Module P2.1 Introducing motion

1 Opening items

1.1 Module introduction

Many of the most interesting problems in physics involve motion; the movement of planets and satellites for example, or the motion of high speed electrons as they are whirled around in a particle accelerator. This module provides an introduction to the study of motion. Its main aim is to enable you to describe and analyse simple examples of motion in an exact and concise way.

The module starts with a brief discussion of three-dimensional space and the way in which a *Cartesian coordinate system* can be used to fix the location of an object in space. During this discussion *position*, *displacement* and *distance* are defined and the distinction between a *vector* and a *scalar* is explained. Having established the three-dimensional nature of space and the difficulties of working in three dimensions, Subsection 2.3 introduces the simplifying concept of *linear motion*, i.e. motion along a straight line. By concentrating in the rest of the module on examples of linear motion, such as objects falling under gravity or cars accelerating along straight roads, we are able to explore some of the fundamental features of motion in a one-dimensional context, without having to make use of vectors. In particular, we show how *position–time graphs* can be used to describe linear motion. By considering such graphs we are led, in Section 3, to important concepts such as *uniform motion*, *average velocity* and *instantaneous velocity*.

Section 4 continues this graphical approach by using *velocity–time graphs* to describe examples of linear motion in which the velocity is changing with time and introduces the concept of *acceleration*, giving specific definitions of *average acceleration* and *instantaneous acceleration*.

Section 5 shows how the linear motion of a *particle* may be described algebraically, so that information can be obtained from the manipulation of equations rather than by drawing graphs. The equations involved in this process are generally called *equations of motion*, and special attention is paid to the *uniform motion equations* (when the particle has a constant velocity) and the *uniform acceleration equations* (when the particle has constant acceleration). By starting in three dimensions and then particularizing to one, we lay a firm foundation for later studies and avoid many of the pitfalls that open-up when one-dimensional motion is treated in isolation without due regard to the real world of three-dimensional space.

The mathematical techniques of *calculus* are not used in this module. However, the *notation* of the calculus *is* introduced so that you will have an opportunity to become familiar with its physical significance before being called upon to use it in a mathematical context. The following problem illustrates the sort of question you will be able to answer by the end of this module. (The solution is given in Subsection 5.3.)

> According to the Highway Code, a car travelling along a straight road at 30 mph (i.e. about $13.3\,\mathrm{m\,s^{-1}}$, read as 13.3 metres per second) can stop within 23 metres of the point at which the driver sees a hazard. This is known as the stopping distance. If the driver always takes 0.70 s to react to a hazard and apply the brakes, what is the stopping distance at 70 mph (i.e. about $31.1\,\mathrm{m\,s^{-1}}$) assuming the same deceleration as at 30 mph?

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*?. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 6.1) and the *Achievements* listed in Subsection 6.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 6.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. However, *if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module*.

Question F1

A car accelerates uniformly along a straight road, so that its speed increases from $15\,\mathrm{m\,s^{-1}}$ to $25\,\mathrm{m\,s^{-1}}$ in $9.0\,\mathrm{s}$. Calculate its acceleration.

Question F2

The graph in Figure 1 represents the motion of a parachutist.

(a) For how long did the parachutist fall before opening the parachute?

(b) What was the acceleration during this time?

(c) How far did the parachutist fall before opening the parachute?

(d) What was the approximate acceleration one second after opening the parachute?

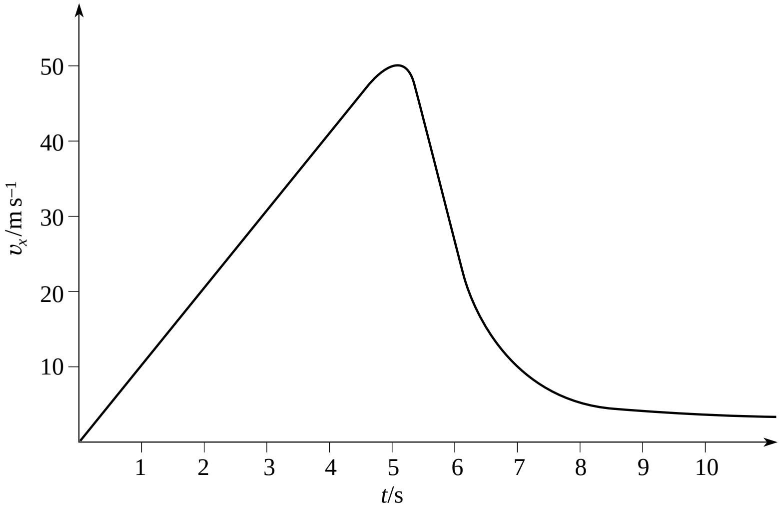

Figure 1 See Question F2.

