Differential Equations

DIRECT INTEGRATION

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A Tutorial Module introducing ordinary differential equations and the method of direct integration

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1. Introduction

$$\boxed{\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = x^7}$$ is an example of an **ordinary** differential equation (o.d.e.) since it contains only **ordinary** derivatives such as $\frac{dy}{dx}$ and not **partial** derivatives such as $\frac{\partial y}{\partial x}$.

The dependent variable is y while the independent variable is x (an o.d.e. has only one independent variable while a partial differential equation has more than one independent variable).

The above example is a **second order** equation since the highest order of derivative involved is **two** (note the presence of the $\frac{d^2y}{dx^2}$ term).

An o.d.e. is **linear** when each term has y and its derivatives only appearing to the power one. The appearance of a term involving the product of y and $\frac{dy}{dx}$ would also make an o.d.e. **nonlinear**.

In the above example, the term $\left(\frac{dy}{dx}\right)^3$ makes the equation **nonlinear**.

The **general solution** of an n^{th} order o.d.e. has $\boldsymbol{n}$ arbitrary constants that can take any values.

In an **initial value problem**, one solves an $\boldsymbol{n^{th}}$ order o.d.e. to find the general solution and then applies $\boldsymbol{n}$ **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

2. Theory

An ordinary differential equation of the following form:

$$\frac{dy}{dx} = f(x)$$

can be solved by integrating both sides with respect to x:

$$y = \int f(x)\, dx \, .$$

This technique, called **DIRECT INTEGRATION**, can also be applied when the left hand side is a higher order derivative.

In this case, one integrates the equation a sufficient number of times until y is found.

3. Exercises

Click on EXERCISE links for full worked solutions (there are 8 exercises in total)

EXERCISE 1.

Show that $y = 2e^{2x}$ is a particular solution of the ordinary differential equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

EXERCISE 2.

Show that $y = 7\cos 3x - 2\sin 2x$ is a particular solution of $\frac{d^2y}{dx^2} + 2y = -49\cos 3x + 4\sin 2x$

Exercise 3.

Show that $y = A\sin x + B\cos x$, where A and B are arbitrary constants, is the general solution of $\dfrac{d^2y}{dx^2} + y = 0$

Exercise 4.

Derive the general solution of $\dfrac{dy}{dx} = 2x + 3$

Exercise 5.

Derive the general solution of $\dfrac{d^2y}{dx^2} = -\sin x$

Exercise 6.

Derive the general solution of $\dfrac{d^2y}{dt^2} = a$, where $a =$ constant

Exercise 7.

Derive the general solution of $\frac{d^3y}{dx^3} = 3x^2$

Exercise 8.

Derive the general solution of $e^{-x}\frac{d^2y}{dx^2} = 3$

4. Answers

1. HINT: Work out $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and then substitute your results, along with the given form of y, into the differential equation ,
2. HINT: Show that $\frac{d^2y}{dx^2} = -63\cos 3x + 8\sin 2x$ and substitute this, along with the given form of y, into the differential equation ,
3. HINT: Show that $\frac{d^2y}{dx^2} = -A\sin x - B\cos x$,
4. $y = x^2 + 3x + C$,
5. $y = \sin x + Ax + B$,
6. $y = \frac{1}{2}at^2 + Ct + D$,
7. $y = \frac{1}{20}x^5 + C'x^2 + Dx + E$,
8. $y = 3e^x + Cx + D$.

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{g'(x)}{g(x)}$	$\ln \lvert g(x) \rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \lvert \cos x \rvert$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left\lvert \tan \frac{x}{2} \right\rvert$	$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \lvert \sin x \rvert$	$\coth x$	$\ln \lvert \sinh x \rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

6. Tips on using solutions

● When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

● Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

We need:

$$\begin{aligned}
\frac{dy}{dx} &= 2 \cdot 2e^{2x} = 4e^{2x} \\
\frac{d^2y}{dx^2} &= 2 \cdot 4e^{2x} = 8e^{2x}. \\
\therefore \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= 8e^{2x} - 4e^{2x} - 2 \cdot e^{2x} \\
&= (8-8)e^{2x} \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

Return to Exercise 1

Exercise 2.

