficulties when deriving a numerical scheme for the Bed-Updating Equation. In some cases, the Sediment Transport Flux cannot be written analytically and is calculated by using a black box approach where the flux is deduced form discrete data. In this report, we will only consider the most basic form of the Sediment Transport Flux,

$$q(u) = Au|u|^{m-1}. \tag{1.4}$$

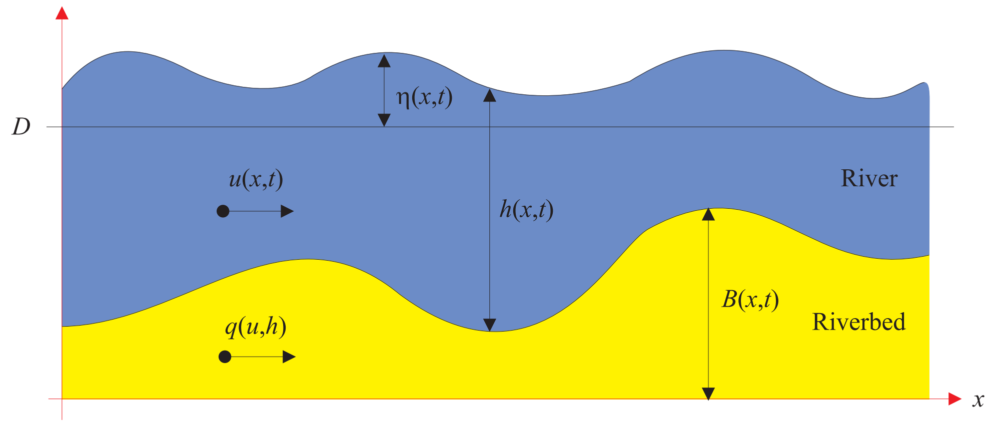

Figure 1.1: The variables $h(x,t)$, $u(x,t)$, $\eta(x,t)$, $q(u,h)$ and $B(x,t)$.

The value of A usually depends on the depth of the river and m is usually chosen such that $1 \leq m \leq 4$. The difficulty with using (1.4) is that it cannot be differentiated with respect to u. However, if we only consider positive values of u then (1.4) becomes

$$q(u) = Au^m \quad \text{where} \quad u \geq 0, \tag{1.5}$$

which can now be differentiated with respect to u. A variety of other Sediment Transport formulae can be found in Soulsby[10], including the Sediment Transport formulae of van Rijn[12].

In Section 2, we will discuss five different formulations that can be used to numerically approximate (1.1), (1.2) and (1.3).

In Section 3, we will discuss some numerical techniques that we can use to approximate the different formulations. Most of the numerical schemes discussed in Section 3 are for the Sediment Transport Flux (1.5), but they could be extended to use any Sediment Transport Flux, including the black box approach.

In Section 4, two test problems will be discussed and the three different formulations will be used to numerically approximate the test problems. An overall comparison will be made to see which approach is the most accurate.

2 Different Formulations

There are numerous ways we can re-write the system (1.1), (1.2) and (1.3) but we require a formulation of the system that can be used to obtain an accurate numerical approximation. All of the formulations discussed in this section will be derived using Sediment Transport Flux (1.5), thus we also require that

$$h(x,t) > 0, \quad u(x,t) \geq 0 \quad \text{and} \quad -\infty < B(x,t) < \infty.$$

All systems discussed in this section can be written in the form

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{w})}{\partial x} = \mathbf{R}, \tag{2.1}$$

where $\mathbf{R}$ is the source term and $\mathbf{F}$ is the flux function, which is called a non-homogeneous conservation law with source term present and whose Jacobian matrix is $\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{w}}$. Some of the numerical schemes discussed in Section 3 require the eigenvalues and eigenvectors of the Jacobian matrix, and so we also derive these here.

2.1 Formulation A

One approach we could use is to re-write (1.1) and (1.2) in system form,

$$\begin{bmatrix} h \\ uh \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ -ghB_x \end{bmatrix}, \tag{2.2}$$

which is called the Shallow Water Equations and is written in conservative variable form. We can then calculate the Bed-Updating Equation,

$$B_t + \xi q_x = 0,$$

separately. This approach we call **Formulation A** and it can be used in three different ways.

1. **Formulation A-NC**: We numerically approximate the Shallow Water Equations and the Bed-Updating Equation using the same time step.

2. **Formulation A-CV**: We converge the Shallow Water Equations to a steady state solution and then we update the riverbed. The overall time step of this formulation is the morphological time step of the Bed-Updating Equation and the Shallow Water Equations are converged to a steady state solution every time the riverbed is updated.

3. **Formulation A-SF**: We can re-write **Formulation A** in system form,

$$\begin{bmatrix} h \\ uh \\ B \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \\ \xi q \end{bmatrix}_x = \begin{bmatrix} 0 \\ -ghB_x \\ 0 \end{bmatrix} \tag{2.3}$$

   and numerically approximate the whole system.

Unfortunately, all three variations of **Formulation A** have a source term present, which may cause difficulties to numerically approximate, especially if the source term is stiff.

The Jacobian matrix will also be required for each variation of **Formulation A**. For

1. **Formulation A-NC** and **Formulation A-CV**, the Shallow Water Equations Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix},$$

   where $c = \sqrt{gh}$, which has eigenvalues

$$\lambda_1 = u - c \quad \text{and} \quad \lambda_2 = u + c,$$

   with corresponding eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ u - c \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ u + c \end{bmatrix}.$$

2. **Formulation A-SF**, if we use the Sediment Transport Flux (1.5) then the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ -ud & d & 0 \end{bmatrix},$$

where $c = \sqrt{gh}$ and $d = \frac{\xi}{h} A m u^{m-1}$ for $u \geq 0$. Notice that this Jacobian matrix is singular, which we might expect to create difficulties when implementing a numerical scheme for this formulation. The eigenvalues of the Jacobian matrix are

$$\lambda_1 = u - c, \quad \lambda_2 = 0 \quad \text{and} \quad \lambda_3 = u + c,$$

whose corresponding eigenvectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ u - c \\ \dfrac{-cd}{u - c} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 1 \\ u + c \\ \dfrac{cd}{u + c} \end{bmatrix}.$$

2.2 Formulation B

Another approach that we can use is to re-write the equation of conservation of momentum (1.2) as

$$\frac{\partial u}{\partial t} + \frac{\partial \left[\frac{1}{2}u^2 + g(h + B)\right]}{\partial x} = 0, \tag{2.4}$$

by using (1.1) and then combine (1.1), (2.4) and (1.3) into system form to obtain **Formulation B**,

$$\begin{bmatrix} h \\ u \\ B \end{bmatrix}_t + \begin{bmatrix} uh \\ \frac{1}{2}u^2 + g(h + B) \\ \xi q \end{bmatrix}_x = 0. \tag{2.5}$$

Notice that this formulation does not have a source term present, hence, **Formulation B** will be easier to numerically approximate. However, **Formulation B** is not in conservative variable form and so, shocks may propagate at incorrect speeds.

If we use the Sediment Transport Flux (1.5) then the Jacobian matrix of **Formulation B** is

$$\mathbf{J} = \begin{bmatrix} u & h & 0 \\ g & u & g \\ 0 & d & 0 \end{bmatrix},$$

where $d = A\xi m u^{m-1}$ for $u \geq 0$. Notice that the Jacobian matrix of **Formulation B** is not singular. The eigenvalues, λ, of the Jacobian matrix cannot be easily written analytically since they are the roots of the polynomial

$$P(\lambda, \mathbf{w}) = \lambda^3 - 2u\lambda^2 + \left[u^2 - g(h+d)\right]\lambda + gud = 0.$$

However, we can prove that the roots of $P(\lambda, \mathbf{w})$ are always real for

$$h(x,t) > 0, \quad u(x,t) \geq 0 \quad \text{and} \quad -\infty < B(x,t) < \infty$$

by using formulae for the roots of a cubic, see Spiegel & Liu[9]. For a cubic equation,

$$P(x) = x^3 + a_1x^2 + a_2x + a_3 = 0,$$

if we let

$$Q = \frac{1}{9}(3a_2 - a_1^2) \quad \text{and} \quad R = \frac{1}{54}(9a_1a_2 - 27a_3 - 2a_1^3),$$

then the *discriminant* is $D = Q^3 + R^2$ and if

1. $D > 0$ then one root is real and two are complex;
2. $D = 0$ then all roots are real and two are equal;
3. $D < 0$ then all roots are real and unequal.

If $D < 0$ then the roots of $P(x)$ are

$$x_1 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta) - \frac{1}{3}a_1, \tag{2.6a}$$

$$x_2 = 2\sqrt{-Q}\cos(\frac{1}{3}(\theta + 2\pi)) - \frac{1}{3}a_1 \tag{2.6b}$$

and

$$x_3 = 2\sqrt{-Q}\cos(\frac{1}{3}(\theta + 4\pi)) - \frac{1}{3}a_1 \tag{2.6c}$$

where $\cos\theta = \frac{R}{\sqrt{-Q^3}}$.

Now, by using the above approach, for **Formulation B**

$$a_1 = -2u, \quad a_2 = u^2 - g(h+d) \quad \text{and} \quad a_3 = gud$$

which implies that

$$Q = -\frac{1}{9}(u^2 + 3g(h+d)) \quad \text{and} \quad R = \frac{u}{54}(9g(2h-d) - 2u^2).$$

Hence,

$$D = \frac{g}{108}[8gu^2h^2 - u^2(4hu^2 + gd(d+20h)) - 4g^2(h^3 + d^3 + 3hd(h+d))]$$

and for all three roots to be real and unequal,

$$8gu^2h^2 < u^2(4hu^2 + gd(d+20h)) + 4g^2(h^3 + d^3 + 3hd(h+d)),$$

which is always satisfied since $h(x,t) > 0$ and $u(x,t) \geq 0 \Rightarrow d \geq 0$. Hence, the roots of $P(\lambda, \mathbf{w})$ are always real and unequal for

$$h(x,t) > 0, \quad u(x,t) \geq 0 \quad \text{and} \quad -\infty < B(x,t) < \infty.$$

It can also be shown that $P(\lambda, \mathbf{w})$ has one negative root and two positive roots, i.e.

$$\lambda_1 < 0 < \lambda_2 < \lambda_3,$$

where λ_2 is the eigenvalue associated with the Bed-Updating Equation.

Now that we have proved that the roots of $P(\lambda)$ are always real, we can obtain the eigenvectors,

$$\mathbf{e}_k = \begin{bmatrix} 1 \\ \frac{1}{h}(\lambda_k - u) \\ \frac{(u-\lambda_k)^2 - gh}{gh} \end{bmatrix},$$

where the λ_k are given by (2.6).

2.3 Formulation C

In order to obtain a formulation that is written in conservative variable form and whose Jacobian matrix is not singular, we can re-write the equation of the conservation of momentum (1.2) as

$$\frac{\partial(uh)}{\partial t} + \frac{\partial\left[hu^2 + \frac{1}{2}gh^2 + ghB\right]}{\partial x} = gBh_x, \tag{2.7}$$

by using the chain rule,

$$\frac{\partial(hB)}{\partial x} = h\frac{\partial B}{\partial x} + B\frac{\partial h}{\partial x},$$

and then combine (1.1), (2.7) and (1.3) into system form to obtain **Formulation C**,

$$\begin{bmatrix} h \\ uh \\ B \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 + ghB \\ \xi q \end{bmatrix}_x = \begin{bmatrix} 0 \\ gBh_x \\ 0 \end{bmatrix}. \tag{2.8}$$

Unfortunately, **Formulation C** has a source term present.

If we use the Sediment Transport Flux (1.5), then the Jacobian matrix of **Formulation C** is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ g(h+B) - u^2 & 2u & gh \\ -ud & d & 0 \end{bmatrix},$$

where $d = \frac{\xi}{h} A m u^{m-1}$ for $u \geq 0$. The eigenvalues, λ, of the Jacobian matrix again cannot be easily written analytically since they are the roots of the polynomial

$$P(\lambda, \mathbf{w}) = \lambda^3 - 2u\lambda^2 + \left[u^2 - g(h + B + hd)\right]\lambda + ghud = 0.$$

However, we can prove that the roots of $P(\lambda, \mathbf{w})$ are always real for

$$h(x,t) > 0, \quad u(x,t) \geq 0 \quad \text{and} \quad -\infty < B(x,t) < \infty$$

by using the approach which was discussed in Section 2.2. For **Formulation C**,

$$a_1 = -2u, \quad a_2 = u^2 - g(h + B + hd) \quad \text{and} \quad a_3 = ghud$$

which implies that

$$Q = -\frac{1}{9}(u^2 + 3g(h + B + hd)) \quad \text{and} \quad R = -\frac{u}{54}(2u^2 - 9g(2h + 2B - hd)).$$

Hence,

$$D = \frac{g}{108}[8u^2 g(h+B)^2 - u^2[4u^2(B+h) + ghd(20(h+B) + hd)] \\ -4g^2(h^3 + B^3 + h^3d^3 + 3h(d+1)(h+B)(hd+B))]$$

and for all three roots to be real and unequal,

$$8u^2 g(h+B)^2 < u^2[4u^2(B+h) + ghd(20(h+B) + hd)] \\ +4g^2(h^3 + B^3 + h^3d^3 + 3h(d+1)(h+B)(hd+B)),$$

which is satisfied if $h(x,t)+B(x,t) > 0$ since $h(x,t) > 0$, $u(x,t) \geq 0 \Rightarrow d \geq 0$ and $-\infty < B(x,t) < \infty$. Hence, the roots of $P(\lambda, \mathbf{w})$ are always real and unequal for

$$h(x,t) > 0, \quad u(x,t) \geq 0 \quad \text{and} \quad -\infty < B(x,t) < \infty$$

if $h(x,t) + B(x,t) > 0$. It can also be shown that $P(\lambda, \mathbf{w})$ has one negative root and two positive roots, i.e.

$$\lambda_1 < 0 < \lambda_2 < \lambda_3,$$

where λ_2 is the eigenvalue associated with the Bed-Updating Equation.

Now that we have proved that the roots of $P(\lambda)$ are always real, we need to determine the eigenvectors. The eigenvectors of the Jacobian matrix are

$$\mathbf{e}_k = \begin{bmatrix} 1 \\ \lambda_k \\ \dfrac{u^2 - g(h+B) + (\lambda_k - 2u)\lambda_k}{gh} \end{bmatrix},$$

where again λ_k are given by (2.6).

Throughout this section, we have discussed five different formulations that can be used to numerically approximate the system (1.1), (1.2) and (1.3). In the next section, we will discuss how these different formulations can be numerically approximated.

3 Numerical Schemes

In this section, we will discuss how we can numerically approximate the formulations discussed in the previous section. We will discuss two numerical approaches: LeVeque & Yee's MacCormack approach (see LeVeque & Yee[7], Yee[13] and Hudson[6]) and an adaptation of Roe's Scheme (see Roe[8], Glaister[3] and Hubbard & Garcia-Navarro[5]). The numerical approaches discussed in this section will be based on (2.1) since all of the formulations can be written in this form. We will also try to ensure that all of the numerical approaches satisfy the Total Variational Diminishing property (see Harten[4] and Sweby[11]). Flux limiter methods and slope limiter methods will be used to try to ensure that the numerical approaches satisfy the TVD property. The numerical approaches will also be adapted specifically for each formulation.

Before we can implement any numerical schemes, we must first define a mesh. In this report, we will be using a variable mesh over the finite region $x_0 \leq x \leq x_I$ and $t_0 \leq t \leq t_N$, which is illustrated in Figure 3.1. The

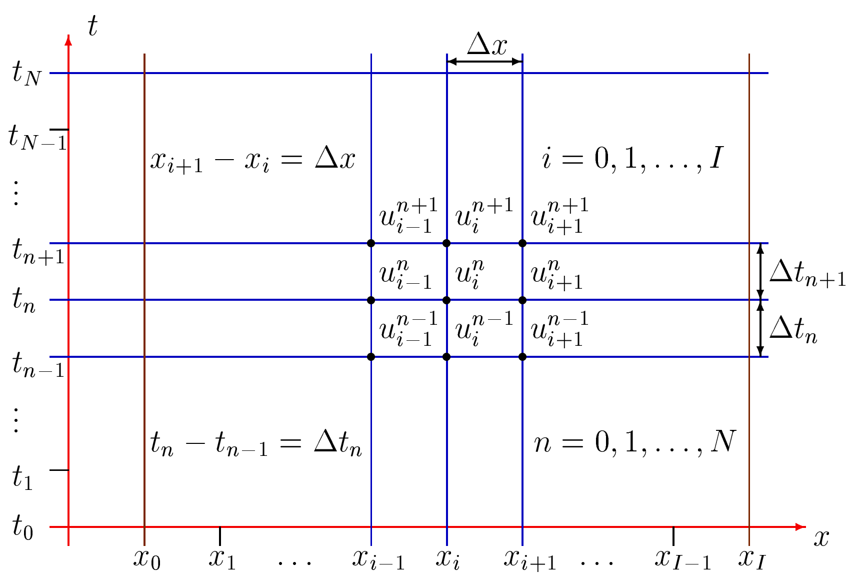

Figure 3.1: The Mesh.

points $x = x_0$ and $x = x_I$ are the spatial boundaries and we will require numerical boundary conditions at these points. The numerical solution is denoted by $u_i^n \approx u(i\Delta x, t_n)$, where $\Delta x = x_{i+1} - x_i$, $t_n = t_0 + \sum_{k=1}^{n} \Delta t_k$ and $\Delta t_k = t_k - t_{k-1}$. The spatial step size, Δx, is fixed and we will use a variable time step, Δt_k, which will be calculated to produce a certain Courant number,

$$\nu = \frac{\Delta t}{\Delta x}\max(|\lambda_k|),$$

in order to ensure that the numerical scheme is stable.

