The State Estimation of Flow Demands in Gas Networks from Sparse Pressure Telemetry

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Abstract

In this paper we present novel observer based techniques for the estimation of flow demands in gas networks from sparse pressure telemetry. We explore a *completely observable* model constructed by incorporating difference equations that assume the flow demands are steady. Since the flow demands usually vary slowly with time, this is a reasonable approximation. Two techniques for constructing *robust* observers are employed: robust eigenstructure assignment and singular value assignment. These techniques help to reduce the effects of the system approximation. Modelling error may be further reduced by making use of known profiles for the flow demands.

The theory is extended to deal successfully with the problem of measurement bias. The pressure measurements available are subject to constant biases which degrade the flow demand estimates, and such biases need to be estimated. This is achieved by constructing a further model variation that incorporates the biases into an augmented state vector, but now includes information about the flow demand profiles in a new form.

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List of symbols

E, A, B^1, B^2 system matrices

C system measurement matrix

$\underline{x}(k)$, $\underline{y}(k)$ system state and measurement vectors

G, H observer feedback matrices

$\underline{\widehat{x}}(k)$, $\underline{e}(k)$ observer state vector and error

Λ set of observer system eigenvalues

λ_i individual observer system eigenvalues

$\underline{y}_i$, $\underline{x}_i$ left and right eigenvectors of observer system

$\kappa(\lambda_i)$ condition number of observer eigenvalues

U, Σ, V matrices from singular value decomposition

σ_1, σ_n largest and smallest singular values

α, β, γ, δ the four model variations

θ arbitrary model parameter between 1/2 and 1

$\underline{p}_1(k)$ vector containing unmeasured nodal pressures

$\underline{p}_2(k)$ vector containing measured nodal pressures

$\underline{p}_3(k)$ upstream pressure

$\underline{d}(k)$ vector containing the flow demand perturbation variables

$\underline{f}(k)$ vector containing discrete jumps in flow demand profiles

$\underline{b}(k)$ vector of measurement biases

$w_k^{demand\ site}$ time-varying flow profile coefficients

$W(k)$ diagonal block containing time-varying profile coefficients, $w_k^{demand\ site}$

$\underline{q}$ vector containing flow demands about which gas network model is linearised

$\underline{l}(k)$ vector containing discrete jumps in flow demand profiles

1 Introduction

For any gas network, it is desirable to have a reasonable estimate of the demand flows. However, flow meters are much more expensive than pressure sensors to install, and so it would be economical to be able to estimate the flow demands from pressure measurements alone. In this paper novel observer based techniques for achieving this are presented. The gas networks considered are linear and consist of a number of pipe sections with a gas source at the upstream end and flow demands at pipe junctions and at the downstream end. For example, a three pipe network would be as in Fig.1. We assume the only measurements of the real gas network available are discrete pressure measurements at all sites of gas inflow (the upstream end) and outflow (the pipe junctions and downstream end). These measurement sites are the natural 'boundaries' of the network, where some data (pressure or flow demand) needs to be specified to drive a network model.

In section 2 we discuss dynamic observer designs capable of estimating the entire system state of discrete dynamical systems that have the property of *complete observability.* Two techniques for constructing *robust* observers are employed: robust eigenstructure assignment and singular value assignment. These techniques help to reduce the effects of modelling error. In section 3 we introduce the basic gas network model which requires the flow demands as inputs. We then investigate ways of rearranging and augmenting such models to produce new model variations that may be used to estimate the flow demands. We explore a *completely observable* model constructed by incorporating difference equations that assume the flow demands are steady. Since the flow demands usually vary slowly with time, this is a reasonable approximation. Such modelling error may be further reduced by making use of known profiles for the flow demands. In section 4 the theory is extended to deal successfully with the problem of measurement bias. The pressure measurements available are subject to constant biases which degrade the flow demand estimates, and such biases need to be estimated. This is achieved by constructing a further model variation that incorporates the biases into an augmented state vector, but now includes information about the flow demand profiles in a new form, which allows the estimation of the measurement biases, as well as the flow demands.

2 Observers

We consider the time-invariant linear descriptor system

$$E\underline{x}(k+1) = A\underline{x}(k) + B^1\underline{u}(k+1) + B^2\underline{u}(k), \quad (1)$$

$$\underline{y}(k) = C\underline{x}(k) \quad for\ k = 0, 1, 2, ..., \quad (2)$$

where $\underline{x}(k)\in\mathbf{R}^n$ is the state vector, $\underline{u}(k)\in\mathbf{R}^m$ is the input vector and $\underline{y}(k)\in\mathbf{R}^g$ is the output vector of measured state variables. It is assumed that E, B^1, B^2 and C are full rank.

The system (1), (2) is defined to be *completely observable* if and only if knowledge of the inputs and measurements over n timesteps is enough to determine uniquely an initial state $\underline{x}(0)$. In practice, the number of available measurements may be small and the challenge of state estimation is to determine the state of the whole system from these measurements over time. If the system is *completely observable* from the available measurements, a dynamic observer may be used for state estimation. We present two observer designs that are constructed using the techniques: robust eigenstructue assignment and singular value assignment. These techniques are used help to make the observers robust, that is, insensitive to modelling and measurement error.

The following theorem gives a necessary and sufficient condition for complete observability, known as the *Hautus condition* (Yip and Sincovec, 1981), (Fletcher, Kautsky and Nichols, 1986), (Kautsky, Nichols and Chu, 1989).