Question F3

What is the difference between a vector and a scalar? What is meant by the magnitude of a vector, and what is wrong with the statement $|\boldsymbol{a}| = -3$?

Study comment

Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items*.

1.3 Ready to study?

Study comment To begin to study this module you will need to be familiar with the following terms: *axes*, *Cartesian coordinates*, *gradient of a line*, *graph*, *origin*, *SI units* and *tangent*. If you are unsure of any of these terms you should refer to the *Glossary*, which will also indicate where in *FLAP* they are developed. The following questions will establish whether you need to review some of these topics before beginning to work through this module.

Question R1

Write down the (Cartesian) coordinates of the points A and B on Figure 2 and mark the origin with an O.

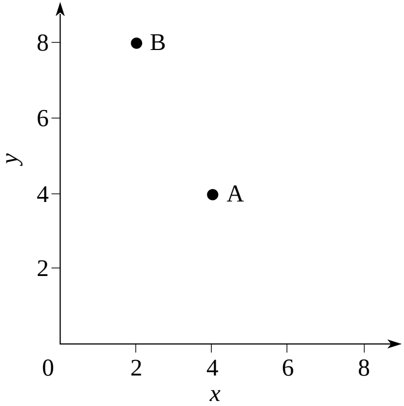

Figure 2 See Question R1.

Question R2

What are the gradients (including the appropriate units) of the two lines in Figure 3?

Question R3

What are the SI units of mass, length and time?

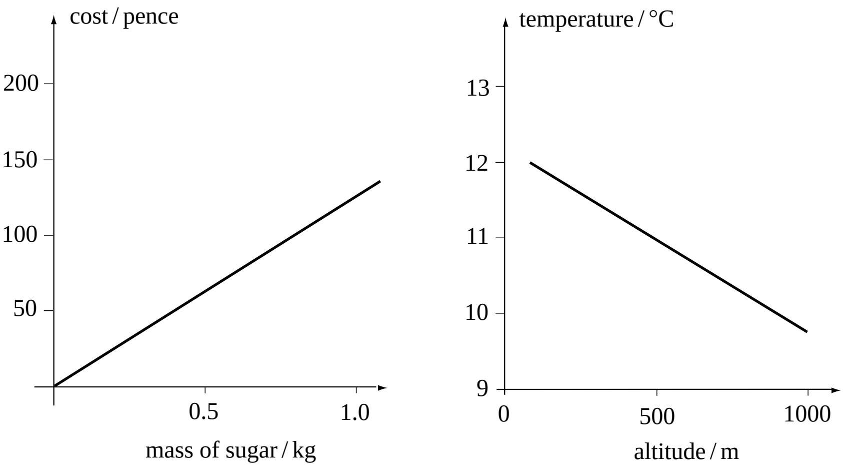

Figure 3 See Question R2.

2 Position, displacement and motion

2.1 Positions and position vectors

It is a basic fact of life that we can only move in three independent directions: to the left and right, up and down, back and forward. Any other movement can always be produced by a suitable combination of these three. We describe this state of affairs by saying that space — which consists of all the possible positions that an object might have — is **three dimensional**. It follows from this that if you want to describe the position of an object fully you must specify its location with respect to *three* independent directions. The most common way of doing this uses a three-dimensional *Cartesian coordinate system*, like that shown in Figure 4.

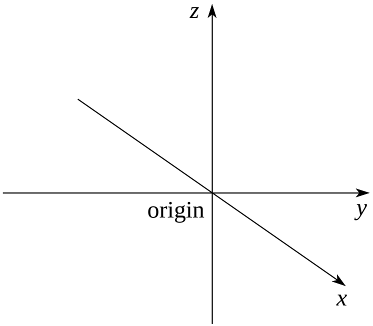

Figure 4 A three-dimensional Cartesian coordinate system. (It should be noted that the z-axis of a three-dimensional coordinate system may be oriented in one of two ways (up or down) relative to the x and y-*axes*. The orientation shown in this figure is the most conventional and constitutes what is known as a *right-handed coordinate system*. Reversing the direction of the z-axis would produce an unconventional *left-handed coordinate system*.)