3.1 MacCormack Approach

One of the most basic numerical schemes we can use to approximate (2.1) is the MacCormack approach as discussed by LeVeque & Yee[7],

$$\mathbf{w}_i^{n+1} = \frac{1}{2}(\mathbf{w}_i^n + \mathbf{w}_i^{(1)}) - \frac{s}{2}(\mathbf{F}_i^{(1)} - \mathbf{F}_{i-1}^{(1)}) + \frac{s}{2}\mathbf{R}_{i-1}^{(1)}, \tag{3.1}$$

where

$$\mathbf{w}_i^{(1)} = \mathbf{w}_i^n - s(\mathbf{F}_{i+1}^n - \mathbf{F}_i^n) + s\mathbf{R}_i^n$$

and $s = \frac{\Delta t}{\Delta x}$. The advantage of using (3.1) is that we do not need to approximate the eigenvalues and eigenvectors of the Jacobian matrix. However, the numerical scheme does not satisfy the TVD property, thus spurious oscillations may occur in the numerical solution. LeVeque & Yee[7] modified (3.1) so that the numerical scheme satisfies the TVD property by using slope limiters

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^{(2)} + (\mathbf{X}_{i+\frac{1}{2}}^{(2)}\mathbf{\Phi}_{i+\frac{1}{2}}^{(2)} - \mathbf{X}_{i-\frac{1}{2}}^{(2)}\mathbf{\Phi}_{i-\frac{1}{2}}^{(2)}) \tag{3.2}$$

where $\mathbf{X}$ is a matrix containing the right eigenvectors $\mathbf{e}_k$ of the Jacobian, $\mathbf{w}_i^{(2)}$ is the numerical approximation derived from (3.1), i.e.

$$\mathbf{w}_i^{(2)} = \frac{1}{2}(\mathbf{w}_i^n + \mathbf{w}_i^{(1)}) - \frac{s}{2}(\mathbf{F}_i^{(1)} - \mathbf{F}_{i-1}^{(1)}) + \frac{s}{2}\mathbf{R}_{i-1}^{(1)},$$

where

$$s = \frac{\Delta t}{\Delta x}, \quad \mathbf{\Phi}_{i+\frac{1}{2}}^{(2)} = \begin{bmatrix} \phi_{i+\frac{1}{2}}^1 \\ \vdots \\ \phi_{i+\frac{1}{2}}^p \end{bmatrix}, \quad \begin{bmatrix} \alpha_{i+\frac{1}{2}}^1 \\ \vdots \\ \alpha_{i+\frac{1}{2}}^p \end{bmatrix} = \mathbf{X}_{i+\frac{1}{2}}^{-1}(\mathbf{w}_{i+1}^n - \mathbf{w}_i^n)$$

$$\phi_{i+\frac{1}{2}}^k = \frac{1}{2}\big[(|\nu| - \nu^2)(\alpha - Q)\big]_{i+\frac{1}{2}}^k, \quad \nu_{i+\frac{1}{2}}^k = \frac{\Delta t}{\Delta x}\lambda_{i+\frac{1}{2}}^k,$$

$$Q_{i+\frac{1}{2}}^k = \text{minmod}(\alpha_{i-\frac{1}{2}}^k, \alpha_{i+\frac{1}{2}}^k, \alpha_{i+\frac{3}{2}}^k)$$

and

$$\text{minmod}(a,b,c) = \begin{cases} d\,\min(|a|,|b|,|c|) & \text{if } d = \text{sgn}(a) = \text{sgn}(b) = \text{sgn}(c), \\ 0 & \text{otherwise.} \end{cases}$$

Here, p is the number of components of the system, $k = 1, 2, \ldots, p$ represents the k^{th} component of the system and λ, $\mathbf{e}$ and α represents the eigenvalues, eigenvectors and wave strengths associated with the system (2.1) respectively.

Alternatively, we could calculate λ, $\mathbf{e}$ and α by using Roe's decomposition

$$\Delta\mathbf{F} = \sum_{k=1}^{p} \tilde{\alpha}_k \tilde{\lambda}_k \tilde{\mathbf{e}}_k = \tilde{\mathbf{A}}\Delta\mathbf{w}.$$

Variables which have $\tilde{}$ represent the Roe Average and $\tilde{\mathbf{A}}$ is the Jacobian matrix evaluated at Roe's average state. We will discuss the method of decomposition in Section 3.2.

The MacCormack approach requires that

$$\frac{\Delta t}{\Delta x}\max(|\lambda_k|) \leq 1$$

so that the numerical scheme will remain stable. Hence, for all of the formulations we will use

$$\Delta t = \frac{\Delta x}{2\max(|\lambda_k^n|)}, \tag{3.3}$$

to calculate the time step, which will ensure that the numerical scheme does not become unstable.

In the next few sections, we will discuss how we can apply LeVeque & Yee's MacCormack approach to the different formulations discussed in Section 2.

3.1.1 Formulation A-NC

In order to use LeVeque & Yee's MacCormack approach to approximate **Formulation A-NC**, we must apply (3.2) to the Shallow Water Equations and the Bed-Updating Equation separately. For

1. The Shallow Water Equations, we use

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n h_i^n \end{bmatrix}, \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_i^n$$

   and

$$\mathbf{R}_i^n = \begin{bmatrix} 0 \\ -\frac{g}{2}(h_{i+1}^n + h_i^n)(B_{i+1}^n - B_i^n) \end{bmatrix}.$$

2. The Bed-Updating Equation, we can re-write (3.2) as

$$B_i^{n+1} = B_i^{(2)} + (\Phi_{i+\frac{1}{2}}^{(2)} - \Phi_{i-\frac{1}{2}}^{(2)}) \tag{3.4}$$

   where $B_i^{(2)}$ is the numerical approximation derived from

$$B_i^{(2)} = \frac{1}{2}(B_i^n + B_i^{(1)}) - \frac{\xi s}{2}(q_i^{(1)} - q_{i-1}^{(1)}),$$

   where

$$B_i^{(1)} = B_i^n - \xi s(q_{i+1}^n - q_i^n),$$

$$\mathbf{\Phi}_{i+\frac{1}{2}}^{(2)} = \frac{1}{2}\big[(|\nu| - \nu^2)(\Delta B - Q)\big]_{i+\frac{1}{2}}, \quad \nu_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x}\lambda_{i+\frac{1}{2}},$$

$$Q_{i+\frac{1}{2}} = \text{minmod}(\Delta B_{i-\frac{1}{2}}, \Delta B_{i+\frac{1}{2}}, \Delta B_{i+\frac{3}{2}}) \quad \text{and} \quad \Delta B_{i+\frac{1}{2}} = B_{i+1}^n - B_i^n.$$

   In order to be able to use (3.4), we need to be able to obtain an accurate approximation of the wave speed, λ, which can be very difficult to obtain since the Sediment Transport Flux is not a direct function of B. A variety of approaches that can be used to obtain an approximation of the wave speed are discussed in Section 3.3 where the advantages and disadvantages of each approach are discussed as well.

For **Formulation A-NC**, the time step of the Shallow Water Equations and the morphological time step of the Bed-Updating Equation are the same and are calculated using (3.3).

3.1.2 Formulation A-CV

For **Formulation A-CV**, we use LeVeque & Yee's MacCormack approach in exactly the same way we did for **Formulation A-NC**, which is discussed in Section 3.1.1. The only difference is that we converge the Shallow Water

Equations to a steady state solution first and then update the riverbed. A steady state solution is obtained if

$$\frac{\partial \mathbf{w}}{\partial t} = 0.$$

Thus, in order to determine if the steady state solution has been obtained, we set a tolerance level, tol, such that

$$|\mathbf{w}_i^{n+1} - \mathbf{w}_i^n| \leq \text{tol} \quad \text{for} \quad i = 0, 1, 2, \ldots, I.$$

The morphological time step of the Bed-Updating Equation is the overall time step of this formulation and is calculated separately from the time step of the Shallow Water Equations. The Shallow Water Equations must be converged to the steady state solution each time the riverbed is updated.

3.1.3 Formulation A-SF

Since **Formulation A-SF** is in system form, LeVeque & Yee's MacCormack approach is easier to implement and is used with

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n h_i^n \\ B_i^n \end{bmatrix}, \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ \xi q \end{bmatrix}_i^n$$

and

$$\mathbf{R}_i^n = \begin{bmatrix} 0 \\ -\frac{g}{2}(h_{i+1}^n + h_i^n)(B_{i+1}^n - B_i^n) \\ 0 \end{bmatrix}.$$

3.1.4 Formulation B

We can also use LeVeque & Yee's MacCormack approach to approximate **Formulation B** by using

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n \\ B_i^n \end{bmatrix} \quad \text{and} \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ \frac{1}{2}u^2 + g(h+B) \\ \xi q \end{bmatrix}_i^n.$$

Also, since no source term is present in **Formulation B**,

$$\mathbf{R}_i^n = 0.$$

3.1.5 Formulation C

LeVeque & Yee's MacCormack approach can be used to approximate **Formulation C** by using

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n h_i^n \\ B_i^n \end{bmatrix}, \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + ghB \\ \xi q \end{bmatrix}_i^n$$

and

$$\mathbf{R}_i^n = \begin{bmatrix} 0 \\ \frac{g}{2}(B_{i+1}^n + B_i^n)(h_{i+1}^n - h_i^n) \\ 0 \end{bmatrix}.$$

3.2 Roe's Scheme

Roe[8] derived an approach which approximates systems of conservation laws,

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0, \tag{3.5}$$

by using a piecewise constant approximation in each cell

$$\mathbf{w}(x, t_n) = \begin{cases} \mathbf{w}_L & \text{if } x_L - \frac{\Delta x}{2} < x < x_L + \frac{\Delta x}{2}, \\ \mathbf{w}_R & \text{if } x_R - \frac{\Delta x}{2} < x < x_R + \frac{\Delta x}{2}. \end{cases}$$

where $\mathbf{w}_L$ and $\mathbf{w}_R$ represent the piecewise constant states at t_n, and determining the exact solution of a linearised Riemann problem, which is related to (3.5)

$$\frac{\partial \mathbf{w}}{\partial t} + \tilde{\mathbf{A}}(\mathbf{w}_L, \mathbf{w}_R)\frac{\partial \mathbf{w}}{\partial x} = 0,$$

where $\tilde{\mathbf{A}}(\mathbf{w}_L, \mathbf{w}_R) \approx \frac{\partial \mathbf{F}}{\partial \mathbf{w}}$ is the linearised Jacobian matrix and $\tilde{}$ is called the Roe average. The eigenvalues and eigenvectors of $\tilde{\mathbf{A}}$ are $\tilde{\lambda}$ and $\tilde{\mathbf{e}}$ respectively and are determined from the decomposition

$$\Delta \mathbf{F} = \sum_{k=1}^{p} \tilde{\alpha}_k \tilde{\lambda}_k \tilde{\mathbf{e}}_k = \tilde{\mathbf{A}} \Delta \mathbf{w}.$$

where $\Delta \mathbf{w} = \mathbf{w}_R - \mathbf{w}_L$, p is the number of components in the system and $\tilde{\alpha}$ represents the wavestrengths, i.e. $\tilde{\alpha}_k = \Delta \mathrm{w}_k$. Once the eigenvalues,

Name of Flux-limiter	$\Phi(\theta)$
Minmod	$\max(0, \min(1, \theta))$
Roe's Superbee	$\max(0, \min(2\theta, 1), \min(\theta, 2))$
van Leer	$\dfrac{\lvert\theta\rvert + \theta}{1 + \lvert\theta\rvert}$
van Albada	$\dfrac{\theta^2 + \theta}{1 + \theta^2}$

Table 3.1: Some Flux-limiters

eigenvectors and wavestrengths associated with the linearised Riemann problem have been obtained, Roe's Scheme [8] can be used

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - s(\mathbf{F}^*_{i+\frac{1}{2}} - \mathbf{F}^*_{i-\frac{1}{2}}) + s\mathbf{R}_i^*, \tag{3.6}$$

where

$$\mathbf{F}^*_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}^n_{i+1} + \mathbf{F}^n_i) - \frac{1}{2}\sum_{k=1}^{p}[\tilde{\alpha}_k|\tilde{\lambda}_k|(1 - \Phi_k(1 - |\nu_k|))\tilde{\mathbf{e}}_k]_{i+\frac{1}{2}},$$

$$s = \frac{\Delta t}{\Delta x}, \quad \nu_k = s\tilde{\lambda}_k, \quad \theta_k = \frac{(\tilde{\alpha}_k)_{I+\frac{1}{2}}}{(\tilde{\alpha}_k)_{i+\frac{1}{2}}}, \quad I = i - \operatorname{sgn}(\nu_k)_{i+\frac{1}{2}},$$

and Φ_k can be any of the flux-limiters listed in Table 3.1. For the source term approximation, $\mathbf{R}_i^*$, we can use

1. A pointwise approach,

$$\mathbf{R}_i^* = \frac{1}{2}(\mathbf{R}^n_{i+\frac{1}{2}} + \mathbf{R}^n_{i-\frac{1}{2}}). \tag{3.7}$$

2. A decomposed approach, where we decompose the source term in a similar way as for the flux terms, i.e.

$$\frac{1}{\Delta x}\sum_{k=1}^{p}\tilde{\beta}_k\tilde{\mathbf{e}}_k = \tilde{\mathbf{R}}.$$

   Here, $\tilde{\beta}_k$ are the coefficients of the decomposition of the source term onto the eigenvectors of the characteristic decomposition (see Glaister[3], Hubbard & Garcia-Navarro[5] and Hudson[6]). Once the values of $\tilde{\beta}_k$ have been obtained, we can approximate the source term by using

$$\mathbf{R}_i^* = \mathbf{R}_{i+\frac{1}{2}}^- + \mathbf{R}_{i-\frac{1}{2}}^+, \tag{3.8}$$

where

$$\mathbf{R}_{i+\frac{1}{2}}^{\pm} = \frac{1}{2}\sum_{k=1}^{p}[\tilde{\beta}_k \tilde{\mathbf{e}}_k(1 \pm \text{sgn}(\tilde{\lambda}_k)(1 - \Phi_k(1 - |\nu_k|)))]_{i+\frac{1}{2}}.$$

The advantage of using the decomposed approach to numerically approximate the source term is that the numerical scheme satisfies the C-property of Bermudez & Vazquez[1]. For a numerical scheme to satisfy the C-property of Bermudez & Vazquez[1], the scheme must be either exact or second order accurate when applied to the quiescent flow case, i.e. $u \equiv 0$ and $h \equiv H$, where $H \equiv D - B$. Since Roe's Scheme with source term approximation (3.8) satisfies the C-property of Bermudez & Vazquez[1], the numerical scheme will not produce any waves in a region that should remain steady (see Bermudez & Vazques[1] and Hudson[6] for more details). Unfortunately, if we use the decomposed approach with the flux-limited second order version of (3.6), then spurious oscillations may occur in the numerical results when the source term is stiff (see Hudson[6] and Hubbard & Garcia-Navarro[5] for more details). If the spurious oscillations do overpower the numerical solution, then we can reduce them by using a smaller Courant number, but sometimes the Courant number required can be very small resulting in the numerical scheme being impractical due to long computational run times.

Roe's Scheme requires that

$$\frac{\Delta t}{\Delta x}\max(|\lambda_k|) \leq 1$$

so that the numerical scheme will remain stable. Hence, for all of the formulations we will use

$$\Delta t = \frac{\Delta x}{2\max(|\lambda_k^n|)}, \tag{3.9}$$

to calculate the time step, which will ensure that the numerical scheme does not become unstable.

In the next few sections, we will discuss how we can apply (3.6) to the different formulations discussed in Section 2.

3.2.1 Formulation A-NC

In order to use (3.6) to approximate **Formulation A-NC**, we must apply the numerical scheme to the Shallow Water Equations and the Bed-Updating Equation separately. For

1. The Shallow Water Equations, here we use

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n h_i^n \end{bmatrix} \quad \text{and} \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}_i^n.$$

The eigenvalues are

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{c} \quad \text{and} \quad \tilde{\lambda}_2 = \tilde{u} + \tilde{c},$$

whose corresponding eigenvectors are

$$\tilde{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{c} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{e}}_2 = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{c} \end{bmatrix},$$

and the wavestrengths are

$$\tilde{\alpha}_1 = \frac{1}{2}\Delta h - \frac{1}{2\tilde{c}}(\Delta(hu) - \tilde{u}\Delta h)$$

and

$$\tilde{\alpha}_2 = \frac{1}{2}\Delta h + \frac{1}{2\tilde{c}}(\Delta(hu) - \tilde{u}\Delta h),$$

where the Roe averages are

$$\tilde{u} = \frac{\sqrt{h_R}u_R + \sqrt{h_L}u_L}{\sqrt{h_R} + \sqrt{h_L}} \quad \text{and} \quad \tilde{c} = \sqrt{\frac{g}{2}(h_R + h_L)}.$$

For the source term approximation, we can use either the pointwise approach,

$$\mathbf{R}_{i+\frac{1}{2}}^n = \begin{bmatrix} 0 \\ -\frac{g}{2}(h_{i+1}^n + h_i^n)(B_{i+1}^n - B_i^n) \end{bmatrix},$$

or the decomposed approach,

$$\tilde{\beta}_1 = \frac{\tilde{c}\Delta B}{2} \quad \text{and} \quad \tilde{\beta}_2 = -\frac{\tilde{c}\Delta B}{2}.$$

2. The Bed-Updating Equation, we can re-write Roe's Scheme as

$$B_i^{n+1} = B_i^n - s(q^*_{i+\frac{1}{2}} - q^*_{i-\frac{1}{2}}), \tag{3.10}$$

where

$$q^*_{i+\frac{1}{2}} = \frac{\xi}{2}(q^n_{i+1} + q^n_i) - \frac{1}{2}|\lambda_{i+\frac{1}{2}}|(1 - \Phi_{i+\frac{1}{2}}(1 - |\nu_{i+\frac{1}{2}}|))(B^n_{i+1} - B^n_i),$$

$$s = \frac{\Delta t}{\Delta x}, \quad \nu_{i+\frac{1}{2}} = s\lambda_{i+\frac{1}{2}}, \quad \theta_{i+\frac{1}{2}} = \frac{\Delta B_{I+\frac{1}{2}}}{\Delta B_{i+\frac{1}{2}}}, \quad I = i - \operatorname{sgn}(\nu_k)_{i+\frac{1}{2}},$$

$$\Delta B_{i+\frac{1}{2}} = B^n_{i+1} - B^n_i, \quad \lambda_{i+\frac{1}{2}} = \xi \left[\frac{\partial q}{\partial B}\right]_{i+\frac{1}{2}},$$

and Φ_k can be any of the flux-limiters listed in Table 3.1. In order to be able to use (3.10), we need to be able to obtain an accurate approximation of the wave speed, λ, which can be very difficult to obtain since the Sediment Transport Flux is not a direct function of B. A variety of approaches that can be used to obtain an approximation of the wave speed are discussed in Section 3.3 where the advantages and disadvantages of each approach are discussed as well.

For **Formulation A-NC**, the time step of the Shallow Water Equations and the morphological time step of the Bed-Updating Equation are the same and are calculated using (3.9).