Theorem 2.1 *A system of the form (1), (2), where the matrix E is non-singular, is completely observable if and only if we have the following condition.*

For all $\mu \in \mathbf{C}$

$$(A - \mu E)\underline{v} = \underline{0} \quad , \quad C\underline{v} = \underline{0} \iff \underline{v} = \underline{0} \tag{3}$$

where $\underline{v} \in \mathbf{R}^n$.

2.1 Design 1 : The Standard Dynamic Observer

The first design takes the form

$$E\hat{\underline{x}}(k+1) = (A - GC)\hat{\underline{x}}(k) + B^1\underline{u}(k+1) + B^2\underline{u}(k) + G\underline{y}(k). \tag{4}$$

Our aim is to construct the matrix G so that $\hat{\underline{x}}(k) \longrightarrow \underline{x}(k)$ as $k \longrightarrow \infty$ regardless of the true value of $\underline{x}(0)$, which we assume to be unknown. If we define the error between the two systems (1) and (4) at time level k to be

$$\underline{e}(k) = \underline{x}(k) - \hat{\underline{x}}(k), \tag{5}$$

then subtracting system (4) from system (1), gives

$$E\underline{e}(k+1) = (A - GC)\underline{e}(k). \tag{6}$$

We require $\underline{e}(k) \longrightarrow \underline{0}$ as $k \longrightarrow \infty$; and for this we require the eigenvalues of the matrix pencil $\lambda E - (A - GC)$ to have modulus less than unity. Hence, designing our observer system is equivalent to finding a matrix G such that all the eigenvalues of the matrix pencil $\lambda E - (A - GC)$ lie strictly within the unit circle. If the model, (1), (2), has an invertible matrix E and is completely observable, then we can find such a matrix G (Bunse-Gerstner, Mehrmann and Nichols, 1992), (Fletcher, Kautsky and Nichols, 1986), (Kautsky, Nichols and Chu, 1989).

Let us assume that we have a set Λ of n distinct real eigenvalues that we wish to assign to the observer system where $\Lambda = \{\lambda_1, ...\lambda_n\}$ with $|\lambda_i| < 1$, $for\ i = 1, ..., n$. We assume that the left and right eigenvectors $\underline{y}_i$ and $\underline{x}_i$, respectively, satisfy

$$(A - GC)\underline{x}_i = \lambda_i E\underline{x}_i \quad , \quad \underline{y}_i^T(A - GC) = \lambda_i \underline{y}_i^T E. \tag{7}$$

In a non-defective pencil, perturbations of $O(\epsilon)$ in A, E, C and G cause purturbations of $O(\epsilon\kappa(\lambda_i))$ in a simple eigenvalue, where $\kappa(\lambda_i)$ is the condition number

$$\kappa(\lambda_i) = \|\underline{y}_i\|_2 \|\underline{x}_i\|_2 \ / \ |\underline{y}_i^T E \underline{x}_i|. \tag{8}$$

The robust eigenstructure assignment technique (Kautsky, Nichols and Van Dooran, 1985), (Kautsky, Nichols and Chu, 1989), seeks to find a feedback matrix, G, such that the condition numbers (8) of the matrix pencil $\lambda E - (A - GC)$ are made small through appropriate assignment of the eigenvectors. Then the observer system is robust in the sense that the eigenvalues are insensitive to small perturbations in the system matrices.

2.2 Design 2 : The Dynamic Observer with Feedback at the Present Time Level

The second observer design takes the form

$$(E - HC)\hat{\underline{x}}(k+1) = (A - GC)\hat{\underline{x}}(k) + B^1\underline{u}(k+1) + B^2\underline{u}(k) - H\underline{y}(k+1) + G\underline{y}(k), \quad (9)$$

where the matrices H and G are to be chosen. The term in H represents feedback at the present time level, and the term in G represents the familiar feedback at the previous time level. If we define the error, $\underline{e}(k)$, between the two systems (1) and (9) at time level k to be as in equation (5), then subtracting equation (9) from equation (1) gives

$$(E - HC)\underline{e}(k+1) = (A - GC)\underline{e}(k). \quad (10)$$

If H can be chosen such that $(E - HC)$ is nonsingular, then its inverse can be calculated to give an explicit expression for $\underline{e}(k+1)$ from equation (10). We notice that $(E-HC)^{-1}$ multiplies terms on the right hand side of equation (10). Therefore, if $\|(E-HC)^{-1}\|$ can be made small for a suitable norm, then the effects of certain forms of modelling error may be reduced. It is also advisable to ensure that the condition number $cond(E - HC)$ is not too large, since this matrix has to be inverted implicitly in equation (9) to calculate $\hat{\underline{x}}(k+1)$.

Once the matrix H has been calculated, the observer design is completed by using robust eigenstructure assignment to calculate a matrix G to assign a set of eigenvalues, all less than unity in modulus, to the matrix pencil $\lambda(E - HC) - (A - GC)$. Then from equation (10), we can readily see that the error, $\underline{e}(k)$, between the observer state estimate, $\hat{\underline{x}}(k)$, and the true system state, $\underline{x}(k)$, tends to zero.