Such a system is obtained quite simply by adding a third axis (the *z-axis*) at right angles to the x and *y-axes* that make up the two-dimensional Cartesian coordinate systems with which you are already familiar. As in the two-dimensional case, the point at which the axes intersect continues to be called the *origin*.

Both the point in space at which the origin of a particular coordinate system is located, and the orientation of that system (i.e. which way the mutually perpendicular axes point) can be chosen arbitrarily. However, once such choices have been made it is important to stick to them so that the location of any point in space can be specified by its three **position coordinates** with respect to that coordinate system. The way in which the position coordinates of a point are determined is probably familiar to you — the process is illustrated in Figure 5 for the sake of completeness.

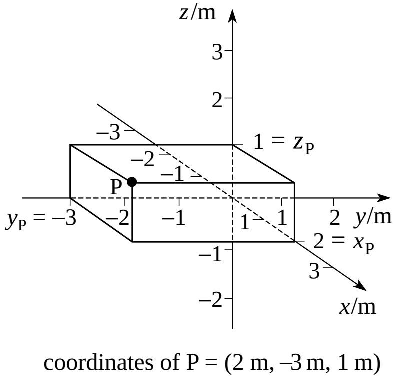

Figure 5 Determining the position coordinates of a point P.

The position coordinates of a point are usually written in the order x, y, z and enclosed in brackets (parentheses) to avoid confusing them. Thus the coordinates of point P in Figure 5 would normally be written

$$(x_P, y_P, z_P) = (2\,\text{m}, -3\,\text{m}, 1\,\text{m})$$

Note that each position coordinate may be positive or negative according to which side of the origin the relevant perpendicular intersects the corresponding axis.

An alternative way of describing the three-dimensional position of a point such as P is in terms of its **position vector**. This can be thought of as an arrow ☞ stretching from the origin of the coordinate system to the point in question (Figure 6). Now arrows have length *and* direction, so they are quite different from physical quantities such as mass and energy that have no particular direction associated with them. For this reason it makes good sense to use a special symbol to distinguish a position vector (or any other directed quantity) from quantities that have no direction. It is conventional to emphasize the special nature of directed quantities — which are generally called **vectors** — by using bold letters to represent them . Thus the position vector of point P might be denoted $\boldsymbol{r}_{\rm P}$.

$$\boldsymbol{r}_{\rm P} = \text{position vector of point P}$$

Quantities such as mass and energy that have no direction are called **scalars** and are usually represented by ordinary (non-bold) letters such as m and E.

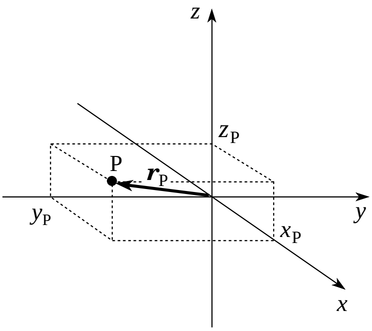

Figure 6 The position vector $r_{\rm P}$ of a point P.

Since a position vector is an alternative way of specifying the location of a point with respect to a given coordinate system there must be an intimate relationship between the position vector of a point and the position coordinates of that point. In fact, a position vector $\boldsymbol{r}$ of a point with position coordinates (x, y, z) is often specified by an equation of the following form

$$\boldsymbol{r} = (x, y, z) \tag{1}$$

In this context x, y and z, the position coordinates of the point in question, are referred to as the x, y and z **components** of the position vector $\boldsymbol{r}$. Note that each of the components is a scalar quantity which may be positive or negative. The vector $\boldsymbol{r}$ may therefore be represented by an ordered arrangement of three scalar quantities, its three components.