To show that $y = 7\cos 3x - 2\sin 2x$ is a particular solution of $\frac{d^2y}{dx^2} + 2y = -49\cos 3x + 4\sin 2x$, work out the following:

$$\begin{aligned}\frac{dy}{dx} &= -21\sin 3x - 4\cos 2x \\ \frac{d^2y}{dx^2} &= -63\cos 3x + 8\sin 2x\end{aligned}$$

$$\begin{aligned}\therefore \quad \tfrac{d^2y}{dx^2} + 2y &= -63\cos 3x + 8\sin 2x + 2(7\cos 3x - 2\sin 2x) \\ &= (-63+14)\cos 3x + (8-4)\sin 2x \\ &= -49\cos 3x + 4\sin 2x \\ &= \text{RHS}\end{aligned}$$

<u>Notes</u>
- The equation is <u>second</u> order, so the general solution would have two arbitrary (undetermined) constants.
- Notice how similar the particular solution is to the Right-Hand-Side of the equation. It involves the same functions but they have different coefficients i.e.

y is of the form

$$\text{“}a\cos 3x + b\sin 2x\text{”} \qquad \left(\begin{array}{c} a = 7 \\ b = -2 \end{array}\right)$$

Return to Exercise 2

Exercise 3.

$$
\begin{aligned}
\text{We need:} \qquad \frac{dy}{dx} &= A\cos x + B\cdot(-\sin x) \\
\frac{d^2y}{dx^2} &= -A\sin x - B\cos x \\
\therefore \quad \frac{d^2y}{dx^2} + y &= (-A\sin x - B\cos x) + (A\sin x + B\cos x) \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

Since the differential equation is <u>second</u> order and the solution has <u>two</u> arbitrary constants, this solution is the general solution.

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Exercise 4.

This is an equation of the form $\frac{dy}{dx} = f(x)$, and it can be solved by direct integration.

Integrate both sides with respect to x:

$$\begin{aligned} \int \frac{dy}{dx} dx &= \int (2x+3)dx \\ \text{i.e.} \quad \int dy &= \int (2x+3)dx \\ \text{i.e.} \quad y &= 2 \cdot \frac{1}{2}x^2 + 3x + C \\ \text{i.e.} \quad y &= x^2 + 3x + C, \end{aligned}$$

where C is the (combined) arbitrary constant that results from integrating both sides of the equation. The general solution must have one arbitrary constant since the differential equation is first order.

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Exercise 5.

This is of the form $\boxed{\frac{d^2y}{dx^2} = f(x)}$, so we can solve for y by direct integration.

Integrate both sides with respect to x:

$$\begin{aligned} \frac{dy}{dx} &= -\int \sin x dx \\ &= -(-\cos x) + A \end{aligned}$$

Integrate again:

$$y = \sin x + Ax + B$$

where A, B are the two arbitrary constants of the general solution (the equation is second order).

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Exercise 6.

Integrate both sides with respect to t:

$$\frac{dy}{dt} = \int a\,dt$$

$$\text{i.e. } \frac{dy}{dt} = at + C$$

Integrate again:

$$y = \int (at + C)dt$$

$$\text{i.e. } y = \frac{1}{2}at^2 + Ct + D\,,$$

where C and D are the two arbitrary constants required for the general solution of the second order differential equation.

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Exercise 7.

Integrate both sides with respect to x:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \int 3x^2 dx \\ \text{i.e. } \frac{d^2y}{dx^2} &= 3 \cdot \frac{1}{3}x^3 + C \\ \text{i.e. } \frac{d^2y}{dx^2} &= x^3 + C \end{aligned}$$

Integrate again:

$$\begin{aligned} \frac{dy}{dx} &= \int (x^3 + C)dx \\ \text{i.e. } \qquad \frac{dy}{dx} &= \frac{x^4}{4} + Cx + D \end{aligned}$$

Integrate again:

$$y = \int \left(\frac{x^4}{4} + Cx + D \right) dx$$

i.e. $$y = \frac{1}{4} \cdot \frac{1}{5} x^5 + C \cdot \frac{1}{2} x^2 + Dx + E$$

i.e. $$y = \frac{1}{20} x^5 + C' x^2 + Dx + E$$

where $C' = \frac{C}{2}$, D and E are the required three arbitrary constants of the general solution of the third order differential equation.

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Exercise 8.

Multiplying both sides of the equation by e^x gives:

$$e^x \cdot e^{-x} \frac{d^2y}{dx^2} = e^x \cdot 3$$

$$\text{i.e.} \quad \frac{d^2y}{dx^2} = 3e^x$$

This is now of the form $\frac{d^2y}{dx^2} = f(x)$, where $f(x) = 3e^x$, and the solution y can be found by direct integration.

Integrating both sides with respect to x:

$$\frac{dy}{dx} = \int 3e^x dx$$

$$\text{i.e.} \quad \frac{dy}{dx} = 3e^x + C\,.$$

Integrate again:

$$y = \int (3e^x + C)dx$$

$$\text{i.e.} \quad y = 3e^x + Cx + D\,,$$

where C and D are the two arbitrary constants of the general solution of the original second order differential equation.

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