In order to calculate H, we use singular value assignment. We know that if

$$E - HC = U\Sigma V^T \quad (11)$$

is the singular value decomposition of $(E - HC)$, where $\Sigma = diag\{\sigma_i\}$ and the σ_i are arranged in nonincreasing order, then

$$\|E - HC\|_2 = \sigma_1 \quad \text{and} \quad \|(E - HC)^{-1}\|_2 = \sigma_n^{-1}$$

and the 2-norm condition number of $E - HC$ is given by

$$cond_2(E - HC) = \sigma_1/\sigma_n.$$

Hence, the the problem reduces to calculating the matrix H to make σ_n as large as possible whilst keeping σ_1 as small as possible. This would satisfy all three conditions: $(E - HC)$ nonsingular, $\|(E - HC)^{-1}\|_2$ small, and $cond(E - HC)$ small. The matrix H is computed using the singular value assignment technique of (Pearson, 1988), (Bunse-Gerstner, Mehrmann and Nichols, 1992).

3 The Base Gas Network Model and Estimation of its Flow Demand Inputs

3.1 The Base α Model

The base gas network model (which we term the α model) is a linear time invariant descriptor system of form (1). The model is given by (Pearson, 1988) and is obtained by linearising the original differential equations about a steady state. All pressure and flow variables are thus perturbations away from that steady state. The linearised differential equations are discretised using the θ method for $1/2 \leq \theta \leq 1$, with $\theta = 1/2$ giving a Crank-Nicolson method with the highest order of accuracy.

The state vector contains the pressure perturbations at the discrete mesh points in the network (except at the far upstream end) and $\underline{u}(k)$ contains the upstream pressure perturbation and the flow demand perturbations at the sites of gas outflow. The base α model for a linear network with g pipes with a single upstream pressure perturbation input and g flow demand perturbation inputs, can be partitioned as

$$\begin{bmatrix} E_1 & E_2 \end{bmatrix} \begin{bmatrix} \underline{p}_1(k+1) \\ \underline{p}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \end{bmatrix} + \begin{bmatrix} B_1^1 & B_2^1 \end{bmatrix} \begin{bmatrix} \underline{p}_3(k+1) \\ \underline{d}(k+1) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1^2 & B_2^2 \end{bmatrix} \begin{bmatrix} \underline{p}_3(k) \\ \underline{d}(k) \end{bmatrix}, \tag{12}$$

where $\underline{p}_2(k)$ is a g dimensional vector containing measured pressure perturbation state variables at the sites of flow demand, $\underline{p}_1(k)$ is a $n-g$ dimensional vector containing the remaining pressure perturbation state variables along the pipes, $\underline{p}_3(k)$ is the upstream pressure input (assumed known), and $\underline{d}(k)$ is a g dimensional vector containing the flow demand perturbation input variables that we wish to estimate.

We have g measurements of the state variables

$$\underline{y}_\alpha(k) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \end{bmatrix} = \underline{p}_2(k) \quad for\ k = 0, 1, 2, ... \tag{13}$$

The model (12), (13), is our general base α model, for which we wish to estimate the g flow demand perturbation inputs, $\underline{d}(k)$. Such a model is useless for estimating the elements of $\underline{d}(k)$, since they are needed as inputs to drive the system.

We have the following two theorems for α models.

Theorem 3.1 *An α model has system eigenvalues within the unit circle, and hence is asymptotically stable, if $(1/2) \leq \theta \leq 1$.*

Theorem 3.2 *If $\theta > 0$, an α model with pressure measurements available at the sites of flow demand is completely observable.*

For proofs of these two theorems, see (Stringer, 1993).

One approach to estimating the flow inputs, $\underline{d}(k)$, is to form a new pressure driven model, which we term a β model, by simply swapping over the g input variables $\underline{d}(k)$ with the g measured state variables $\underline{p}_2(k)$, along with their corresponding matrix coefficients. The system matrices, E and A, of the new β system then have the forms $[E_1\ ,\ -B_2^1]$ and $[A_1\ ,\ B_2^2]$ respectively. If the β model is asymptotically stable, its system state tends to the state of the true dynamical system as $k \longrightarrow \infty$, hence giving an estimate for $\underline{d}(k)$. However, we cannot arbitrarily control the eigenvalues of a β system, and such a system may not be asymptotically stable. Hence we look at a new model variation which is completely observable, around which we can then construct an asymptotically convergent observer design.

3.2 Observable γ Models

New *observable* models, which we term γ models, can be constructed assuming the flow demands are fairly steady, i.e.

$$flow\ demand_{k+1}^{demand\ site} \;=\; flow\ demand_{k}^{demand\ site}.$$

These scalar equations may be written as the system

$$\underline{q} + \underline{d}(k+1) \;=\; \underline{q} + \underline{d}(k),$$

where $\underline{q}$ is a vector containing the steady flow demands, about which the gas network model is linearised. Hence, we derive the following trivial difference equations

$$\underline{d}(k+1) = \underline{d}(k). \tag{14}$$

To form a γ model, we start from a base α model and move the g variables, $\underline{d}(k)$, from the input vector to the state vector. We then introduce g new trivial difference equations of the form (14) into the new system. The new $n+g$ dimensional γ system has the form

$$\begin{bmatrix} E_1 & E_2 & -B_2^1 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k+1) \\ \underline{p}_2(k+1) \\ \underline{d}(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & B_2^2 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \\ \underline{d}(k) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1^1 \\ 0 \end{bmatrix} \underline{p}_3(k+1) + \begin{bmatrix} B_1^2 \\ 0 \end{bmatrix} \underline{p}_3(k). \tag{15}$$

The only input required for the γ model is $\underline{p}_3(k)$ which is assumed known. The g measurements of $\underline{p}_2(k)$ are not needed as inputs to the γ model, and are in fact measurements of its state variables

$$\underline{y}_\gamma(k) = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \\ \underline{d}(k) \end{bmatrix} = \underline{p}_2(k) \quad for\ k = 0, 1, 2, \ldots.. \tag{16}$$

available for use in an observer. We have the following observability theorem, which, in fact, holds for general descriptor systems with the same block structures.