Aside Distinguishing vectors by a bold typeface is fine in print, but it presents a problem for those using pens or pencils. How are *you* going to show that $\boldsymbol{r}$ is a vector when you write it down? Fortunately there is a simple way of indicating vectors in handwritten work. When authors are preparing material to be printed they show that an item should be set in boldface type by putting a wavy underline beneath it. What is written as $\underset{\sim}{r}$ will be read and printed as $\boldsymbol{r}$. Unless you have been instructed to distinguish vectors from scalars in some other way, you should adopt this convention in all your written work.

2.2 Displacement and distance

It should be clear from Figure 6 that the **distance** from the origin of the coordinate system to the point P is nothing other than the *length* of the position vector $\boldsymbol{r}_{\rm P}$. When dealing with vectors of any kind, including position vectors, it is customary to use the term **magnitude** when referring to their length, so we can say, quite generally:

> The *distance* from the origin of a coordinate system to any particular point is given by the *magnitude* of the position vector of that point.

It is worth noting that the magnitude of a vector cannot be negative. (A length may be positive or zero, but it can't possibly be less than zero!) Because of this it is usual to denote the magnitude of any position vector $\boldsymbol{r}$ by $|\boldsymbol{r}|$, since mathematicians use a similar notation to indicate the non-negative value they call the *modulus* or *absolute value* of any ordinary number. Of course, $|\boldsymbol{r}|$ is just a scalar quantity so it is also common to see it represented by the usual scalar symbol r. Using these notations for the magnitude of a vector we can say

> $r = |\boldsymbol{r}|$, the magnitude of the position vector $\boldsymbol{r}$ of a point, represents the distance from the origin of a coordinate system to that point. ☞

Positions are always measured with respect to the origin of a coordinate system, but quite often we are more interested in the location of one point relative to another rather than in the location of either relative to the origin. For instance, if an object moves from a point P to a point Q so that its position vector changes from $\boldsymbol{r}_{\rm P}$ to $\boldsymbol{r}_{\rm Q}$ (see Figure 7), we may be more interested in knowing the distance and direction from P to Q than in specifying the position of either P or Q relative to the origin. The quantity that describes the difference in the two positions is called the **displacement** from P to Q and may be denoted $\boldsymbol{s}_{\rm PQ}$. As its bold symbol indicates, the displacement $\boldsymbol{s}_{\rm PQ}$ is another vector quantity since both a magnitude *and* a direction are required for its specification. This information is often provided by expressing the displacement in terms of its scalar components, as in the case of a position vector. Thus you will often see a general displacement $\boldsymbol{s}$ defined by an equation of the form:

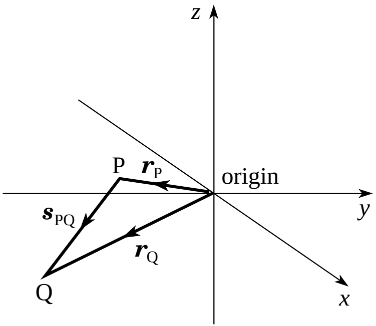

Figure 7 The displacement $\boldsymbol{s}_{\rm PQ}$ from P to Q.

$$\boldsymbol{s} = (s_x, s_y, s_z)$$

where s_x, s_y and s_z are, respectively, called the x, y and z-components of the displacement.

We shall have more to say about the determination of these components later but for the present we shall just note two important properties of displacement:

- Position vectors are a special class of displacements — they are displacements from the origin to the points in question.
- The magnitude of the displacement from one point to another represents the distance between those two points. Thus, in general:

> $s = |\boldsymbol{s}|$, the magnitude of the displacement from one point to another, is the distance between those two points.

✧ If $\boldsymbol{s}$ is the displacement from one point to another, what is wrong with the statement $|\boldsymbol{s}| = -3\,\text{m}$?

2.3 Linear motion — from three dimensions to one

So far our discussion has been fully three-dimensional, just as much of physics has to be. However, in the rest of this module we shall be mainly concerned with **linear motion** — i.e. the motion of an object along a straight line. Although the line may point along any direction in three-dimensional space, the motion itself is really one-dimensional. If we choose to locate the origin of a coordinate system on the line and to orientate one of the axes, the x-axis say, along the line, then all the possible positions of the moving object can be specified by values of the single position coordinate x — and that's what characterizes one-dimensional motion. By confining our attention to one dimension we will be able to explore some of the basic features of motion without having to make much use of vector notation. On the other hand, because we have started from a fully three-dimensional (vector) point of view the results we obtain will be easy to generalize to three dimensions.