3.2.2 Formulation A-CV

For **Formulation A-CV**, Roe's Scheme with source term approximation is used in exactly the same way as **Formulation A-NC**, which is discussed in Section 3.2.1. The only difference is that the Shallow Water Equations are converged to a steady state solution and then the riverbed updated. In order to determine if the steady state solution has been obtained, we set a tolerance level, tol, such that

$$|\mathbf{w}_i^{n+1} - \mathbf{w}_i^n| \leq \text{tol} \quad \text{for} \quad i = 0, 1, 2, \ldots, I.$$

The morphological time step of the Bed-Updating Equation is the overall time step of this formulation and is calculated separately from the time step of the Shallow Water Equations. The Shallow Water Equations must be converged to the steady state solution each time the riverbed is updated.

3.2.3 Formulation A-SF

We can use (3.6) to approximate **Formulation A-SF** by using

$$\mathbf{w}_i^n = \begin{bmatrix} h_i^n \\ u_i^n h_i^n \\ B_i^n \end{bmatrix} \quad \text{and} \quad \mathbf{F}_i^n = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ \xi A u^m \end{bmatrix}_i^n .$$

The eigenvalues are

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{c}, \quad \tilde{\lambda}_2 = 0 \quad \text{and} \quad \tilde{\lambda}_3 = \tilde{u} + \tilde{c},$$

whose corresponding eigenvectors are

$$\tilde{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{c} \\ \dfrac{-\tilde{c}\tilde{d}}{\tilde{u} - \tilde{c}} \end{bmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{e}}_3 = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{c} \\ \dfrac{\tilde{c}\tilde{d}}{\tilde{u} + \tilde{c}} \end{bmatrix},$$

and the wavestrengths are

$$\tilde{\alpha}_1 = \frac{1}{2\tilde{c}}((\tilde{u} + \tilde{c})\Delta h - \Delta(uh)),$$

$$\tilde{\alpha}_2 = \Delta B - \frac{\tilde{d}(\tilde{u}\Delta(uh) + 2\tilde{u}\Delta h)}{(\tilde{u} - \tilde{c})(\tilde{u} + \tilde{c})}$$

and

$$\tilde{\alpha}_3 = \frac{1}{2\tilde{c}}(\Delta(uh) - (\tilde{u} - \tilde{c})\Delta h),$$

where the Roe averages are

$$\tilde{u} = \frac{\sqrt{h_R}u_R + \sqrt{h_L}u_L}{\sqrt{h_R} + \sqrt{h_L}}, \quad \tilde{c} = \sqrt{\frac{g}{2}(h_R + h_L)}$$

and

$$\tilde{d} = \frac{\xi \Delta(Au^m)}{\Delta(uh) - \tilde{u}\Delta h}.$$

If the Sediment Transport Flux (1.5) is used with $A = 1$ and m is an integer, then we can use

$$\tilde{d} = \frac{\xi(\sqrt{h_R} + \sqrt{h_L})}{\sqrt{h_L}h_R + \sqrt{h_R}h_L} \sum_{k=0}^{m-1} (u_R)^k (u_L)^{m-(1+k)}.$$

For the source term approximation, we can use either the pointwise approach,

$$\mathbf{R}^n_{i+\frac{1}{2}} = \begin{bmatrix} 0 \\ -\frac{g}{2}(h^n_{i+1} + h^n_i)(B^n_{i+1} - B^n_i) \\ 0 \end{bmatrix},$$

or the decomposed approach,

$$\tilde{\beta}_1 = \frac{\tilde{c}\Delta B}{2}, \quad \tilde{\beta}_2 = \frac{\tilde{d}\tilde{u}\tilde{c}^2\Delta B}{(\tilde{u}-\tilde{c})(\tilde{u}+\tilde{c})} \quad \text{and} \quad \tilde{\beta}_3 = -\frac{\tilde{c}\Delta B}{2}.$$

3.2.4 Formulation B

Formulation B can also be approximated by using (3.6) with

$$\mathbf{w}^n_i = \begin{bmatrix} h^n_i \\ u^n_i \\ B^n_i \end{bmatrix} \quad \text{and} \quad \mathbf{F}^n_i = \begin{bmatrix} hu \\ \frac{1}{2}u^2 + g(h+B) \\ \xi A u^m \end{bmatrix}^n_i.$$

The Roe averaged eigenvalues are obtained by finding the roots of

$$\tilde{P}(\tilde{\lambda}) = \tilde{\lambda}^3 - 2\tilde{u}\tilde{\lambda}^2 + [\tilde{u}^2 - g(\tilde{h} + \tilde{d})]\tilde{\lambda} + g\tilde{u}\tilde{d} = 0.$$

The roots of $\tilde{P}(\tilde{\lambda})$ are determined by using the approach which was discussed in Section 2.2. Once the values of the eigenvalues have been obtained, they are used to determine the values of the corresponding eigenvectors

$$\tilde{\mathbf{e}}_k = \begin{bmatrix} 1 \\ \dfrac{\tilde{\lambda}_k - \tilde{u}}{\tilde{h}} \\ \dfrac{(\tilde{u} - \tilde{\lambda}_k)^2 - g\tilde{h}}{g\tilde{h}} \end{bmatrix}$$

and the wavestrengths

$$\tilde{\alpha}_k = \frac{((\tilde{u} - \tilde{\lambda}_a)(\tilde{u} - \tilde{\lambda}_b) + g\tilde{h})\Delta h + (2\tilde{u} - \tilde{\lambda}_a - \tilde{\lambda}_b)\Delta u + g\tilde{h}\Delta B}{(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)}$$

where $a \neq k \neq b$. The Roe averages are

$$\tilde{u} = \frac{1}{2}(u_R + u_L), \quad \tilde{h} = \frac{1}{2}(h_R + h_L) \quad \text{and} \quad \tilde{d} = \frac{\xi\Delta(Au^m)}{\Delta u}.$$

If the Sediment Transport Flux (1.5) is used with $A = 1$ and m is an integer, then we can use

$$\tilde{d} = \xi \sum_{k=0}^{m-1} (u_R)^k (u_L)^{m-(1+k)}.$$

Also for **Formulation B**, there is no source term present so

$$\mathbf{R}^n_{i+\frac{1}{2}} = 0 \quad \text{and} \quad \tilde{\beta}_k = 0.$$

3.2.5 Formulation C

Formulation C can also be approximated by using (3.6) with

$$\mathbf{w}^n_i = \begin{bmatrix} h^n_i \\ u^n_i h^n_i \\ B^n_i \end{bmatrix} \quad \text{and} \quad \mathbf{F}^n_i = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + ghB \\ \xi A u^m \end{bmatrix}^n_i.$$

The Roe averaged eigenvalues are obtained by finding the roots of

$$\tilde{P}(\tilde{\lambda}) = \tilde{\lambda}^3 - 2\tilde{u}\tilde{\lambda}^2 + [\tilde{u}^2 - g(\tilde{h} + \tilde{B} + \tilde{h}\tilde{d})]\tilde{\lambda} + g\tilde{h}\tilde{u}\tilde{d} = 0.$$

The roots of $\tilde{P}(\tilde{\lambda})$ are determined by using the approach which was discussed in Section 2.2. Once the values of the eigenvalues have be obtained, they are used to determine the values of the corresponding eigenvectors

$$\tilde{\mathbf{e}}_k = \begin{bmatrix} 1 \\ \tilde{\lambda}_k \\ \dfrac{\tilde{u}^2 - g(\tilde{h} + \tilde{B}) + (\tilde{\lambda}_k - 2\tilde{u})\tilde{\lambda}_k}{g\tilde{h}} \end{bmatrix}$$

and the wavestrengths

$$\tilde{\alpha}_k = \frac{(\tilde{\lambda}_a\tilde{\lambda}_b + g(\tilde{h} + \tilde{B}) - \tilde{u}^2)\Delta h + (2\tilde{u} - \tilde{\lambda}_a - \tilde{\lambda}_b)\Delta(uh) + g\tilde{h}\Delta B}{(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)},$$

where $a \neq k \neq b$. The Roe averages are

$$\tilde{u} = \frac{\sqrt{h_R}u_R + \sqrt{h_L}u_L}{\sqrt{h_R} + \sqrt{h_L}}, \quad \tilde{h} = \frac{1}{2}(h_R + h_L), \quad \tilde{B} = \frac{1}{2}(B_R + B_L)$$

and

$$\tilde{d} = \frac{\xi\Delta(Au^m)}{\Delta(uh) - \tilde{u}\Delta h}.$$

If the Sediment Transport Flux (1.5) is used with $A = 1$ and m is an integer, we can use

$$\tilde{d} = \frac{\xi(\sqrt{h_R} + \sqrt{h_L})}{\sqrt{h_L}h_R + \sqrt{h_R}h_L} \sum_{k=0}^{m-1} (u_R)^k (u_L)^{m-(1+k)}.$$

For the source term approximation, we can use a pointwise approach,

$$\mathbf{R}^n_{i+\frac{1}{2}} = \left[\begin{array}{c} 0 \\ \frac{g}{2}(B^n_{i+1} + B^n_i)(h^n_{i+1} - h^n_i) \\ 0 \end{array} \right],$$

or a decomposed approach:

$$\tilde{\beta}_k = \frac{g\tilde{B}(2\tilde{u} - \tilde{\lambda}_a - \tilde{\lambda}_b)\Delta h}{(\tilde{\lambda}_k - \tilde{\lambda}_a)(\tilde{\lambda}_k - \tilde{\lambda}_b)} \quad \text{where} \quad a \neq k \neq b.$$

3.3 Approximating the Wave Speed of the Bed-Updating Equation

Most of the numerical schemes discussed for **Formulation A-NC** and **Formulation A-CV** require a numerical approximation of the wave speed of the Bed-Updating Equation,

$$\frac{\partial B}{\partial t} + \xi \frac{\partial q}{\partial x} = 0,$$

whose wave speed is $\lambda = \xi \frac{\partial q}{\partial B}$. Unfortunately, the Sediment Transport Flux, q, is not a direct function of B, which can create difficulties in obtaining an accurate approximation of the wave speed.

One approach we can use to approximate the wave speed is to use a finite difference approach,

$$\lambda_{i+\frac{1}{2}} = \xi \frac{q_{i+1} - q_i}{B_{i+1} - B_i} \quad \text{if} \quad B_{i+1} - B_i \neq 0. \tag{3.11}$$

Unfortunately, the finite difference approach can only be used when $B_{i+1} - B_i \neq 0$ and can also produce an inaccurate approximation of the wave speed when the gradient of the riverbed changes sign. In Section 2, we proved that if we use $q(u) = Au^m$ for $u \geq 0$, then the wave speed of the Bed-Updating

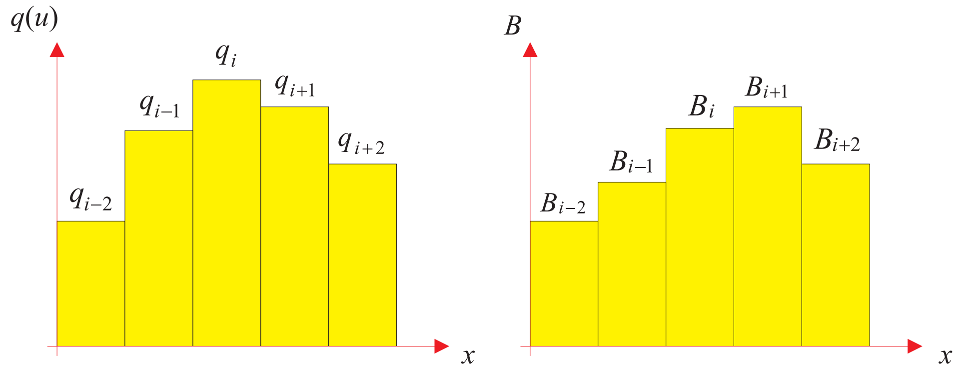

Figure 3.2: The Occurrence of a Negative Wave Speed.

Equation is always positive. However the finite difference approach may produce negative wave speeds, as illustrated in Figure 3.2, where it can be seen that $B_{i+1} - B_i > 0$ but $q_{i+1} - q_i < 0$, which implies that for the illustrated data

$$\lambda_{i+\frac{1}{2}} = -\xi \frac{|q_{i+1} - q_i|}{|B_{i+1} - B_i|}.$$

Hence, the finite difference approach has produced a negative wave speed but the analytic wave speed is positive.

Another approach we can use is an analytical approach, see Chesher *et al*[2], but can only be used for problems where the height of the riverbed is considerably smaller than the height of the river. The approach assumes that the height of the river is constant, i.e. $h(x,t) \approx D - B$ and then rewrites the Sediment Transport Flux so that it may be differentiated with respect to B. As an example, consider $q(u) = u^m$ for $u \geq 0$. Now, by letting $Q = uh$, we may obtain

$$\lambda = \xi \frac{\partial(u^m)}{\partial B} = \xi \frac{\partial(Q^m h^{-m})}{\partial B},$$

and since $h \approx D - B$,

$$\lambda \approx \xi \frac{\partial(Q^m (D-B)^{-m})}{\partial B} = \xi Q^m \frac{\partial((D-B)^{-m})}{\partial B} + \xi (D-B)^{-m} \frac{\partial(Q^m)}{\partial B}$$

$$\Rightarrow \quad \lambda \approx \xi Q^m \frac{\partial((D-B)^{-m})}{\partial B} = \xi m Q^m (D-B)^{-m-1} = \frac{\xi m}{h} Q^m h^{-m},$$

and since $u = Qh^{-1}$,

$$\lambda \approx \frac{\xi m}{h} u^m.$$

Hence, we can use

$$\lambda_{i+\frac{1}{2}} = \frac{\xi m}{h^n_{i+\frac{1}{2}}} [u^m]^n_{i+\frac{1}{2}}, \quad \text{where} \quad h^n_{i+\frac{1}{2}} = \frac{1}{2}(h^n_{i+1} + h^n_i) \tag{3.12}$$

as an approximation of the wave speed. Unfortunately, the analytical approach can only be used if the Sediment Transport Flux can be written analytically and can only be used for test problems where the height of the riverbed is significantly smaller than the height of the river.

There are numerical schemes that we can use to approximate the Bed-Updating Equation that do not require a value of the wave speed to be calculated. For example, the Lax-Friedrichs scheme,

$$B^{n+1}_i = \frac{1}{2}(B^n_{i+1} + B^n_{i-1}) - \xi \frac{s}{2}(q^n_{i+1} - q^n_{i-1}), \tag{3.13}$$

can be used to approximate the Bed-Updating Equation but the Lax-Friedrichs scheme suffers from dissipation resulting in a very small Courant number being required. Alternatively, since we know that the sign of the wave speed of the Bed-Updating Equation must be positive if we use $q(u) = Au^m$ where $u \geq 0$, we can use the first order Upwind scheme,

$$B^{n+1}_i = B^n_i - \xi s(q^n_i - q^n_{i-1}), \tag{3.14}$$

to approximate the Bed-Updating Equation. Alternatively LeVeque & Yee's MacCormack approach (3.4) with $\Phi^{(2)}_{i+\frac{1}{2}} = 0$ does not require an approximation of the wave speed either and is second order accurate.

4 Numerical Results

In this section, we will discuss two different test problems which we will use to compare the different formulations. All of the test problems use the Sediment Transport Flux (1.5) with $A = 1$ and $n = 3$, i.e.

$$q(u) = u^3.$$

The advantage of using this form of the Sediment Transport Flux is that it is differentiable and is valid for all values of u. In addition, the porosity of the bed will be taken as $\epsilon = 0.2$.

We will use LeVeque & Yee's MacCormack approach and Roe's Scheme with the source term decomposed, i.e (3.6) with source term approximation (3.8), to approximate the different formulations. Unless stated, both LeVeque & Yee's MacCormack approach and Roe's Scheme with source term decomposed will be used with a maximum Courant number of $\nu = 0.5$, a spatial step size of $\Delta x = 1$ and boundary conditions

$$\mathbf{w}_{-i}^{n+1} = \mathbf{w}_0^n \quad \text{and} \quad \mathbf{w}_{I+i}^{n+1} = \mathbf{w}_I^n,$$

where $i = 1$ to 5. Roe's Scheme with source term decomposed will be used with the minmod flux-limiter and the final time for both test problems is $t = 500s$. For **Formulation A-CV**, a tolerance level of tol $= 10^{-7}$ will be used to converge the Shallow Water Equations. **Formulation A-NC** and **Formulation A-CV** will require an approximation of the wave speed of the Bed-Updating Equation so the analytical approach (3.12) will be used to approximate the wave speed.

4.1 Test Problems

In order to rigorously test the numerical formulations discussed in this report, we will use two distinct test problems, which consist of a pulse in a riverbed for **Problem A** and a sediment bore for **Problem B** in a 1D Channel. For both test problems, we require initial conditions that are for a moving riverbed and not a fixed riverbed. In order to obtain such initial conditions, we use **Formulation B** with the following dummy initial conditions, which consists of the region $x_0 = 0$ to $x_I = 1000m$ where

$$h^*(x, 0) = 10 - B^*(x, 0), \quad u^*(x, 0) = \frac{10}{h^*(x, 0)}$$

and the bathymetry for

1. **Problem A** is

$$B^*(x,0) = \begin{cases} \sin^2\left(\dfrac{\pi(x-300)}{200}\right) & \text{if } 300 \leq x \leq 500 \\ 0 & \text{otherwise} \end{cases},$$

   which is illustrated in Figure 4.1.

2. **Problem B** is

$$B^*(x,0) = \frac{1}{2}\left[1 - \tanh\left(\frac{1}{10\pi}(x-400)\right)\right],$$

   which is illustrated in Figure 4.2.

These dummy initial conditions are then iterated by using **Formulation B** until $t = 75s$ by which time the numerical data has settled and represents a moving riverbed. Then the initial conditions of the test problems are now $\mathbf{w}(x,0) = \mathbf{w}^*(x,75)$, which are illustrated in Figure 4.3 and Figure 4.4 for **Problem A** and Figure 4.5 and Figure 4.6 for **Problem B**.

4.2 Numerical Results for Problem A

4.2.1 Roe's Scheme with Source Term Decomposed

From Figure 4.7 and Figure 4.8, we can see that **Formulation A-NC** using Roe's Scheme with source term decomposed has produced spurious oscillations in the numerical results. This is due to the Bed-Updating Equation being calculated separately and requires an accurate approximation of the wave speed for the numerical formulation to be accurate. However the spurious oscillations can be reduced by using a smaller Courant number but at the expense of long computational run times.