Theorem 3.3 *γ models are completely observable if the original α model is observable and the corresponding β system has no eigenvalue equal to 1.*

For a proof of this theorem, see appendix B.

Since, for our particular models, it can be shown that the gas network base α model is completely observable for $\theta > 0$, and that, from (Stringer, 1993), the corresponding β model has no eigenvalues equal to 1 for $1/2 \leq \theta \leq 1$, the corresponding gas network γ model is completely observable for $1/2 \leq \theta \leq 1$, and we can construct an observer for such a model.

To estimate the flow demands in the gas network, an observer constructed upon a γ model is run assuming all the pressure and flow perturbations are initially zero. The pressure perturbation measurements are fed in at each time level, and the observer state tends to the state of the gas network with time. Perfect asymptotic convergence is not obtained unless the flow demands do not vary with time, since equations (14) contain modelling error. If the flow demands are changing, although not too rapidly, the observer may still track the state of the gas network fairly well. In practice, the observer state estimates are expected to be fairly accurate since the flow demands in gas networks change only slowly throughout the day.

In fact, the profiles of the flow demands may be fairly well known from other measured demands that change thoughout the day with similar patterns of gas consumption. More accurate γ models may be constructed using specific information available about the flow demand profiles with time. At each timestep, k, the vector $\underline{f}(k)$ is estimated where

$$\underline{d}(k+1) = \underline{d}(k) + \underline{f}(k). \tag{17}$$

The γ models now incorporate trivial difference equations of the form (17) for the flow demands, where the vector, $\underline{f}(k)$, is added to the right hand side of the system. Note that only the *shapes* of the flow demand profiles are needed and not their precise values. Adding $\underline{f}(k)$ to the right hand side of the system does not alter the observability of the γ model.

3.3 Experiments and Discussion

A standard α model of a linear three pipe network is run with the upstream pressure perturbation, junction flow demand perturbations, and downstream flow demand perturbation specified as boundary inputs to the system. This α model is identical to the α model upon which all γ model based observers are constructed. Model parameters for all experiments are given in appendix A.

After the α model is run for a short while, the pressures at the upstream end and at the sites of flow demand begin to be recorded at some timestep taken to be $k = 0$. These pressures are then fed into the γ model based observers and the flow demands predicted by these techniques compared with the true flows used as inputs to the base α model. The observer designs are tested with both 'large' and 'small' system eigenvalues assigned; although graphical results are only given for the latter case.

Graphical results are included for the following experiments
Fig.2 Observer Design 1 with $\theta = 1.0$.
Fig.3 Observer Design 2 with $\theta = 1.0$.

In each figure, the true flow demand profile for the first gas demand in the network is shown as a thick line in graph A and the state estimate is shown as a thin line. The percentage error between the state estimate and its true value is shown in graph B. Since observers produce large errors during the first few timesteps, graphical results can be obscured by bad scaling. Hence, the graphs are presented starting only after the estimation techniques have already run a few timesteps.

3.3.1 Behaviour of Observers Based on γ Models Without Weightings, $\underline{f}(k)$

For $\theta = 1$, Design 2 observers were found to be significantly more accurate than Design 1 observers. This can be understood by considering the γ system (15), which assumes trivial difference equations for the demand flows of the form (14). If the demand flows are changing, then there are errors in the modelling. If we include error terms to the right hand side of the trivial difference equations in the γ system (15), we get a system of the form

$$E\underline{x}(k+1) = A\underline{x}(k) + B^1\underline{u}(k+1) + B^2\underline{u}(k) + \underline{l}(k) \tag{18}$$

where the vector $\underline{l}(k)$ contains the error terms.

The dynamic observers are built around the original γ system (15). If we define the error between (18) and the observer (9) at time level k to be $\underline{e}(k) = \underline{x}(k) - \widehat{\underline{x}}(k)$, then subtracting the observer system (9) from (18) gives

$$(E - HC)\underline{e}(k+1) = (A - GC)\underline{e}(k) + \underline{l}(k) \tag{19}$$

where $H = 0$ for Design 1 observers, and where $\underline{l}(k)$ acts as a forcing term on the errors.

For Design 2 observers, the matrix H is chosen to minimise the 2-norm of $(E-HC)^{-1}$, and this matrix is implicitly multiplied into the forcing term, $\underline{l}(k)$, thus reducing its effects. In Figs. 2 and 3, the 2-norm of the matrix E^{-1} is 1.50, while the 2-norm of $(E - HC)^{-1}$ is 0.56 and this is believed to explain the significant improvement in the accuracy of the state estimate when feedback is included at the present time-level.

It is also found that the state estimates contain less error when smaller eigenvalues are assigned to the observer systems (Stringer, 1993). If the eigenvalues of the observer system are small, then so are the eigenvalues of the error system (19), and hence the errors damp down more quickly.