As a definite example, consider Figure 8 which shows a car moving along a straight line away from a pedestrian. The line has been designated as the x-axis of some coordinate system and the origin located at a fixed point between the car and the pedestrian. The position of the car at any instant is completely specified by a single position coordinate x, which may be positive or negative according to whether the car is to the right or the left of the origin. Similarly, the displacement from one point on the line (the position of the pedestrian, x_1, say) to any other point on the line (such as the position of the car, x_2) can be specified by a single positive or negative quantity — the x-component of displacement s_x given by

$$s_x = x_2 - x_1 \qquad (2)$$

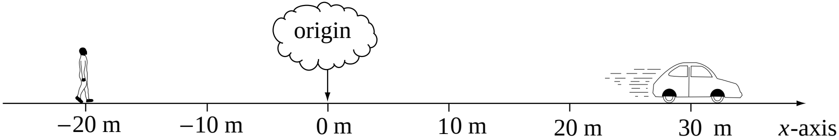

Figure 8 An example of one-dimensional (linear) motion.

In the rest of this module, when dealing with one-dimensional motion along the x-axis, we shall usually refer to the position and displacement components x and s_x simply as ‘the position’ and ‘the displacement’, respectively, since each effectively specifies the corresponding vector quantity even though each is, in reality, only one component of that vector. Of course it’s still the case that the magnitude of a displacement is a distance but even this is simplified in a one-dimensional problem since all you have to do to work out the distance from one point to another is to subtract the lesser value of x from the greater so that you get a positive result.

✦ In Figure 8, what is the displacement from the car to the pedestrian? What is the distance from the car to the pedestrian?

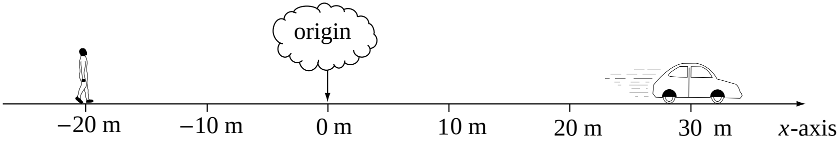

Figure 8 An example of one-dimensional (linear) motion.

2.4 Position–time and displacement–time graphs

To describe the motion of the car in Figure 8, we can determine its position at various times and then display the results in a suitable way. For example, we could choose the origin of time — the moment at which we start the clock and record time $t = 0$ — to be when the car passes the origin. We might then choose to measure the car's position (i.e. its displacement from the origin) at 5-second intervals for 1 minute. The results are shown in Table 1.

Note how the units are shown in such tables. The oblique slash (/) denotes a ratio, thus

$$x/\text{m} = \frac{\text{position coordinate in metres}}{1\ \text{metre}} \tag{3}$$

so the units cancel and the entries in the table are just numbers.

These data could also be displayed visually by plotting them as points on a *graph*. Of course, there are conventions to bear in mind when drawing such a graph:

Table 1 The position coordinate of the car at various times.

Time t/s	Position coordinate x/m
0	0
5	1.7
10	6.8
15	15
20	26
25	39
30	53
35	68
40	84
45	99
50	115
55	131
60	146

- Time, the *independent variable*, should be plotted along the horizontal axis and position, the *dependent variable*, along the vertical axis. ☞
- The axes should be labelled to show what is being plotted and the label should include an oblique slash (/) followed by the appropriate units, as in the table headings.
- *SI units* should be used unless there are good reasons to do otherwise.

Question T1

Plot the values in Table 1 as points on a graph and draw a smooth curve through the points ☞. Label the axes as described above. ❑

The graph you have just drawn in response to Question T1 is known as a **position–time graph**. Using the graph, you can read off the position of the car at any given time, or the time at which the car reaches a given position.

Question T2

Use your graph (from Question T1) to estimate the position of the car after 32 s and the time it takes to travel the first 100 m. ❑

Table 1 The position coordinate of the car at various times.