In Figure 4.9 and Figure 4.10, we can see that **Formulation A-CV** using Roe's Scheme with source term decomposed has increased the height of the pulse in the velocity and height of the river dramatically. This is because by converging the Shallow Water Equations, we have assumed that the riverbed is fixed. We would expect the pulse in the velocity to decrease for a moving riverbed since the velocity is pushing the pulse in the riverbed resulting in a smaller velocity over the pulse in the riverbed. However, by using **Formulation A-CV**, no spurious oscillations are present in the numerical results even thought we have calculated the Bed-Updating Equation separately. In addition, by using **Formulation A-CV**, we can use the finite difference approach (3.11) to approximate the wave speed since the gradients of the riverbed and the Sediment Transport Flux are never of opposite sign.

For **Formulation A-SF** using Roe's Scheme with source term decomposed, Figure 4.11 and Figure 4.12 show that the numerical results do not suffer from spurious oscillations. Surprisingly, even thought the formulation has a singular matrix, the numerical results are still quite accurate.

From Figure 4.13 and Figure 4.14, we can see that **Formulation B** using Roe's Scheme has produced very smooth numerical results with no spurious oscillations present.

Figure 4.15 and Figure 4.16 show that **Formulation C** using Roe's Scheme with source term decomposed has also produced very smooth numerical results with no spurious oscillations present.

In Figures 4.17 to 4.20, an overall comparison is given at $t = 500s$. Here, we can see that the most accurate formulation was **Formulation B**, closely followed by **Formulation C**. **Formulation A-SF** produced quite accurate numerical results but differed slightly to **Formulation B** and **Formulation C**. **Formulation A-NC** gave the least accurate numerical results as far as spurious oscillations is concerned but **Formulation A-CV** moved the pulse in the riverbed at a completely different wave speed than the other approaches. Hence, converging the Shallow Water Equations to a steady state solution before updating the riverbed is incorrect as the approach assumes that the riverbed is fixed.

4.2.2 LeVeque & Yee's MacCormack Approach

For LeVeque & Yee's MacCormack approach, an overall comparison of the different formulations is given at $t = 500s$ in Figures 4.21 to 4.24. Here, we can see that **Formulation A-CV** has again moved the pulse in the riverbed at a completely different wave speed than the other approaches. However, for the MacCormack approach, **Formulation A-SF** has produced spurious oscillations that have overpowered the numerical solution. These spurious oscillations cannot be minimised using a smaller Courant number. By using the second order version of LeVeque & Yee's MacCormack approach, the spurious oscillations are considerably smaller but we require a numerical scheme that satisfies the TVD property. **Formulation A-NC** has not produced spurious oscillations for LeVeque & Yee's MacCormack approach but did for Roe's Scheme with source term decomposed. However, by looking at Figure 4.21, we can see that **Formulation A-NC** has produced slight waves on the right side of the pulse and the maximum height of the pulse differs from the other formulations. As with Roe's Scheme with source term decomposed, **Formulation B** and **Formulation C** have produced the most accurate numerical results for the MacCormack approach.

4.3 Numerical Results for Problem B

4.3.1 Roe's Scheme with Source Term Decomposed

Figure 4.25 and Figure 4.26 show the numerical results of **Formulation A-NC** using Roe's Scheme with source term decomposed. Here, we can see that spurious oscillations have begin to overpower the numerical solution due to the Bed-Updating Equation being approximated separately. However, the spurious oscillations are not as prominent as with **Problem A** and can be minimised by using a smaller Courant number.

For **Formulation A-CV** using Roe's Scheme with source term decomposed, Figure 4.27 and Figure 4.28 show that by converging the Shallow Water Equations, the velocity and the height of the riverbed increase dramatically. However, the numerical results do not suffer from spurious oscillations even though the Bed-Updating Equation is approximated separately.

From Figure 4.29 and Figure 4.30, we can see that **Formulation A-SF** using Roe's Scheme with source term decomposed has produced quite accurate numerical results even though the Jacobian matrix is singular.

Figure 4.31 and Figure 4.32 show the numerical results of **Formulation B** using Roe's Scheme. As with **Problem A**, the numerical results obtained are very smooth and do not suffer from spurious oscillations.

For **Formulation C** using Roe's Scheme with source term decomposed, Figure 4.33 and Figure 4.34 show that the numerical results are also very smooth.

In Figures 4.35 to 4.38, an overall comparison of the different formulations is given at $t = 500s$. Here, we can see that the most accurate formulation was **Formulation B**, closely followed by **Formulation C**. **Formulation A-SF** has again produced quite accurate numerical results but differed slightly to **Formulation B** and **Formulation C**. **Formulation A-NC** gave the least accurate numerical results as far as spurious oscillations is concerned but **Formulation A-CV** moved the pulse in the riverbed at a completely different wave speed than the other approaches. Hence, converging the Shallow Water Equations to a steady state solution before updating the riverbed is incorrect as the approach assumes that the riverbed is fixed.

4.3.2 LeVeque & Yee's MacCormack Approach

For LeVeque & Yee's MacCormack approach, an overall comparison of the different formulations is given at $t = 500s$ in Figures 4.39 to 4.42. As with **Problem A**, **Formulation A-SF** using the MacCormack approach has produced spurious oscillations that have overpowered the numerical solution and can only be minimised by using the second order version of the numerical scheme. **Formulation A-CV** has again moved the sediment bore at a completely different wave speed than the other formulations. **Formulation B** has produced the most accurate numerical results closely followed by **Formulation C**. **Formulation A-NC** has produced accurate results that do not suffer from spurious oscillations but is less accurate than **Formulation C**.

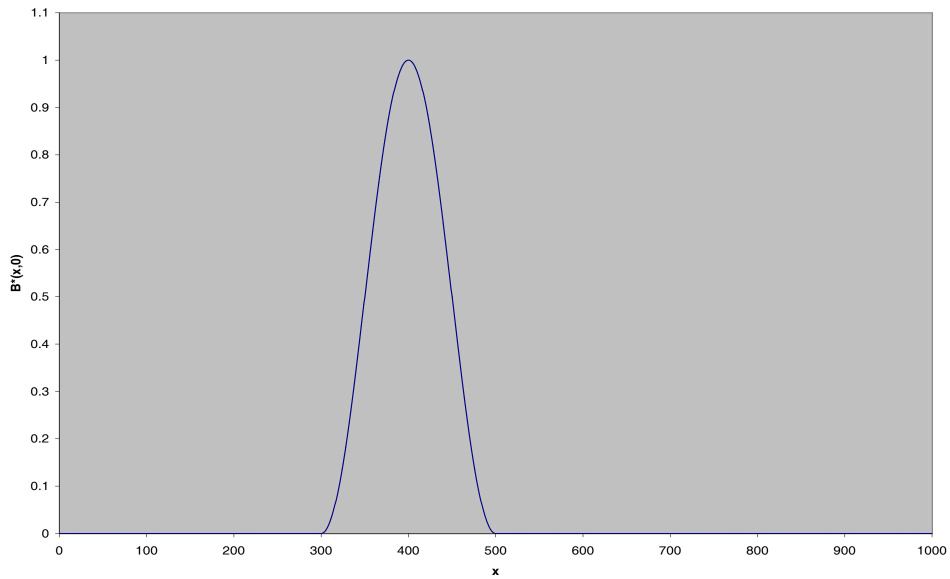

Figure 4.1: The Dummy Initial Condition $B^*(x, 0)$ for **Problem A**.

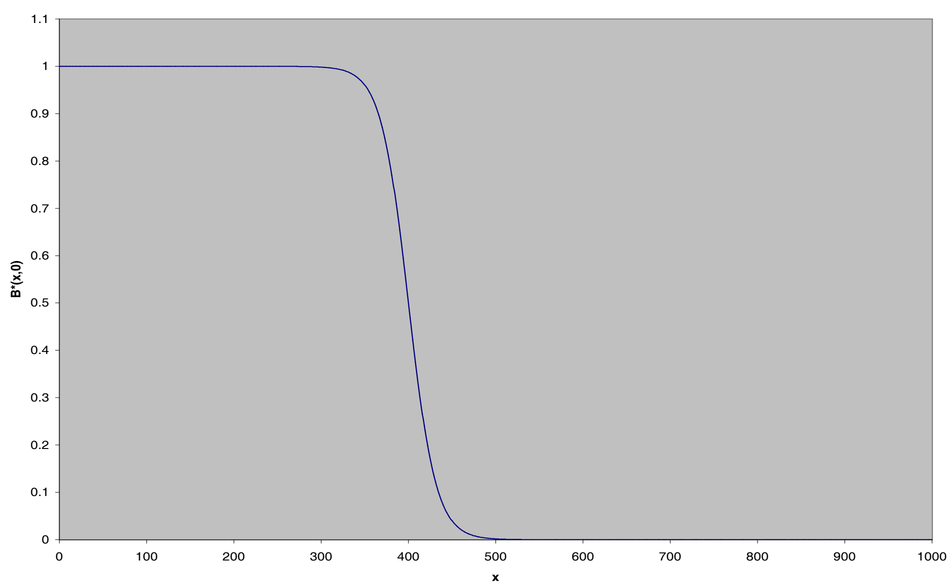

Figure 4.2: The Dummy Initial Condition $B^*(x, 0)$ for **Problem B**.

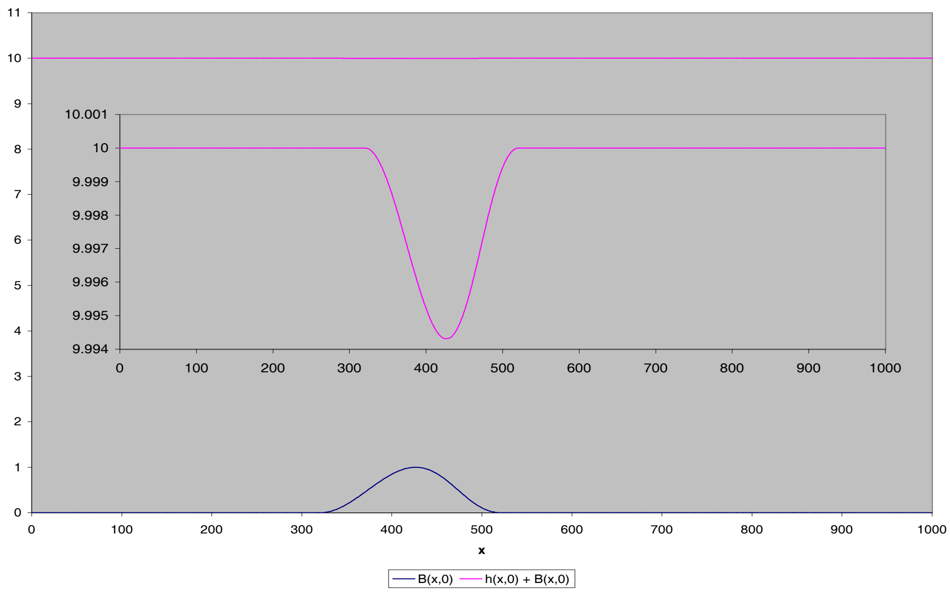

Figure 4.3: The Initial Conditions for **Problem A** (h and B).

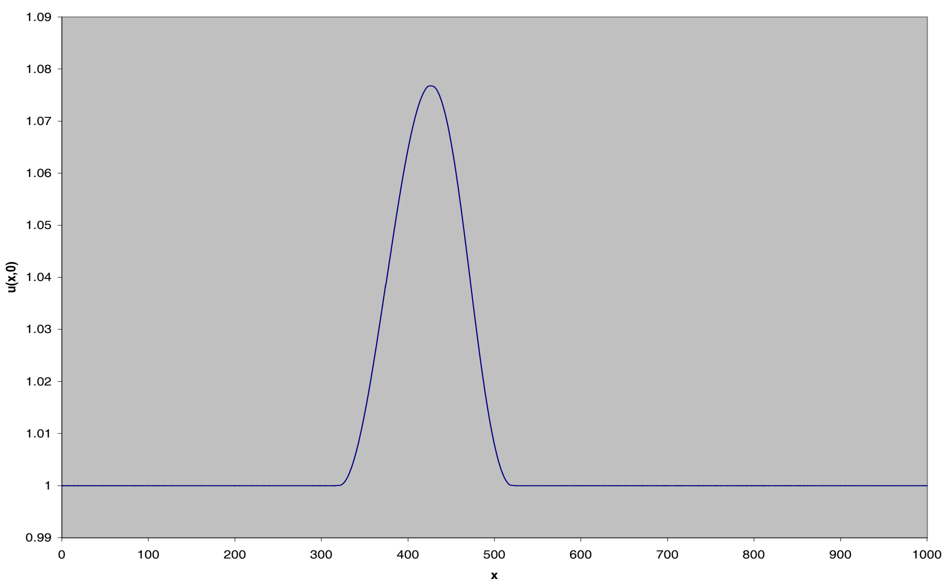

Figure 4.4: The Initial Conditions for **Problem A** (u).

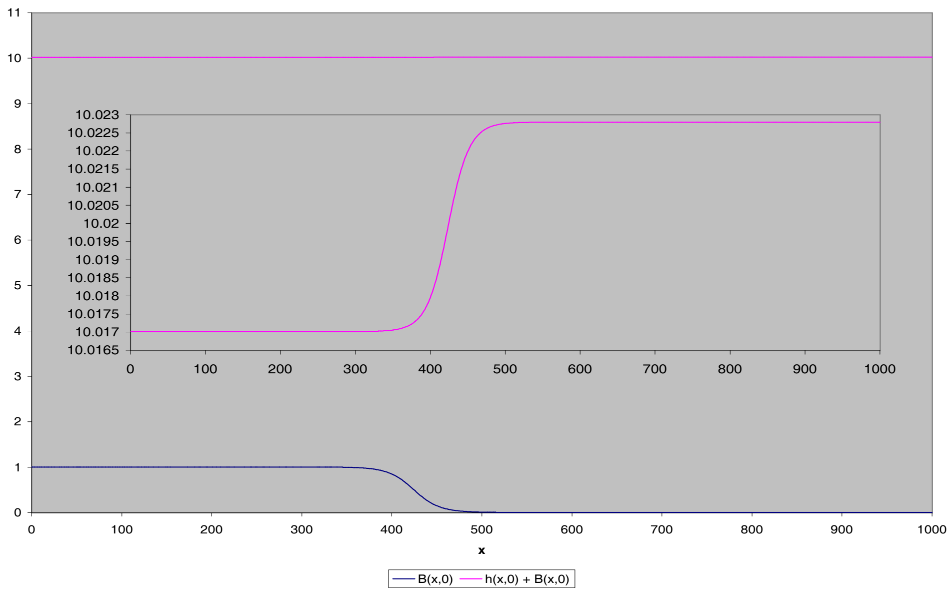

Figure 4.5: The Initial Conditions for **Problem B** (h and B).

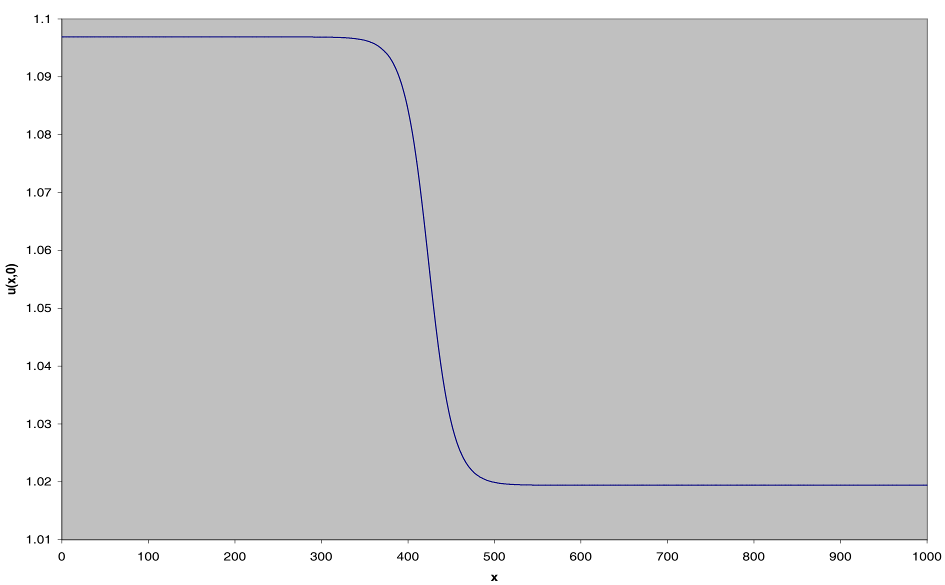

Figure 4.6: The Initial Conditions for **Problem B** (u).

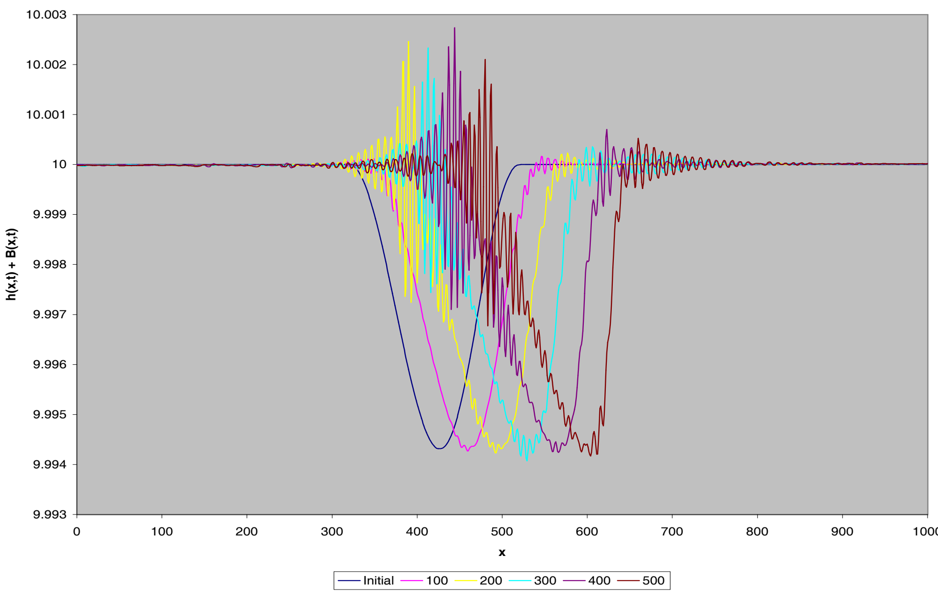

Figure 4.7: The Numerical Results for **Problem A** using **Formulation A-NC** and Roe's Scheme with Source Term Decomposed (h + B).