3.3.2 Behaviour of Observers Based on γ Models With Weightings, $\underline{f}(k)$

When the weightings, $\underline{f}(k)$, are included in the γ model, the state estimates can converge perfectly, and assigning smaller system eigenvalues to the observers gives faster convergence. However, in practice, the $\underline{f}(k)$ would probably be slightly innacurate, giving some small error in the observer state estimate.

4 Measurement Bias and δ Models

It may be the case that the pressure measurements at the sites of flow demand are subject to a constant bias, i.e.

$$\underline{y}(k) = \underline{p}_2(k) + \underline{b}(k) \tag{20}$$

where $\underline{b}(k)$ is a g dimensional vector of constant measurement biases, which are assumed to obey

$$\underline{b}(k+1) = \underline{b}(k). \tag{21}$$

The asymptotic estimates of flow demands from models and observers presented so far can be shown to be sensitive to pressure measurement biases (see Fig.4). This is a serious problem for flow demand estimation and the adverse effects of these pressure measurement biases need to be eliminated. However, in (Stringer, 1993), it is shown that when running a β model, or either type of observer based upon a γ model, any time series of pressure inputs and measurements which have been corrupted by any set of constant pressure sensor biases at the sites of flow demand, are perfectly consistent with some time series of β or γ system states with no pressure input or measurement biases, which will be the actual state estimates given (asymptotically, in the case of a dynamic observer). Hence, as long as the biases are not so gross that these new β or γ system states are unphysical, the biases will be undetectable. Hence, we can see that a new model is required in order to eliminate the adverse effects of the biases.

In order to estimate the biases, we construct a new model variation, which we call a δ model. As with the construction of the earlier γ model, we start from a base α model.

As before we first incorporate the input variables, $\underline{d}(k)$, into the state vector, but this time the δ model assumes trivial difference equations for the flow demands of the form

$$flow\ demand_{k+1}^{demand\ site} = w_k^{demand\ site} \times flow\ demand_k^{demand\ site},$$

where the $w_k^{demand\ site}$ are profile coefficients estimated from other measured flow demands. The above scalar equations may be written as the system

$$\underline{q} + \underline{d}(k+1) = W(k)(\underline{q} + \underline{d}(k)),$$

where $W(k)$ is a diagonal block containing time-varying profile coefficients, $w_k^{demand\ site}$. Hence, the normalised and linearised gas network δ models now contain difference equations of the form

$$\underline{d}(k+1) = W(k)\underline{d}(k) + (W(k) - I)\underline{q}.$$

The term on the far right, $(W(k) - I)\underline{q}$, is included on the right hand side of the δ system as shown below.

Next we incorporate the measurement biases into the new state vector, and incorporate the trivial difference equations (21) into the system. The new $n + 2g$ dimensional δ system has the form

$$\begin{bmatrix} E_1 & E_2 & -B_2^1 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k+1) \\ \underline{p}_2(k+1) \\ \underline{d}(k+1) \\ \underline{b}(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & B_2^2 & 0 \\ 0 & 0 & W(k) & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \\ \underline{d}(k) \\ \underline{b}(k) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1^1 \\ 0 \\ 0 \end{bmatrix} \underline{p}_3(k+1) + \begin{bmatrix} B_1^2 \\ 0 \\ 0 \end{bmatrix} \underline{p}_3(k) + \begin{bmatrix} 0 \\ (W(k) - I)\underline{q} \\ 0 \end{bmatrix}. \quad (22)$$

There are g pressure plus bias measurements of the real gas network corresponding to

$$\underline{y}_\delta(k) = \begin{bmatrix} 0 & I & 0 & I \end{bmatrix} \begin{bmatrix} \underline{p}_1(k) \\ \underline{p}_2(k) \\ \underline{d}(k) \\ \underline{b}(k) \end{bmatrix} = \underline{p}_2(k) + \underline{b}(k) \quad for\ k = 0, 1, 2, ... \quad (23)$$

We construct dynamic observers for such a δ system by finding a new feedback matrix, $G(k)$, at each timestep to assign eigenvalues within the unit circle. Since one of the δ system matrices is time-varying, assigning eigenvalues within the unit circle is not necessarily sufficient to cause the observer to converge asymptotically. However, if the dynamics of the real system are quite slow, from (Gelb, 1974), it follows that assigning sufficiently small eigenvalues can give convergence. We have the following theorem giving sufficient conditions for the Hautus condition to hold. This theorem holds for general descriptor systems with the same block structures.

Theorem 4.1 *For a general δ model, at each individual timestep we have the Hautus condition if the base α system is completely observable and has no eigenvalue equal to 1, and the diagonal elements of $W(k)$ are not equal to either* 1 *or to any of the eigenvalues of the corresponding β model.*

For a proof of this theorem, see appendix B.

When the Hautus condition holds, we can find a feedback matrix $G(k)$ to assign observer eigenvalues arbitrarily at that timestep k. Since it can be shown that the base gas network α model is completely observable for $\theta > 0$, and has no eigenvalue equal to one for $1/2 \leq \theta \leq 1$, the Hautus condition may fail to hold for a δ model with $1/2 \leq \theta \leq 1$ only for a few specific values of the coefficients $w_k^{demand\ site}$. At these particular timesteps we can run the simple δ model (22) without the observer feedback terms. As the δ model based observer steps forward in time, it is intended that, the observer state estimate should converge asymptotically to the true state of the gas network.