Time t/s	Position coordinate x/m
0	0
5	1.7
10	6.8
15	15
20	26
25	39
30	53
35	68
40	84
45	99
50	115
55	131
60	146

The position–time graph of any linear motion provides a very simple description of that motion. It enables you to see the way in which x changes throughout the motion and helps you to appreciate the precise way in which x is determined by t. As a mathematician would say, it shows x as a *function* of t.

It is important to realize that the precise form of a position–time graph depends on the choice of origin and orientation of the x-axis. For example, Figure 9 shows a position–time graph for the moving car described above, but this time the motion is described in terms of a different coordinate system in which the origin has a different location.

❖ Where is the new origin?

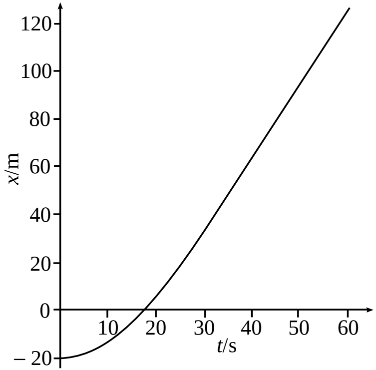

Figure 9 The position–time graph resulting from a change of origin.

Figure 11 represents the same motion again, but an even more radical change of coordinate system has taken place.

Question T3

How is the coordinate system used to produce Figure 11 related to the original system shown in Figure 8? ❏

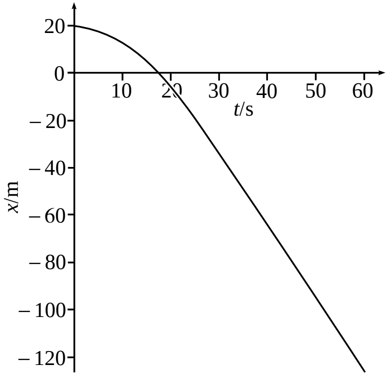

Figure 11 Another position–time graph for the moving car.

In addition to position–time graphs, we should also consider *displacement–time* and *distance–time* graphs as means of representing linear motion.

The **displacement–time graph** for linear motion — a plot of the displacement s_x from some chosen reference point against time t — is often used in preference to the position–time graph. (Indeed, position–time graphs are just a special class of displacement–time graphs in which the reference point from which displacements are measured is the origin.) One of the potential advantages of a displacement–time graph is that the reference point does not have to be fixed.

In the context of Figure 8, for example, it would be quite possible (and of interest to the pedestrian) to represent the motion of the car in terms of its changing displacement from the pedestrian even though both the car *and* the pedestrian are in motion. However, the **distance–time graph** of linear motion — a plot of the distance s from some reference point against time t — is generally less useful because it carries less information. Two different linear motions in opposite directions might well have the same distance–time graph even though their position–time graphs would be very different.

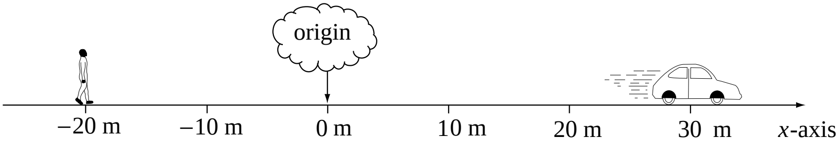

Figure 8 An example of one-dimensional (linear) motion.

3 Velocity and speed

When describing the three-dimensional motion of an object relative to a coordinate system we often want to know how fast the object is travelling and in which direction. This information is given by the **velocity** of the object, which is simply the rate of change of the object's position vector with time. Since changes in position involve both a direction *and* a magnitude it follows that velocity must also be a vector quantity that requires both a magnitude and a direction for its complete specification. For this reason velocity is usually denoted by the bold symbol $\boldsymbol{v}$. The magnitude of an object's velocity is called its **speed** and is denoted $|\boldsymbol{v}|$, though often it is simply written as v. Note that speed, like distance, can never be negative since it is the *magnitude* of a vector and magnitudes are never negative. Indeed, you can think of speed as the rate at which *distance along the path of motion* is being covered, so it can always be described by a value, such as 30 mph (or $13.3\,\mathrm{m\,s^{-1}}$), that can't possibly be less than zero. Velocity on the other hand, since it is a vector quantity and involves direction, is generally specified by an equation of the form

$$\boldsymbol{v} = (v_x, v_y, v_z)$$

where the three components along the x, y and z-axes are each scalar quantities that may be positive or negative.