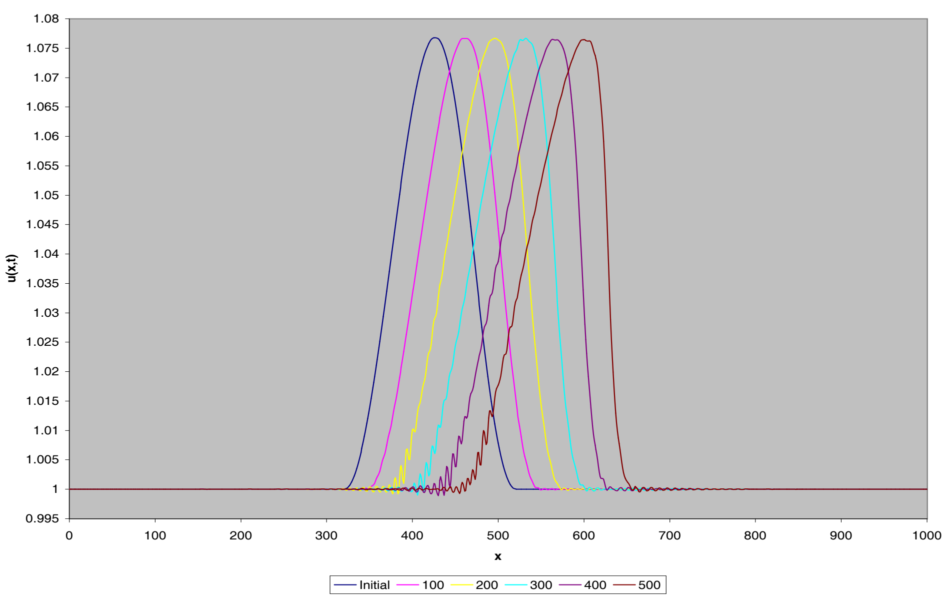

Figure 4.8: The Numerical Results for **Problem A** using **Formulation A-NC** and Roe's Scheme with Source Term Decomposed (u).

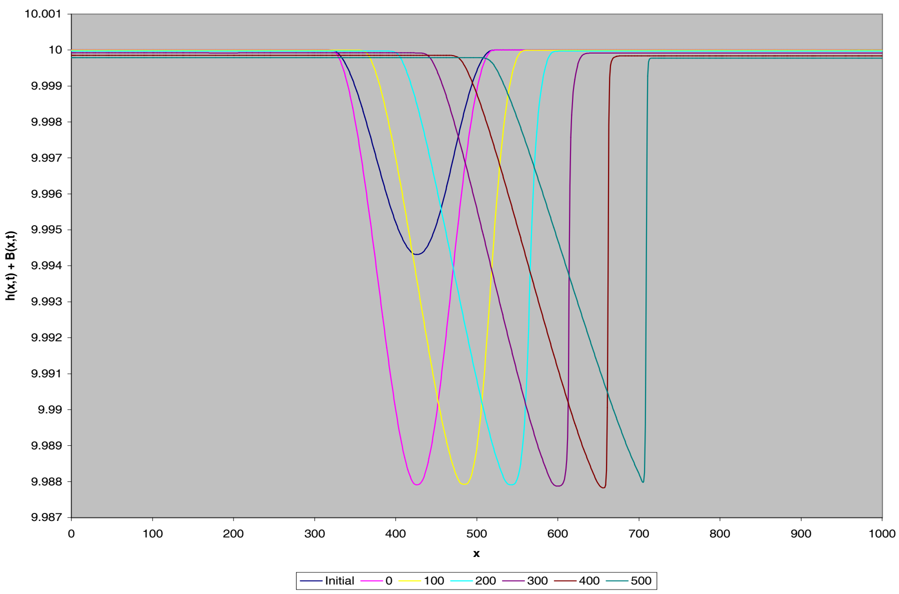

Figure 4.9: The Numerical Results for **Problem A** using **Formulation A-CV** and Roe's Scheme with Source Term Decomposed (h + B) .

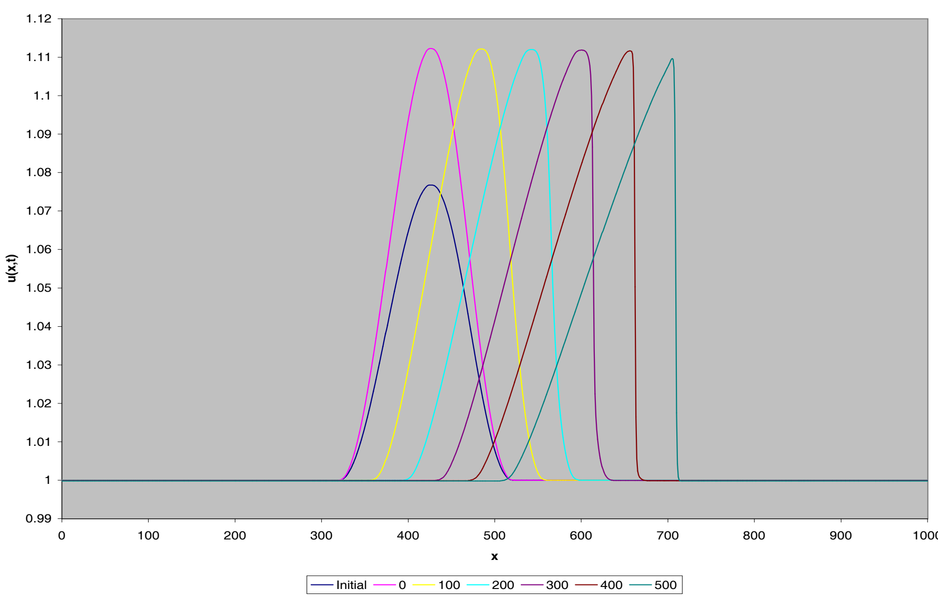

Figure 4.10: The Numerical Results for **Problem A** using **Formulation A-CV** and Roe's Scheme with Source Term Decomposed (u).

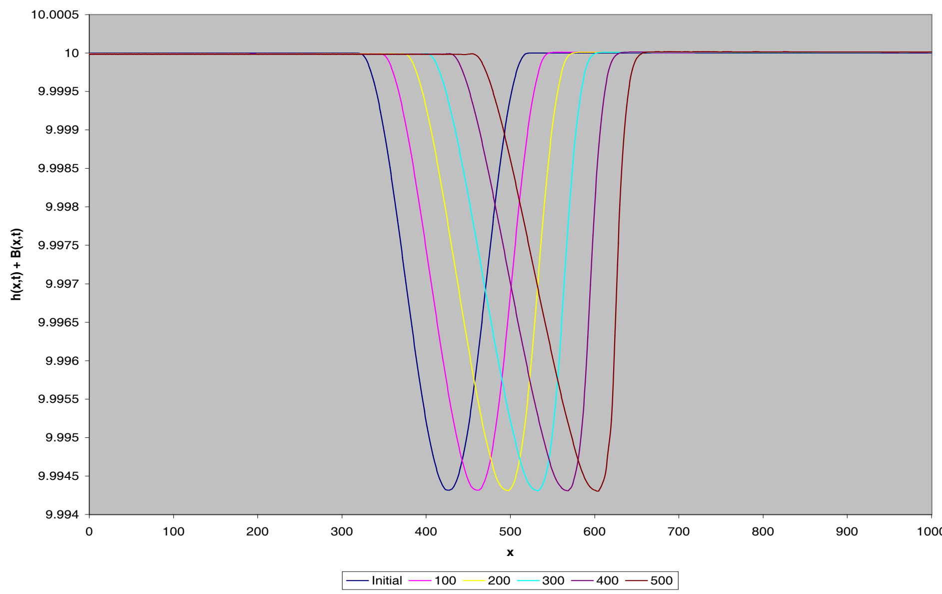

Figure 4.11: The Numerical Results for **Problem A** using **Formulation A-SF** and Roe's Scheme with Source Term Decomposed (h + B).

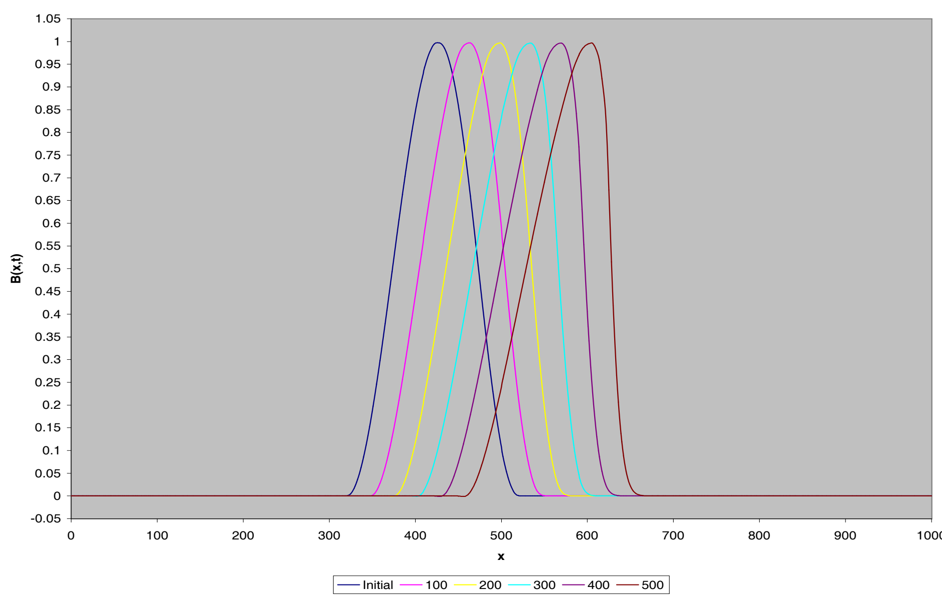

Figure 4.12: The Numerical Results for **Problem A** using **Formulation A-SF** and Roe's Scheme with Source Term Decomposed (B).

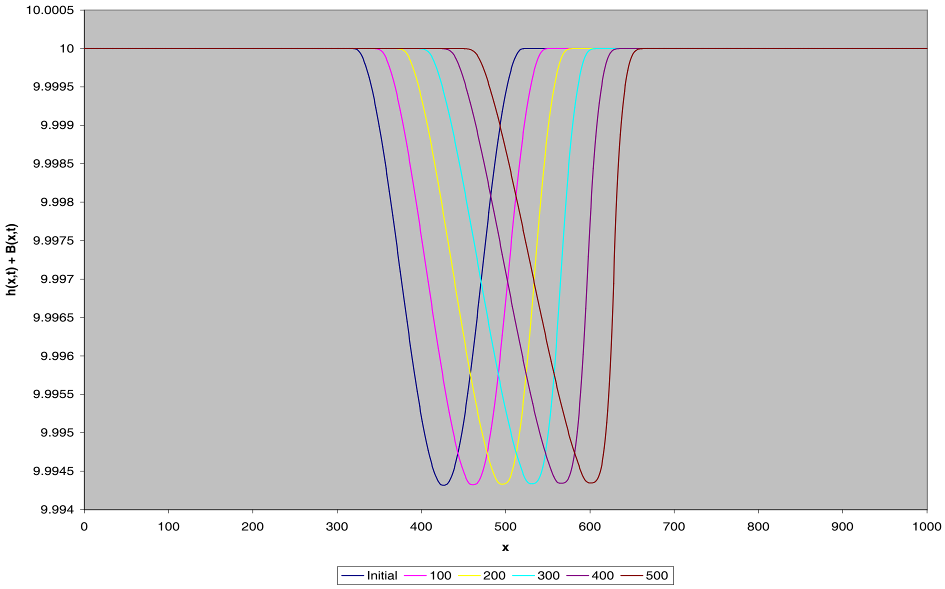

Figure 4.13: The Numerical Results for **Problem A** using **Formulation B** and Roe's Scheme (h+B).

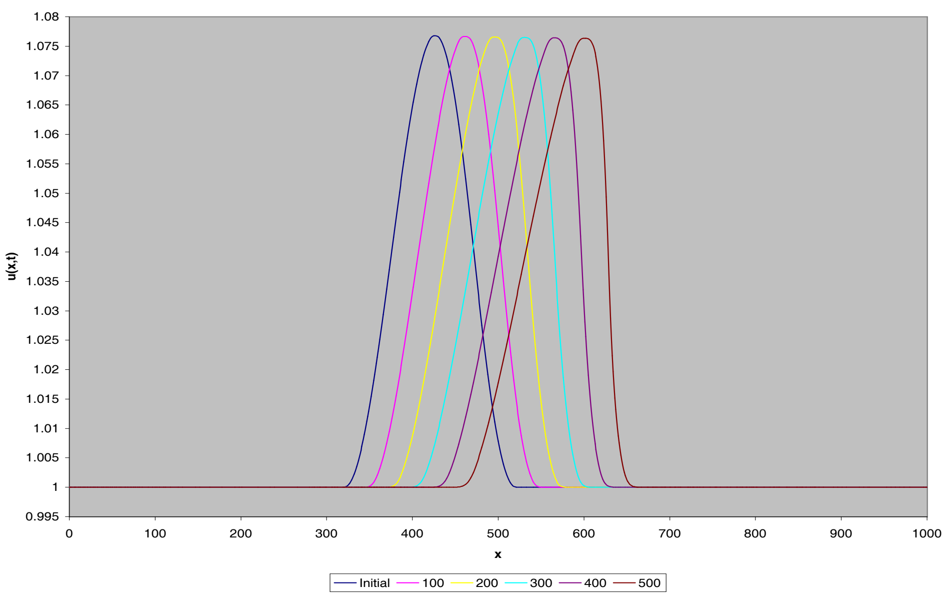

Figure 4.14: The Numerical Results for **Problem A** using **Formulation B** and Roe's Scheme (u).

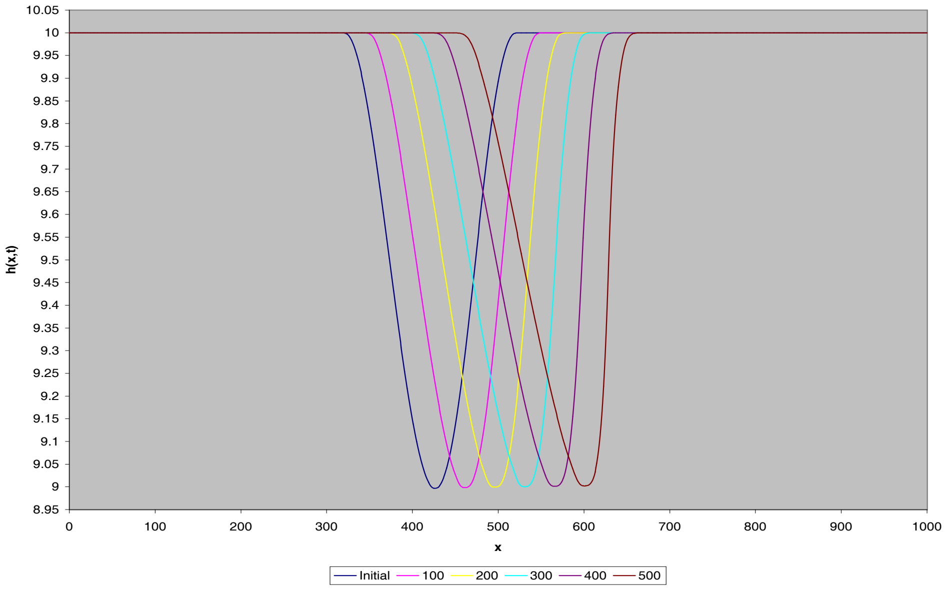

Figure 4.15: The Numerical Results for **Problem A** using **Formulation C** and Roe's Scheme with Source Term Decomposed (h).

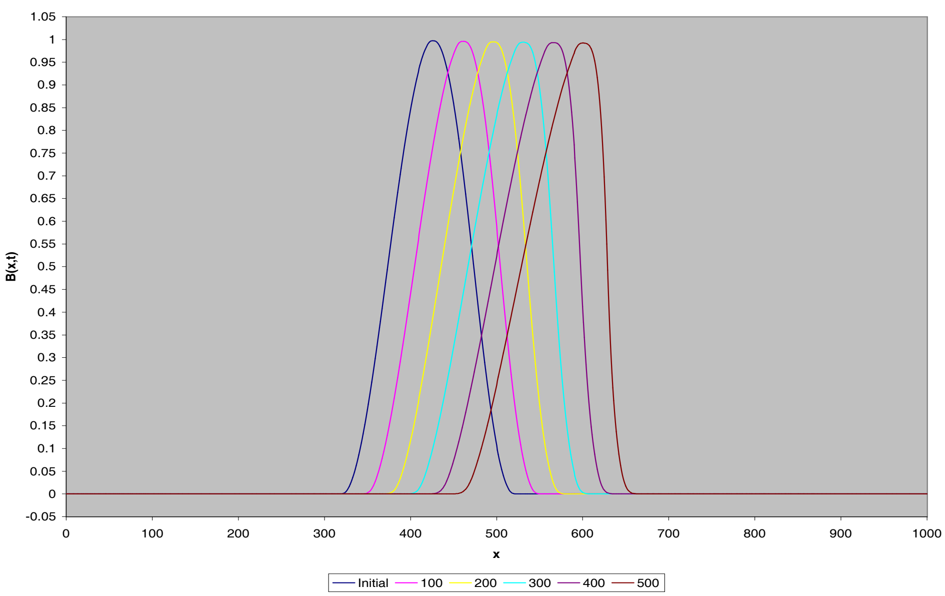

Figure 4.16: The Numerical Results for **Problem A** using **Formulation C** and Roe's Scheme with Source Term Decomposed (B).

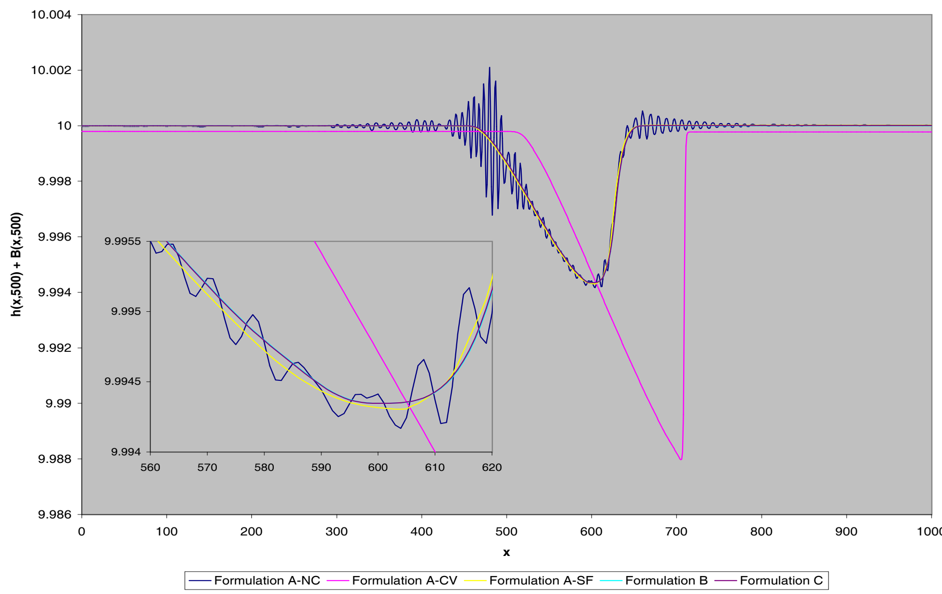

Figure 4.17: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem A** (h+B).