4.1 Experiments and Discussion

In these experiments, the pressure measurements at the three flow demand sites, A/B, B/C and C, are corrupted by constant biases of 1 bar, -1 bar and 1 bar respectively, before being fed into the γ and δ model based observers. Graphical results are given for the following experiments.
Fig.4 γ Model Based Observer Design 2 with $\theta = 1.0$.
Fig.5 δ Model Based Observer Design 1 with $\theta = 1.0$

4.1.1 Behaviour of γ Model Based Observers

The flow demand estimates of both the Design 1 and Design 2 observers based on γ models are completely swamped by the error due to pressure measurement bias, which is demonstrated by Fig.4.

4.1.2 Behaviour of δ Model Based Observers

For $\theta = 1.0$, the Design 1 observer converges successfully giving accurate flow demand estimates, as shown in Fig.5. However, for any value of θ, it has been found that the δ models are much more sensitive to modelling error and measurement noise (Stringer, 1993).

5 Conclusions

In this paper the problem of estimating the flow demand inputs to a base (α) gas network model is investigated using different rearrangements and augmentations to the original base model.

A model variation, (γ), is investigated which is completely observable, and around which we construct an asymptotically convergent observer design. The new models are augmented with trivial difference equations that assume the flows are steady. Because, in practice, the flow demands vary slowly throughout the day, observers constructed upon such models can give fairly good state estimates. Two techniques, robust eigenstructure assignment and singular value assignment, can significantly improve the accuracy of the observer state estimates.

Lastly, the problem of measurement bias is dealt with by the formulation of a further model variation, (δ), which includes knowledge about the flow demand profiles in a special form such that the pressure measurement biases can now be successfully estimated.

It should be emphasised that all the experiments in this paper have been performed using *simulated* pressure data. In fact, although experimental results have not been presented here, all flow estimation techniques are badly affected in practice by measurement noise due to the underlying sensitivity of the flow estimates. This problem has been further explored in (Stringer, 1993) with further model variations that include a *total* flow variable that is the sum of all the individual flow demands. The flow estimates of such models are much less sensitive to measurement noise. Two smoothing methods presented in (Stringer, 1993), also significantly improve the flow estimates adversely affected by measurement noise.

6 References

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7 Appendix A - Model Details

List of symbols

x distance along pipe in metres
t time in hours
$P(x,t)$ gas pressure in bar
$Q(x,t)$ mass flow rate in m.s.c.m.h.
$\mathcal{Q}$, $\mathcal{P}(x)$ steady-state mass flow rate and pressure profile about which models are linearised
$q(x,t)$, $p(x,t)$ perturbations about the steady mass flow rate and pressure profile
$d_k^{z/z+1}$ gas outflow at junction of pipes z and $z+1$
ϵ_1, ϵ_2, ϵ_3, α constants
D_x, D_t derivatives with respect to x and t respectively
τ constant from the linear expression for compressibility, $Z = 1 - \tau P$
$\Omega(x)$, $\Gamma(x)$ known functions of x
δx, δt computational space and time steps respectively
$r = \delta t/(\delta x)^2$
$\Phi^{z/z+1}$ model parameter at pipe junctions
L, A length and cross-sectional area, respectively, of the pipe
α flow exponent
K friction coefficient
T temperature of the gas
T_s, P_s, Z_s temperature, pressure and compressibility, respectively, at standard conditions
Z_0 compressibility at reference conditions

7.1 Construction of Base α Model

For each pipe section, from (Pearson, 1988), we have the momentum balance equation ignoring time variations in Q

$$D_x(P^2) + \epsilon_1(1 - \tau P)Q|Q|^{\alpha - 1} = 0, \tag{24}$$

and the mass balance equation

$$D_t(P) + \epsilon_2(1 - \tau P)^2 D_x(Q) = 0, \tag{25}$$

where $P(x,t)$ is the gas pressure in bar, $Q(x,t)$ is the mass flow rate in m.s.c.m.h., x is distance along pipe in metres, t is time in hours, $\epsilon_1, \epsilon_2, \tau, \alpha$ are constants, and D_x, D_t are derivatives with respect to x and t respectively. τ is used in the linear expression for compressibility, $Z = 1 - \tau P$, which is always positive. These equations are linearised about a constant mass flow rate, $\mathcal{Q}$, and pressure profile, $\mathcal{P}(x)$, in each pipe to obtain

$$D_t(p) = \Omega D_{xx}(\Gamma p), \tag{26}$$

$$q = -\epsilon_3 D_x(\Gamma p), \tag{27}$$

where q and p are perturbations about the mass flow rate and pressure profile, i.e.

$$P(x,t) = \mathcal{P}(x) + p(x,t), \quad Q(x,t) = \mathcal{Q} + q(x,t), \tag{28}$$

and Ω and Γ are known functions of x

$$\Omega = \epsilon_2 \epsilon_3 (1 - \tau \mathcal{P})^2, \quad \Gamma = \mathcal{P}/(1 - \tau \mathcal{P}),$$

and $\epsilon_3 = 2/(\alpha \epsilon_1 \mathcal{Q}^{\alpha - 1})$.

For each pipe section, equation (26) is approximated by the finite difference equation

$$\begin{aligned} p_{i,k+1} - p_{i,k} &= \theta \Omega_i r (\Gamma_{i-1} p_{i-1,k+1} - 2\Gamma_i p_{i,k+1} + \Gamma_{i+1} p_{i+1,k+1}) \\ &+ (1-\theta) \Omega_i r (\Gamma_{i-1} p_{i-1,k} - 2\Gamma_i p_{i,k} + \Gamma_{i+1} p_{i+1,k}) \end{aligned} \tag{29}$$

for $0 \leq \theta \leq 1$, where $r = \delta t / (\delta x)^2$. For a downstream *flow* boundary condition given at node $i = s$ we approximate equation (27) by

$$- \epsilon_3 (\Gamma_{s+1} p_{s+1,k} - \Gamma_{s-1} p_{s-1,k}) = 2 \delta x q_{s,k}. \tag{30}$$

Eliminating $\Gamma_{s+1} p_{s+1,k}$ and $\Gamma_{s+1} p_{s+1,k+1}$ between equations (29) and (30) then gives at node s the difference equation

$$\begin{aligned} &-2\theta \Omega_s r \Gamma_{s-1} p_{s-1,k+1} + (1 + 2\theta \Omega_s r \Gamma_s) p_{s,k+1} + (2\theta \Omega_s r \delta x / \epsilon_3) q_{s,k+1} = \\ &2(1-\theta) \Omega_s r \Gamma_{s-1} p_{s-1,k} + (1 - 2(1-\theta) \Omega_s r \Gamma_s) p_{s,k} - (2(1-\theta) \Omega_s r \delta x / \epsilon_3) q_{s,k}. \end{aligned} \tag{31}$$

An equivalent difference equation may be derived for an upstream flow boundary condition.

For an outflow, $d_k^{z/z+1}$, at the junction of two pipes, denoted by superscripts z and $z+1$ respectively, the pipe end flow boundary equations may be used to derive

$$\begin{aligned} &-(\Phi^{z/z+1} \epsilon_3^z \Gamma_{s^z-1}^z \theta / \delta x^z) p_{s^z-1,k+1}^z + (1 + \Phi^{z/z+1} \epsilon_3^z \Gamma^{z/z+1} \theta / \delta x^z + \Phi^{z/z+1} \epsilon_3^{z+1} \Gamma^{z/z+1} \theta / \delta x^{z+1}) p_{k+1}^{z/z+1} \\ &\qquad - (\Phi^{z/z+1} \epsilon_3^{z+1} \Gamma_1^{z+1} \theta / \delta x^{z+1}) p_{1,k+1}^{z+1} + \Phi^{z/z+1} \theta d_{k+1}^{z/z+1} = \\ &(\Phi^{z/z+1} \epsilon_3^z \Gamma_{s^z-1}^z (1-\theta) / \delta x^z) p_{s^z-1,k}^z + (1 - \Phi^{z/z+1} \epsilon_3^z \Gamma^{z/z+1} (1-\theta) / \delta x^z - \Phi^{z/z+1} \epsilon_3^{z+1} \Gamma^{z/z+1} (1-\theta) / \delta x^{z+1}) p_k^{z/z+1} \\ &\qquad + (\Phi^{z/z+1} \epsilon_3^{z+1} \Gamma_1^{z+1} (1-\theta) / \delta x^{z+1}) p_{1,k}^{z+1} - \Phi^{z/z+1} (1-\theta) d_k^{z/z+1} \end{aligned} \tag{32}$$

where we have defined

$$\Phi^{z/z+1} = \left(\frac{\epsilon_3^z}{2 r^z \delta x^z \Omega_{s^z}^z} + \frac{\epsilon_3^{z+1}}{2 r^{z+1} \delta x^{z+1} \Omega_0^{z+1}} \right)^{-1}$$

and where, because $p_{s^z,k}^z \equiv p_{0,k}^{z+1}$ and $\Gamma_{s^z}^z \equiv \Gamma_0^{z+1}$, at the junction we have denoted the pressure perturbation by $p_k^{z/z+1} \equiv p_{s^z,k}^z \equiv p_{0,k}^{z+1}$, and denoted Γ by $\Gamma^{z/z+1} \equiv \Gamma_{s^z}^z \equiv \Gamma_0^{z+1}$. This equation is the 'connectivity equation', which is used at the junction nodes.

7.2 Model Parameters for Experiments

The values of the model parameters for all pipes used in our experiments are given by

$$\epsilon_1 = \frac{KT}{T_s Z_0}, \quad \epsilon_2 = \frac{10^6 P_s T}{A T_s Z_s},$$

where $L = 7000m$ is the length of the pipe, $A = 1.0m^2$ is the cross-sectional area of the pipe, $\alpha = 1.854$ is the flow exponent, $K = 0.105$ is the friction coefficient, $T = 300^0 K$ is the temperature of the gas, $T_s = 288.7056^0 K$ is the temperature at standard conditions, $P_s = 1.0138 Bar$ is the pressure at standard conditions, $Z_s = 0.9978$ is the compressibility at standard conditions, $Z_0 = 1.0$ is the compressibility at reference conditions, and $\tau = 0.00205946$. In all numerical experiments, the timestep used for the models was 0.1666665 hours (= 10 *minutes*), and each pipe had ten computational nodes. The state variables of the flow estimation models and dynamic observers were initially set to zero.