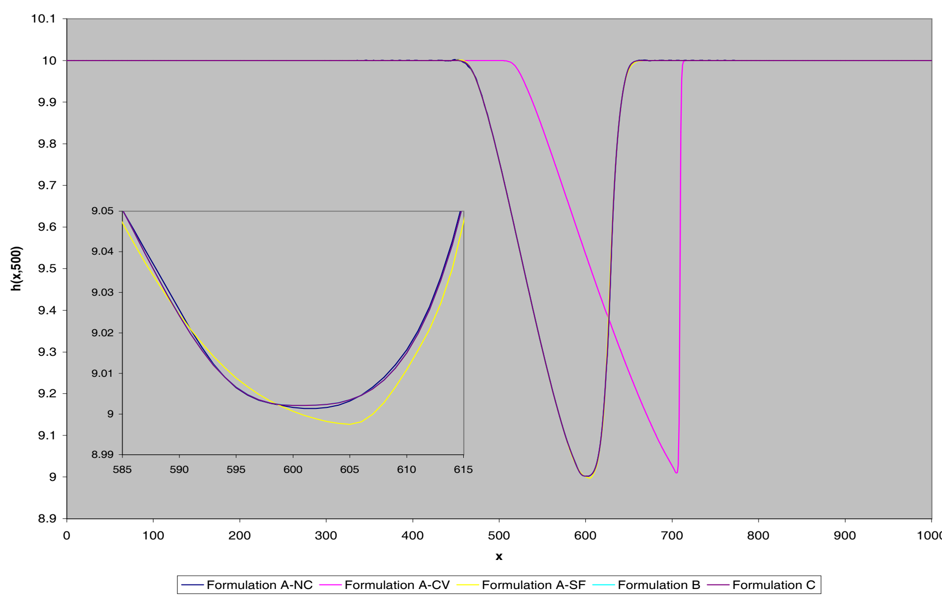

Figure 4.18: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem A** (h).

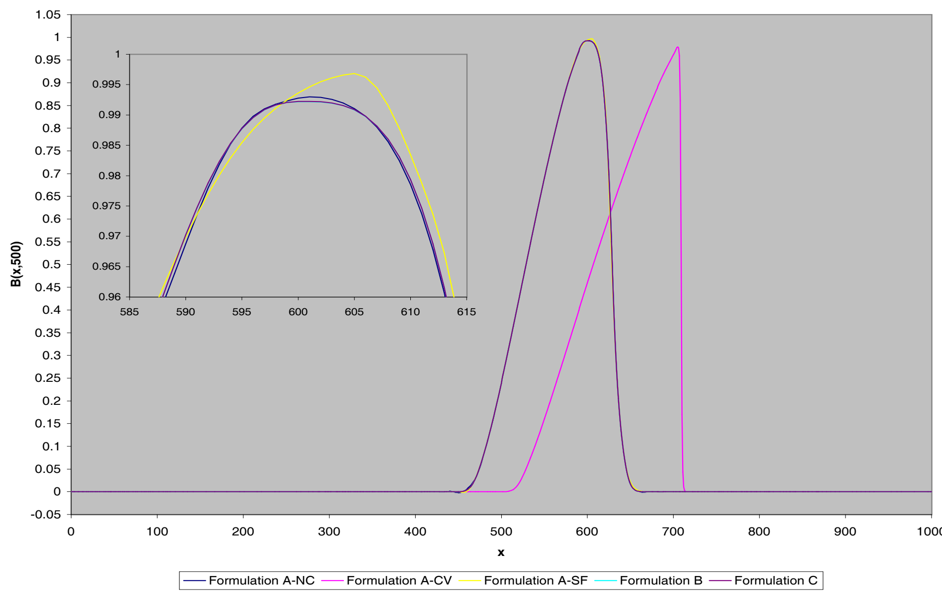

Figure 4.19: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem A** (u).

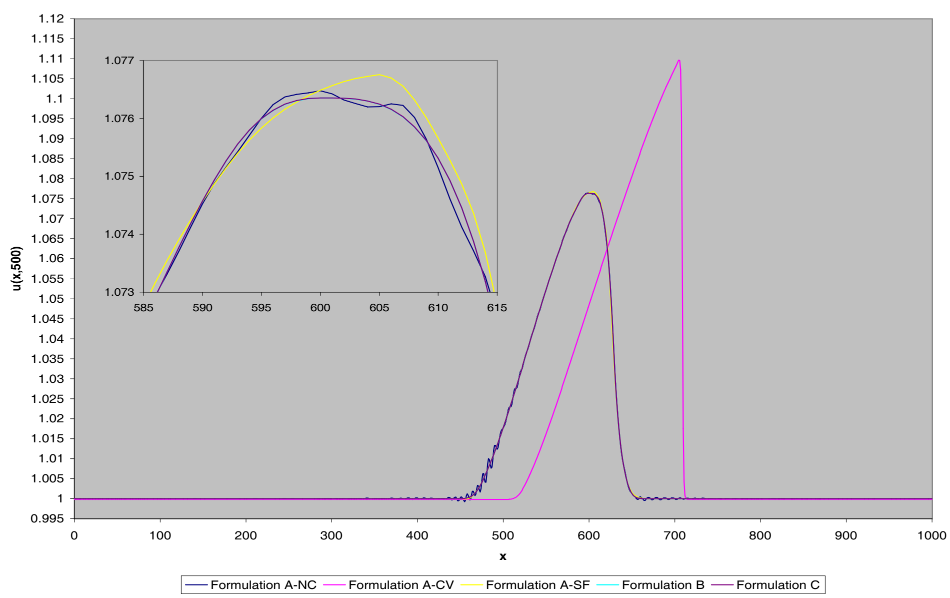

Figure 4.20: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem A** (B).

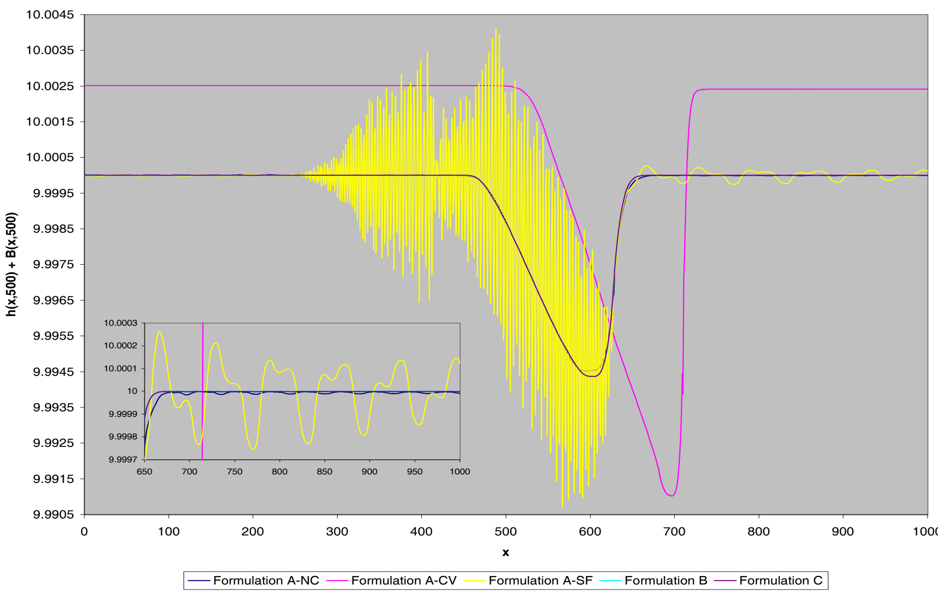

Figure 4.21: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem A** (h+B).

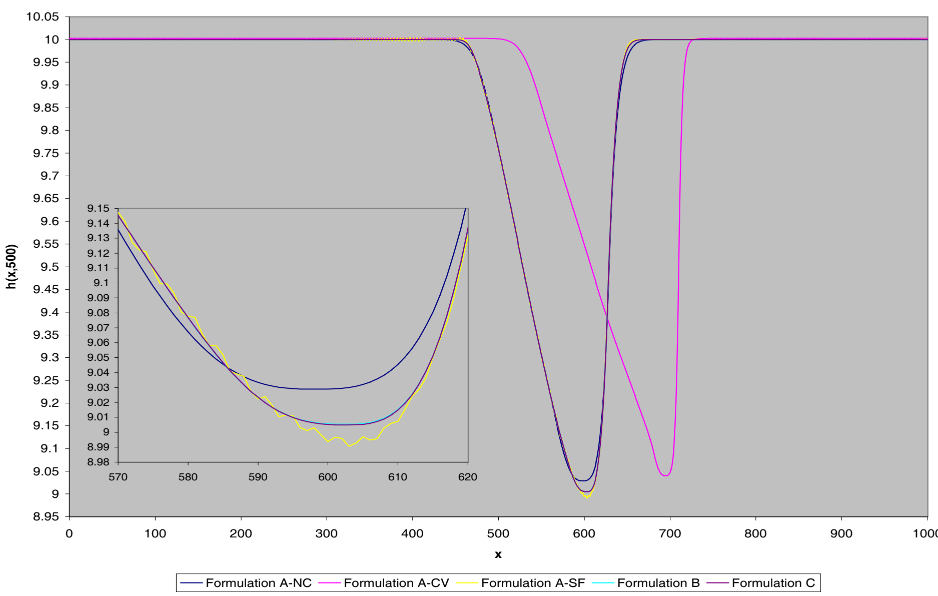

Figure 4.22: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem A** (h).

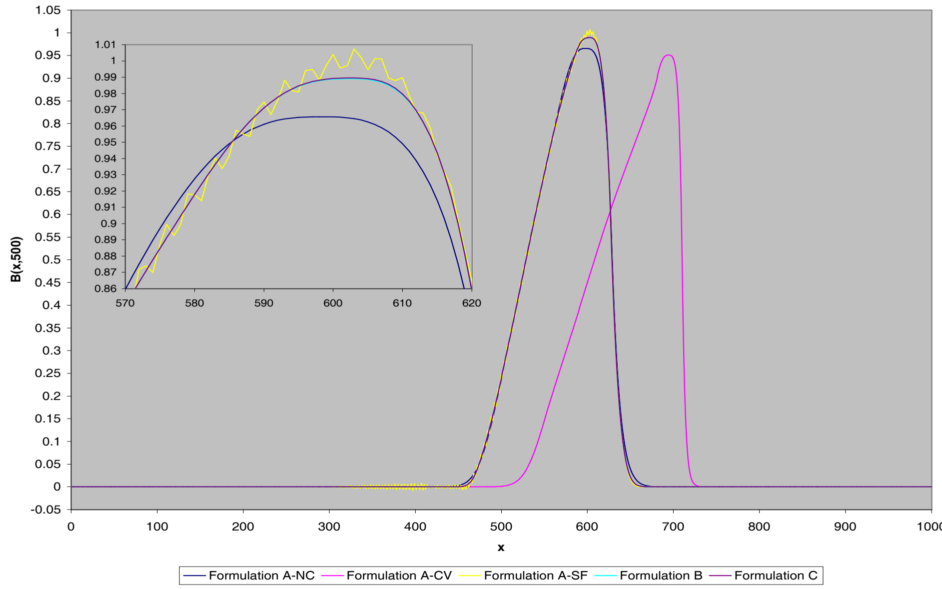

Figure 4.23: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem A** (B).

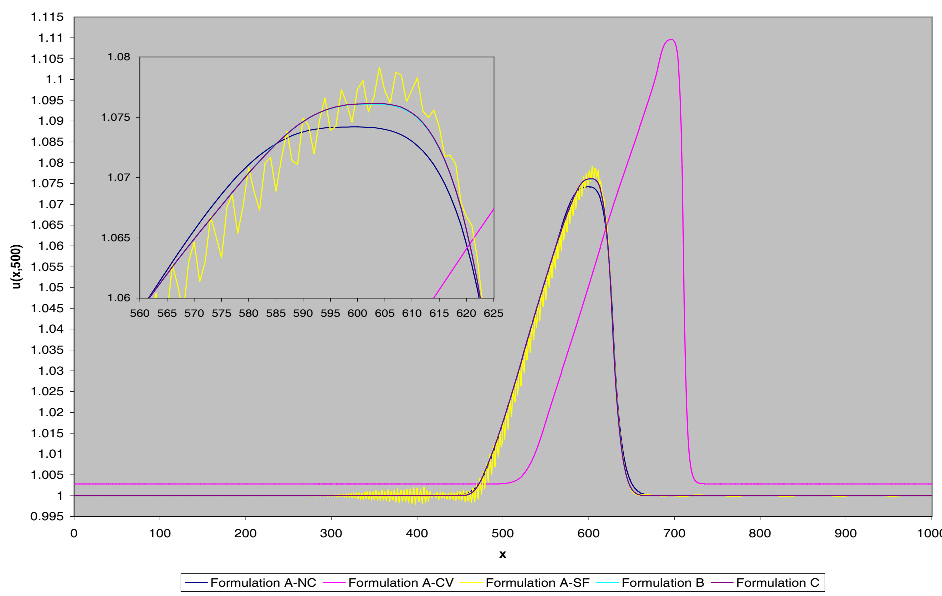

Figure 4.24: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem A** (u).

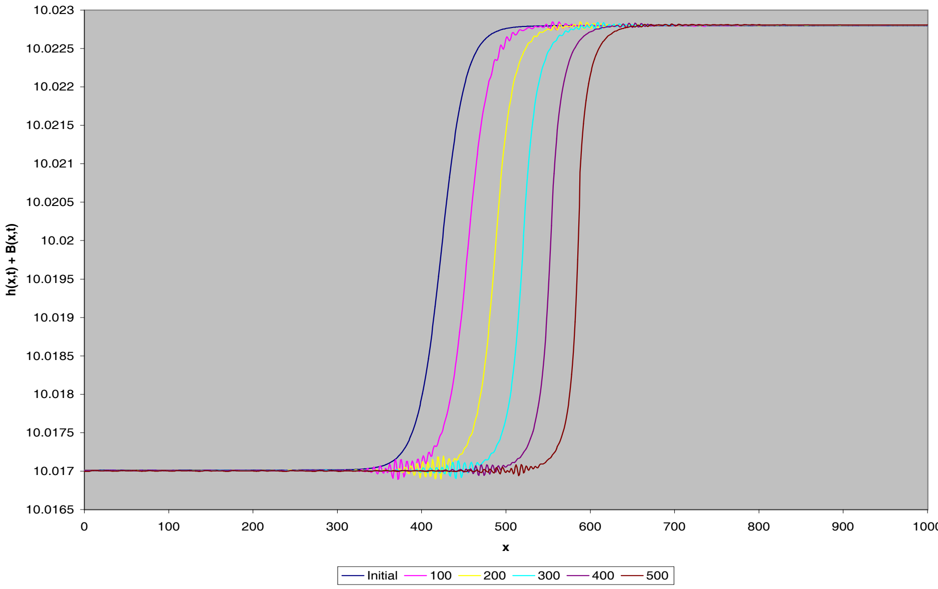

Figure 4.25: The Numerical Results for **Problem B** using **Formulation A-NC** and Roe's Scheme with Source Term Decomposed (h+B).

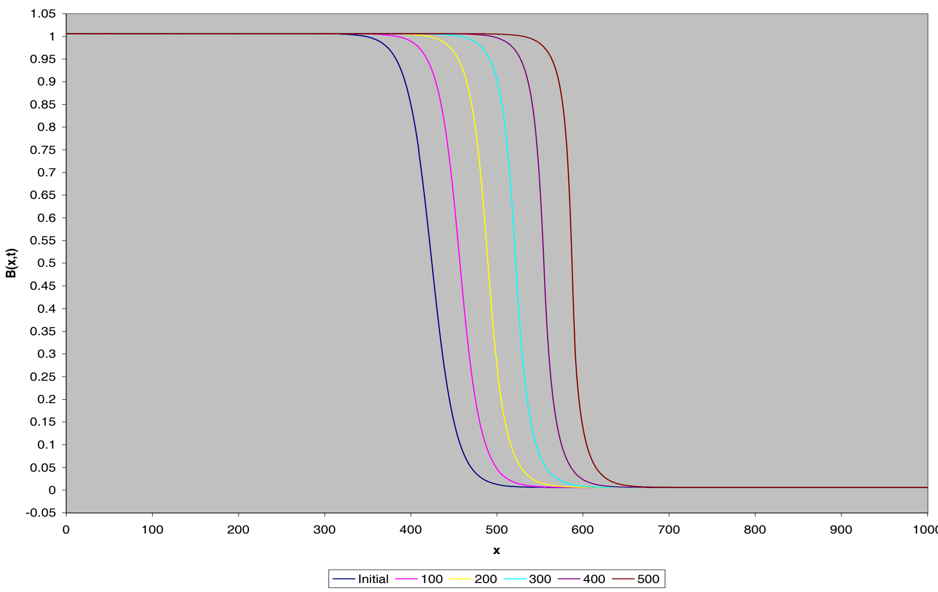

Figure 4.26: The Numerical Results for **Problem B** using **Formulation A-NC** and Roe's Scheme with Source Term Decomposed (B).

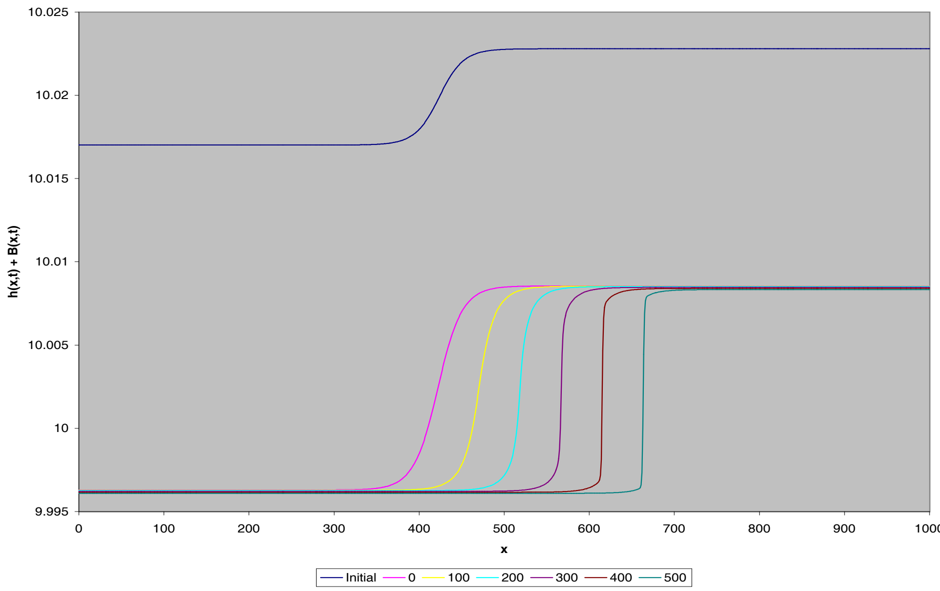

Figure 4.27: The Numerical Results for **Problem B** using **Formulation A-CV** and Roe's Scheme with Source Term Decomposed (h+B).

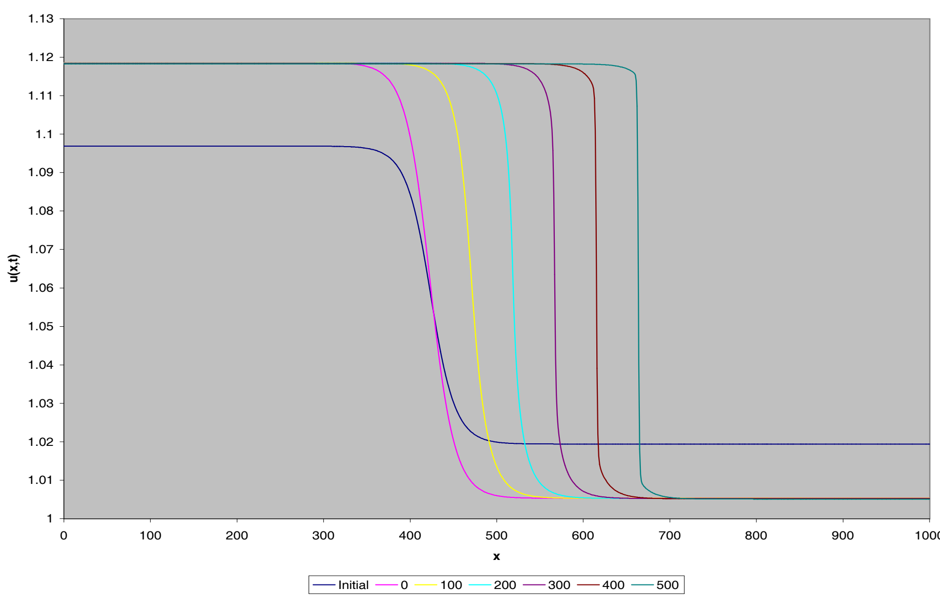

Figure 4.28: The Numerical Results for **Problem B** using **Formulation A-CV** and Roe's Scheme with Source Term Decomposed (u).

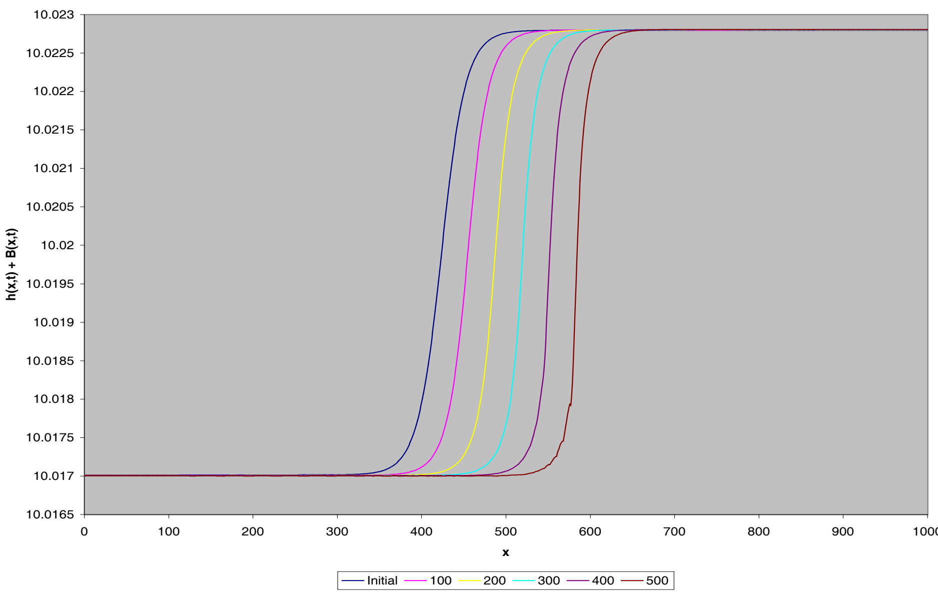

Figure 4.29: The Numerical Results for **Problem B** using **Formulation A-SF** and Roe's Scheme with Source Term Decomposed (h+B).

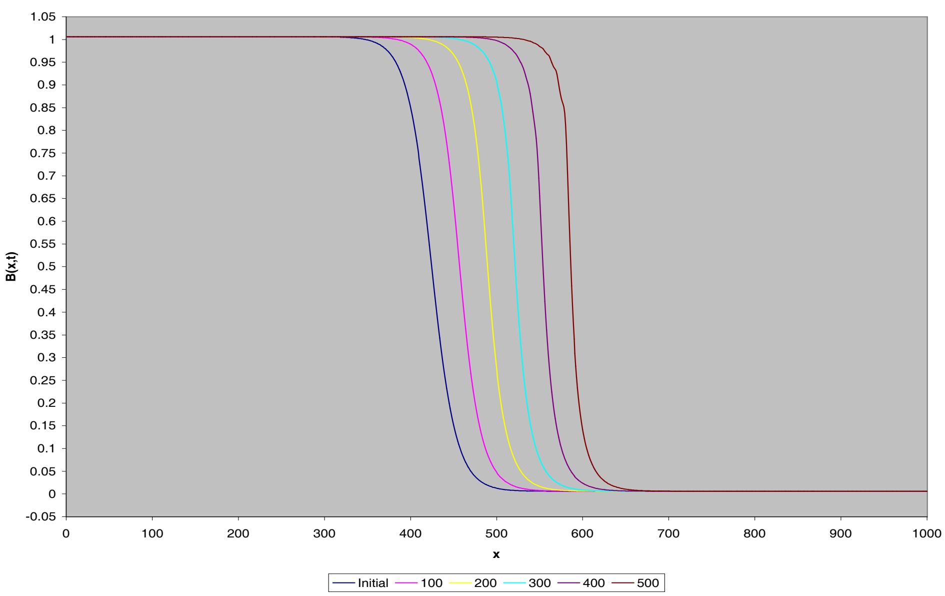

Figure 4.30: The Numerical Results for **Problem B** using **Formulation A-SF** and Roe's Scheme with Source Term Decomposed (B).

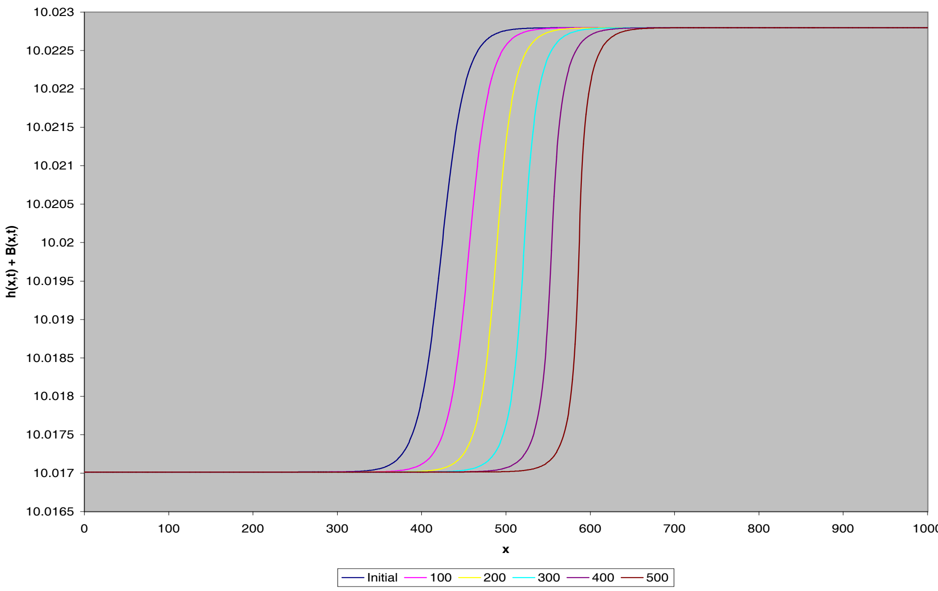

Figure 4.31: The Numerical Results for **Problem B** using **Formulation B** and Roe's Scheme (h+B).

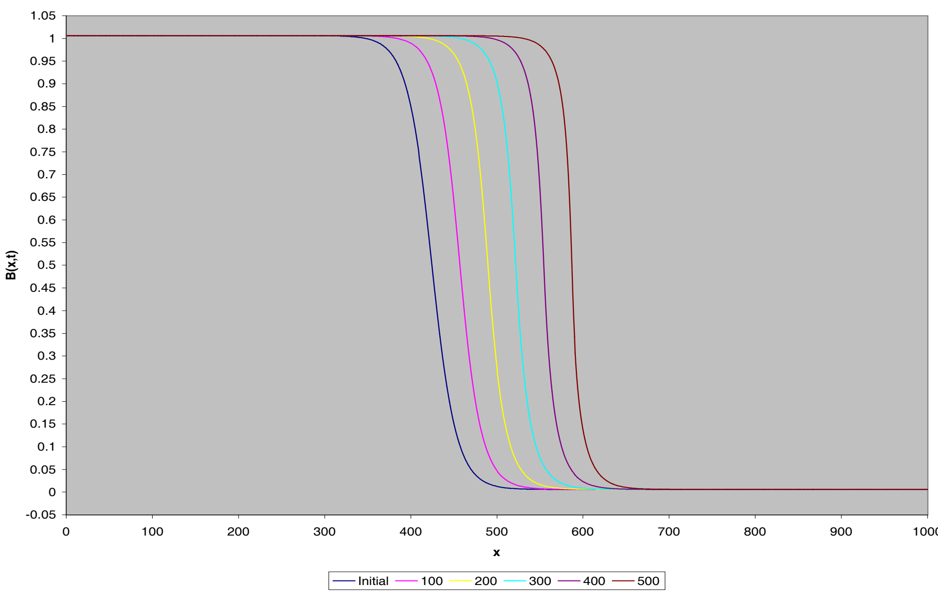

Figure 4.32: The Numerical Results for **Problem B** using **Formulation B** and Roe's Scheme (B).

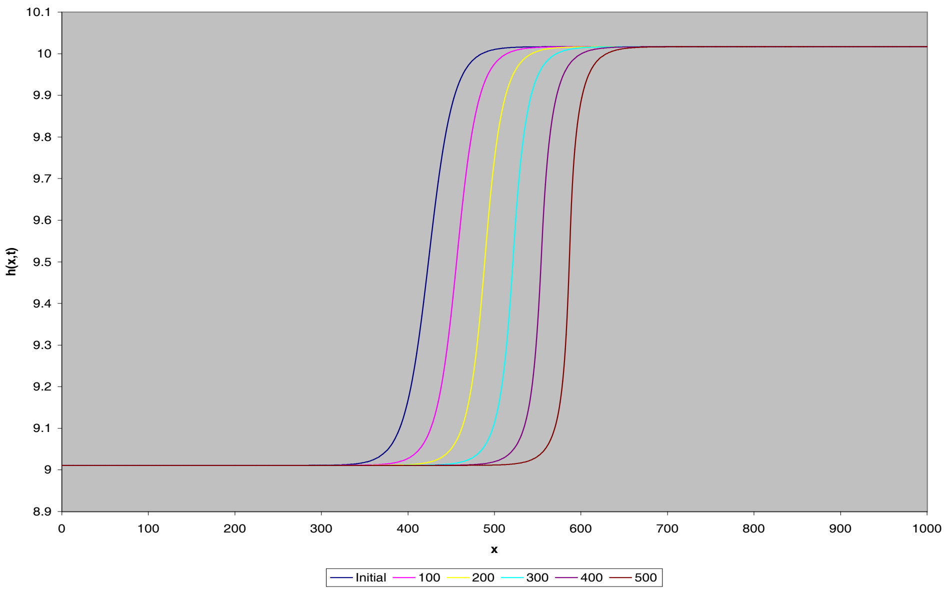

Figure 4.33: The Numerical Results for **Problem B** using **Formulation C** and Roe's Scheme with Source Term Decomposed (h).

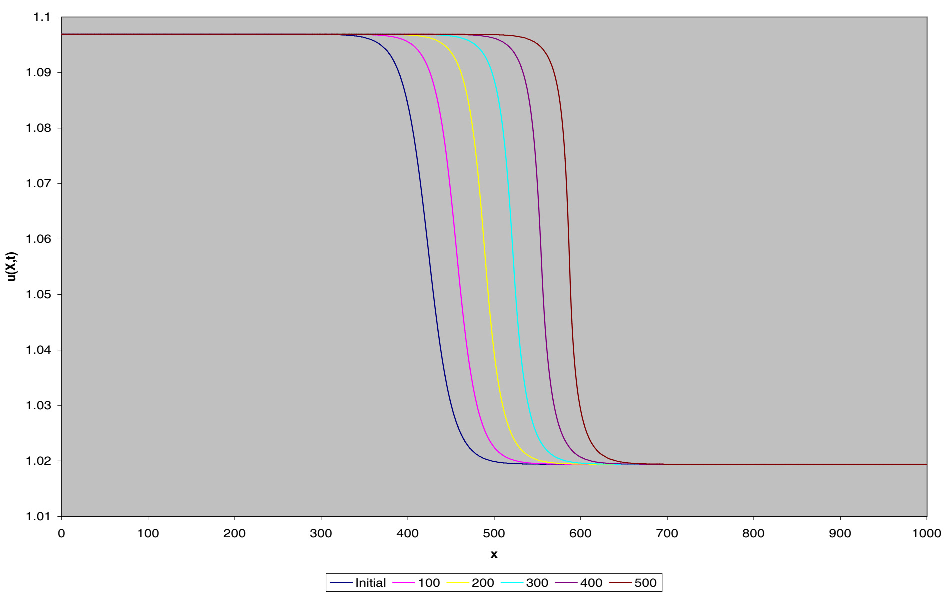

Figure 4.34: The Numerical Results for **Problem B** using **Formulation C** and Roe's Scheme with Source Term Decomposed (u).

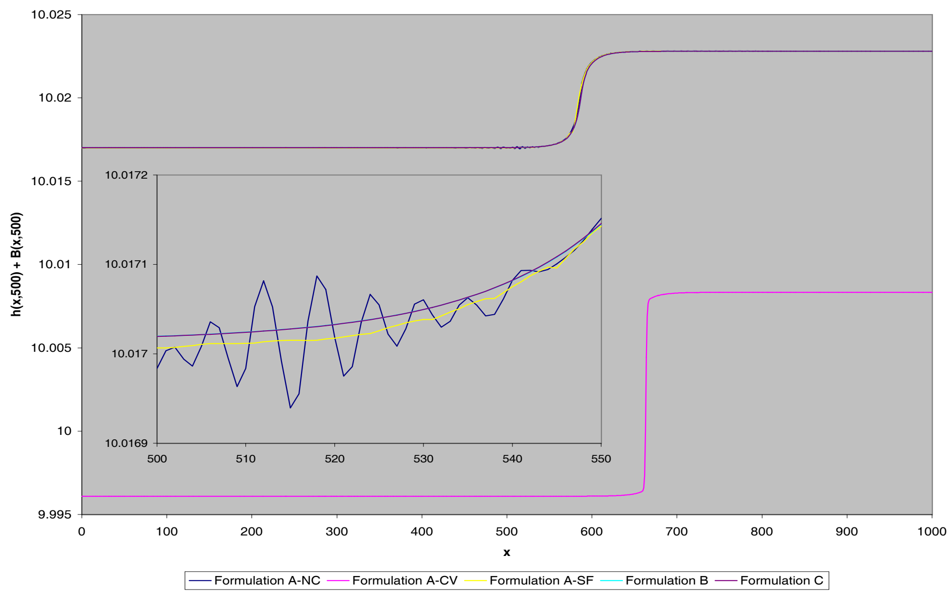

Figure 4.35: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem B** (h+B).

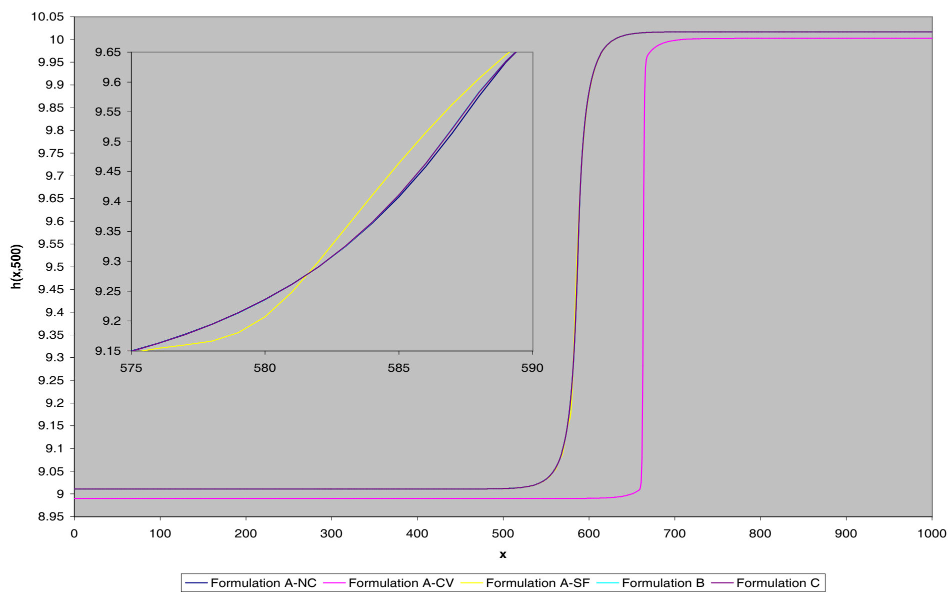

Figure 4.36: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem B** (h).