8 Appendix B - Proofs of Theorems

8.1 Proof of Theorem 3.3

¿From the Hautus condition we have that the γ system is observable if and only if for all $\mu \in \mathbf{C}$

$$[(A_1 - \mu E_1) \quad (A_2 - \mu E_2) \quad (B_2^2 - \mu(-B_2^1))] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^1 \\ \underline{v}_g^2 \end{bmatrix} = \underline{0} \tag{33}$$

$$(1-\mu)\underline{v}_g^2 = \underline{0} \tag{34}$$

$$\underline{v}_g^1 = \underline{0} \tag{35}$$

$$\Longleftrightarrow$$

$$\underline{v}_{n-g} = \underline{0}\ ,\ \underline{v}_g^1 = \underline{0}\ ,\ \underline{v}_g^2 = \underline{0} \tag{36}$$

where $\underline{v}_{n-g} \in \mathbf{R}^{n-g}$ and $\underline{v}_g^i \in \mathbf{R}^g$ for i=1,2.

Consider the case of $\mu \neq 1$. Equation (34) implies $\underline{v}_g^2 = \underline{0}$. Substituting $\underline{v}_g^2 = \underline{0}$ into equation (33) gives

$$[(A_1 - \mu E_1) \quad (A_2 - \mu E_2)] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^1 \end{bmatrix} = \underline{0}. \tag{37}$$

If the α system is completely observable, then the Hautus condition holds for the α system, and equations (35) and (37) imply $\underline{v}_{n-g} = \underline{0}$ and $\underline{v}_g^1 = \underline{0}$.

Consider the case of $\mu = 1$. Removing $\underline{v}_g^1$ from system (33) gives the system

$$[(A_1 - E_1) \quad (B_2^2 - (-B_2^1))] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^2 \end{bmatrix} = \underline{0}. \tag{38}$$

By inspection, if 1 is not an eigenvalue of the corresponding β system, equation (38) implies $\underline{v}_{n-g} = \underline{0}$ and $\underline{v}_g^2 = \underline{0}$. □

8.2 Proof of Theorem 4.1

The Hautus condition holds for the δ system if and only if for all $\mu \in \mathbf{C}$

$$[(A_1 - \mu E_1) \quad (A_2 - \mu E_2) \quad (B_2^2 - \mu(-B_2^1))] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^1 \\ \underline{v}_g^2 \end{bmatrix} = \underline{0} \tag{39}$$

$$(W(k) - \mu I)\underline{v}_g^2 = \underline{0} \tag{40}$$

$$(1-\mu)\underline{v}_g^3 = \underline{0} \tag{41}$$

$$\underline{v}_g^1 + \underline{v}_g^3 = \underline{0} \tag{42}$$

$$\Longleftrightarrow$$

$$\underline{v}_{n-g} = \underline{0}\ ,\ \underline{v}_g^1 = \underline{0}\ ,\ \underline{v}_g^2 = \underline{0}\ ,\ \underline{v}_g^3 = \underline{0} \tag{43}$$

where $\underline{v}_{n-g} \in \mathbf{R}^{n-g}$ and $\underline{v}_g^i \in \mathbf{R}^g$ for i=1,2,3.

We firstly consider the case where $\mu \neq 1$.

Equation (41) implies $\underline{v}_g^3 = \underline{0}$. Substituting $\underline{v}_g^3 = \underline{0}$ into equation (42) gives $\underline{v}_g^1 = \underline{0}$.

If $\mu \neq w_k^{demand\ site}$ for all flow demands, equation (40) gives $\underline{v}_g^2 = \underline{0}$. Substituting $\underline{v}_g^2 = \underline{0}$ into equation (39) gives

$$[(A_1 - \mu E_1) \quad (A_2 - \mu E_2)] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^1 \end{bmatrix} = \underline{0}. \tag{44}$$

If the α system is completely observable, then the Hautus condition holds for the α system, and $\underline{v}_g^1 = \underline{0}$ together with equation (44) imply $\underline{v}_{n-g} = \underline{0}$.

If $\mu = w_k^{demand\ site}$ for any flow demands, removing $\underline{v}_g^1$ from system (39) gives

$$[(A_1 - \mu E_1) \quad (B_2^2 - \mu(-B_2^1))] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^2 \end{bmatrix} = \underline{0}. \tag{45}$$

Then, by inspection, if $\mu = w_k^{demand\ site}$ is not equal to an eigenvalue of the corresponding β system, equation (45) implies $\underline{v}_{n-g} = \underline{0}$ and $\underline{v}_g^2 = \underline{0}$.

We secondly consider the case where $\mu = 1$.

If none of the $w_k^{demand\ site}$ are equal to 1, equation (40) gives $\underline{v}_g^2 = \underline{0}$. Substituting $\underline{v}_g^2 = \underline{0}$ into equation (39) gives

$$[(A_1 - E_1) \quad (A_2 - E_2)] \begin{bmatrix} \underline{v}_{n-g} \\ \underline{v}_g^1 \end{bmatrix} = \underline{0}.$$

If all the eigenvalues of the α model are within the unit circle, then the above equation gives $\underline{v}_{n-g} = 0$ and $\underline{v}_g^1 = 0$. Substituting $\underline{v}_g^1 = \underline{0}$ into equation (42) gives $\underline{v}_g^3 = \underline{0}$. □

List of Captions for Figures

Fig.1 Example three pipe linear gas network

Fig.2 γ Model Based Observer Design 1 with $\theta = 1$

Fig.3 γ Model Based Observer Design 2 with $\theta = 1$

Fig.4 γ Model Based Observer Design 2 with $\theta = 1$ (Pressure measurements corrupted by bias)

Fig.5 δ Model Based Observer Design 1 with $\theta = 1$ (Pressure measurements corrupted by bias)