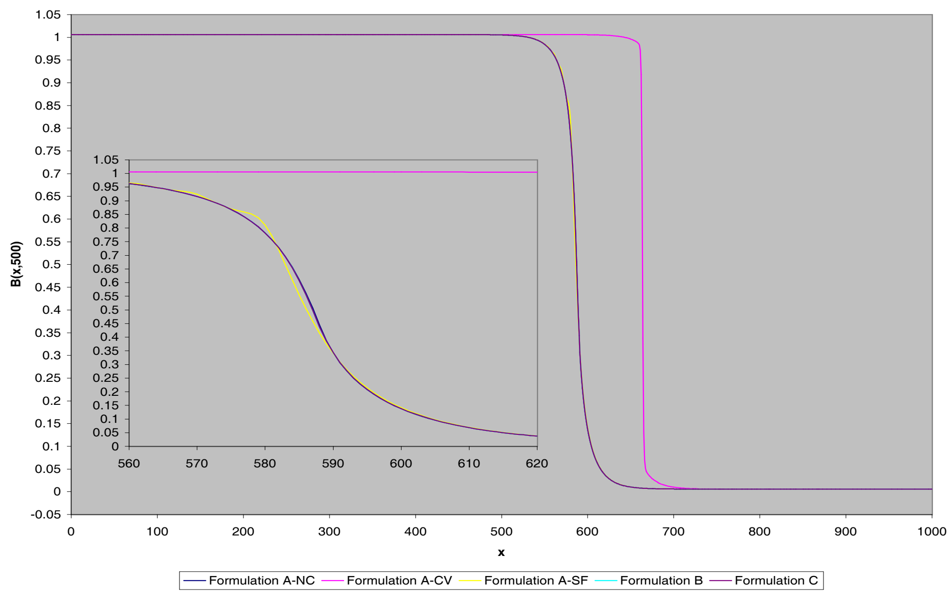

Figure 4.37: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem B** (B).

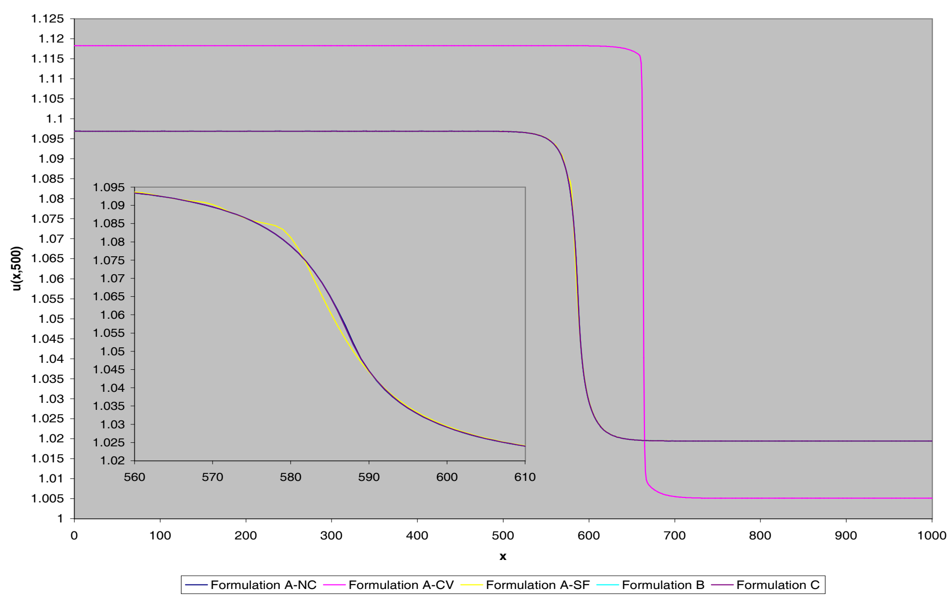

Figure 4.38: Comparison of the Numerical Results of Roe's Scheme with Source Term Decomposed at t = 500 for **Problem B** (u).

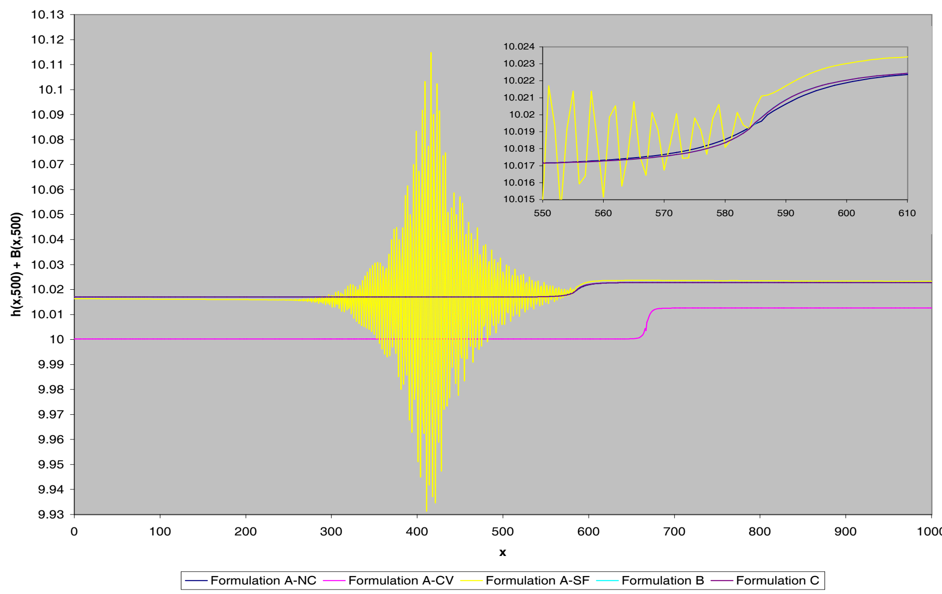

Figure 4.39: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem B** (h+B).

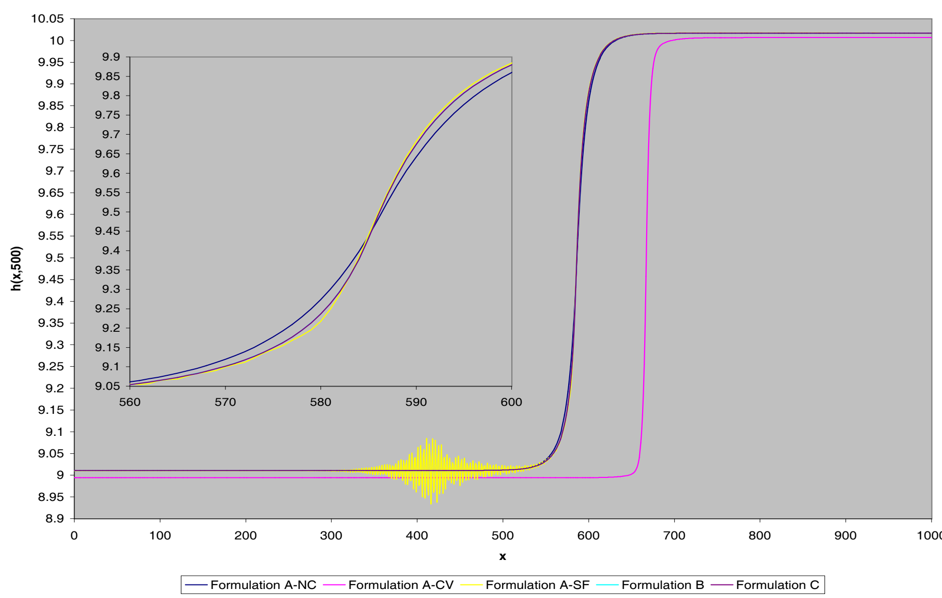

Figure 4.40: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem B** (h).

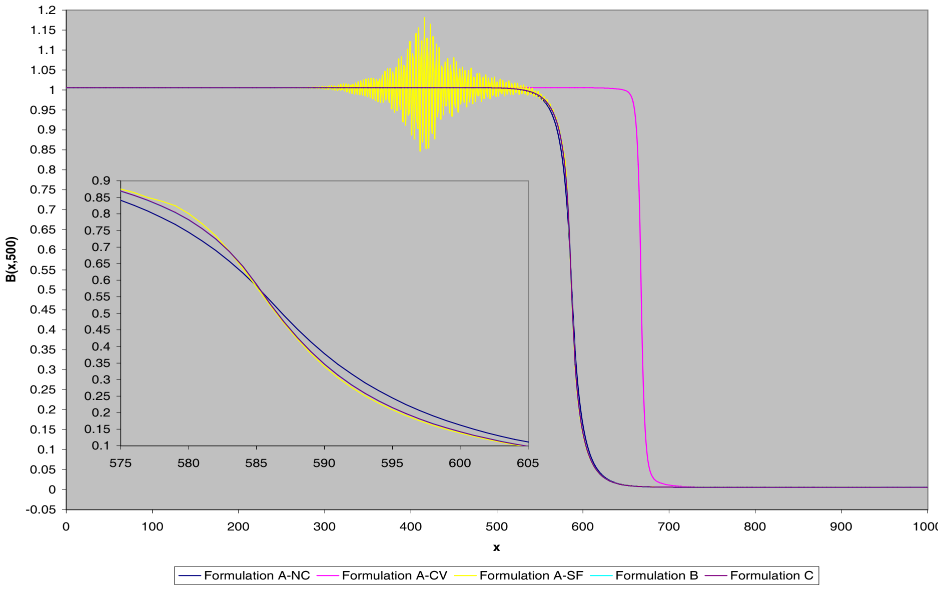

Figure 4.41: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem B** (B).

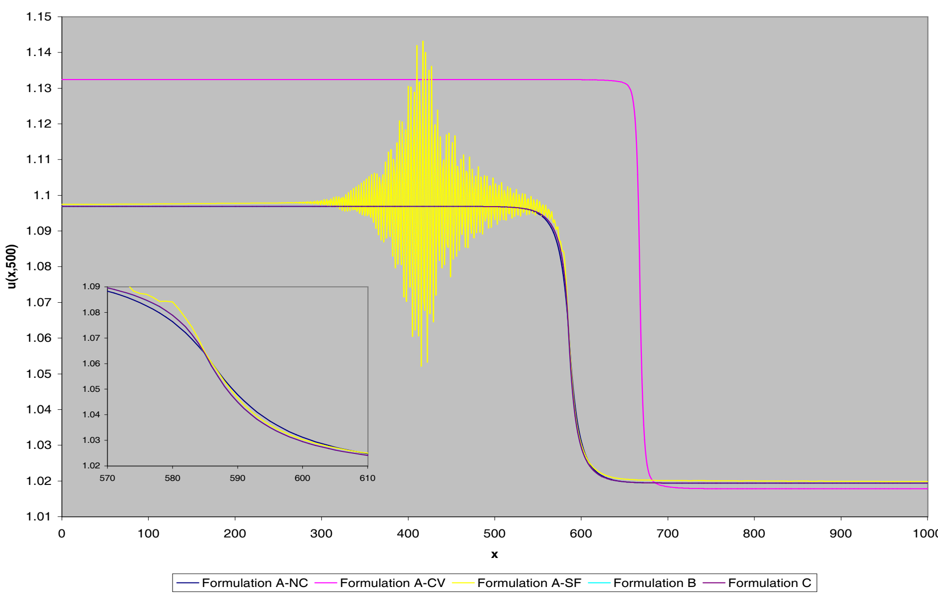

Figure 4.42: Comparison of the Numerical Results of LeVeque & Yee's MacCormack Approach at t = 500 for **Problem B**(u).

5 Conclusion

Throughout this report, we have discussed five different formulations and compared them for two test problems. We have shown that the most accurate formulation is **Formulation B** closely followed by **Formulation C**. The advantage of using **Formulation B** is that a source term is not present, which makes the formulation easier to numerically approximate. However, the numerical results of **Formulation B** may not be so accurate when
considering shocks as the formulation is not written in conservative variable form. **Formulation A-SF** was quite accurate when using Roe's Scheme with source term decomposed but suffered from spurious oscillations when using LeVeque & Yee's MacCormack approach and the numerical results differed slightly to **Formulation B** and **Formulation C**. **Formulation A-CV** gave inaccurate results as the approach moved the pulse in **Problem A** and the sediment bore in **Problem B** at a faster wave speed than any of the other formulations due to the Shallow Water Equations being converged before the riverbed was updated. **Formulation A-CV** also took considerably longer computational run times than any of the other approaches and sometimes took up to 50 times longer making the method very impractical. **Formulation A-NC** produced accurate results when using LeVeque & Yee's MacCormack approach but suffered from spurious oscillations when using Roe's Scheme with source term decomposed. These spurious oscillations are due to the Bed-Updating Equation being calculated separately but can be minimised by using a smaller Courant number. However, by using a smaller Courant number, the formulation may become impractical due to long computational run times. As an alternative, we could use Roe's Scheme with source term decomposed to numerically approximate the Shallow Water Equations with

$$B_i^{n+1} = B_i^n - s\xi(q^*_{i+\frac{1}{2}} - q^*_{i-\frac{1}{2}}), \tag{5.1}$$

where

$$q^*_{i+\frac{1}{2}} = \begin{cases} q_i^n + \frac{1}{2}(1-\nu^n_{i+\frac{1}{2}})(q^n_{i+1} - q^n_i)\Phi^n_i & \text{if } \nu^n_{i+\frac{1}{2}} > 0 \\ q^n_{i+1} - \frac{1}{2}(1+\nu^n_{i+\frac{1}{2}})(q^n_{i+1} - q^n_i)\Phi^n_i & \text{if } \nu^n_{i+\frac{1}{2}} < 0 \end{cases},$$

to numerically approximate the Bed-Updating Equation for **Formulation A-NC**. But the numerical scheme still suffers from spurious oscillations, which are considerably worse than the normal approach (3.10), see Figures

5.1 to 5.2. However, by using a staggered approach,

$$q^*_{i+\frac{1}{2}} = \begin{cases} q_i^{n+1} + \frac{1}{2}(1 - \nu^{n+1}_{i+\frac{1}{2}})(q^{n+1}_{i+1} - q_i^{n+1})\Phi_i^n & \text{if } \nu^{n+1}_{i+\frac{1}{2}} > 0 \\ q_{i+1}^{n+1} - \frac{1}{2}(1 + \nu^{n+1}_{i+\frac{1}{2}})(q^{n+1}_{i+1} - q_i^{n+1})\Phi_i^n & \text{if } \nu^{n+1}_{i+\frac{1}{2}} < 0 \end{cases},$$

we can minimise these spurious oscillations without having to reduce the Courant number, see Figure 5.1 and Figure 5.2.

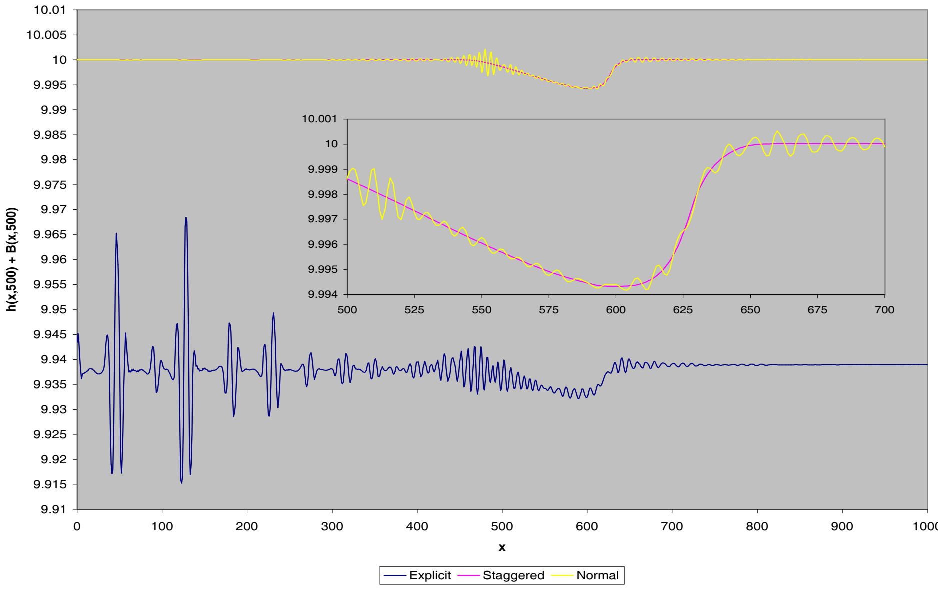

Figure 5.1: Comparison of the numerical results of **Problem A** using **Formulation A-NC** with the alternative numerical approach (h+B).

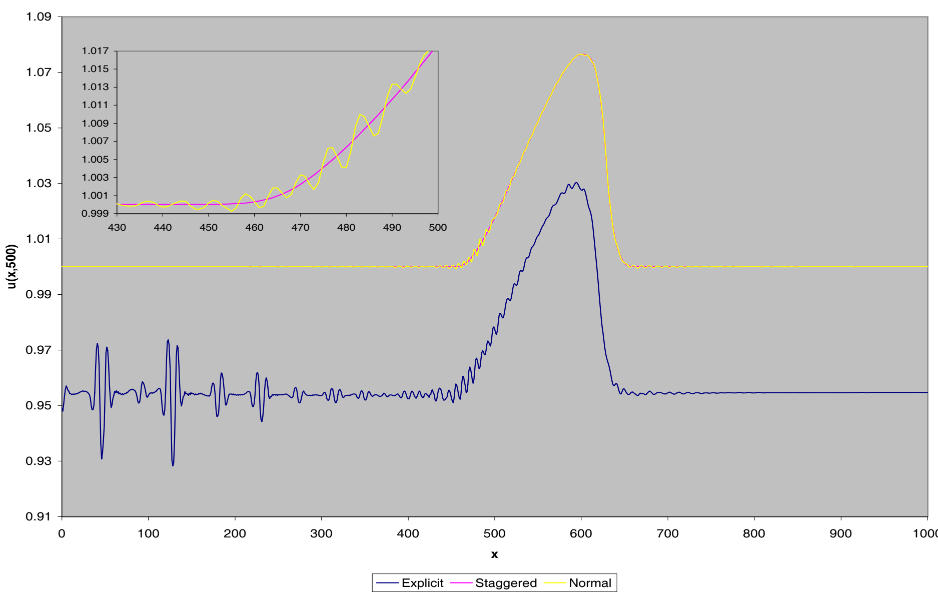

Figure 5.2: Comparison of the numerical results of **Problem A** using **Formulation A-NC** with the alternative numerical approach (u).

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