In what follows we shall once again avoid the difficulties of dealing with velocities in their fully three-dimensional form by restricting our attention to (one-dimensional) linear motion along the *x*-axis of a coordinate system. Under such circumstances the velocity of an object is entirely specified by the *x*-component of its velocity, v_x, and the speed of such an object is given by

$$v = |v_x| \tag{4}$$

☞

3.1 Constant velocity and constant speed

The simplest case of linear motion is that of an object moving in a fixed direction and covering distance at a constant rate. Such an object is said to be moving with **constant velocity** and it is inevitable that such an object must also move with **constant speed**. Constant velocity motion is also known as **uniform motion** and we say that in such motion objects have **uniform velocity** and **uniform speed**. You have already dealt with an object that moves in this way; to see this look again at the graph you drew in response to Question T1. You will notice that after about 30 s the graph appears to be a straight line, i.e. to be *linear* (see Figure 12). Equal changes in position coordinate are therefore occurring in equal intervals of time. This implies that the car is moving with constant velocity and consequently with constant speed.

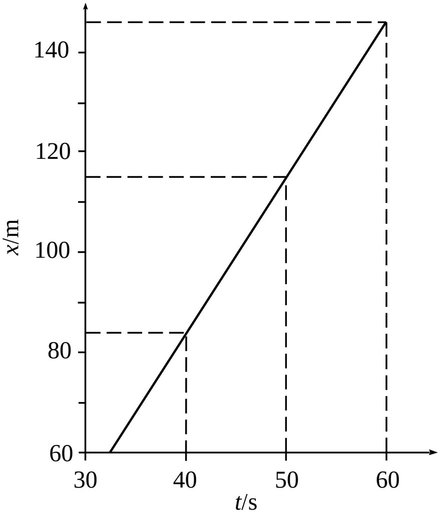

Figure 12 Expanded version of the linear part of the position–time graph for Question T1 (i.e. Figure 21).

If you look at the interval between 40 s and 50 s you can see that the position coordinate of the car changes from 84 m to 115 m at a uniform rate. This uniform rate of change of position is

$$v_x = \frac{(115 - 84)\,\text{m}}{(50 - 40)\,\text{s}} = 3.1\,\text{m}\,\text{s}^{-1}$$

Thus the constant velocity of the car during this part of its journey is $3.1\,\text{m}\,\text{s}^{-1}$.

The same value of the velocity is found by considering the change of position in any other interval of time on the same linear part of the graph.

❖ Calculate the velocity of the car in the above example by repeating the calculation over a different time interval, e.g. 50 s to 60 s.

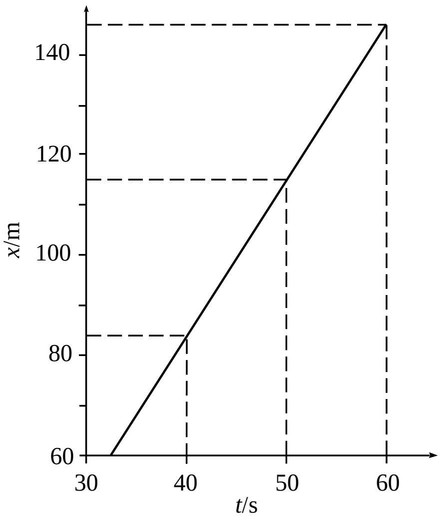

Figure 12 Expanded version of the linear part of the position–time graph for Question T1 (i.e. Figure 21).

✦ In what way does the velocity of $3.1\,\mathrm{m\,s^{-1}}$ that has just been calculated differ from a speed of $3.1\,\mathrm{m\,s^{-1}}$?

✦ Two cars are moving along the same line at the same speed but they have different velocities υ_{x1} and υ_{x2}. How are the velocities related?

✦ In the case of linear motion, how can you tell by looking at a position–time graph that an object is moving at a constant velocity?

Question T4

Figure 13 shows the position–time graphs for four different bodies, each moving with a different constant velocity. If you assume the position and time scales are the same in each case, arrange the bodies in order of increasing speed, indicate which have positive velocities, and arrange the bodies in order of increasing velocity. ❏

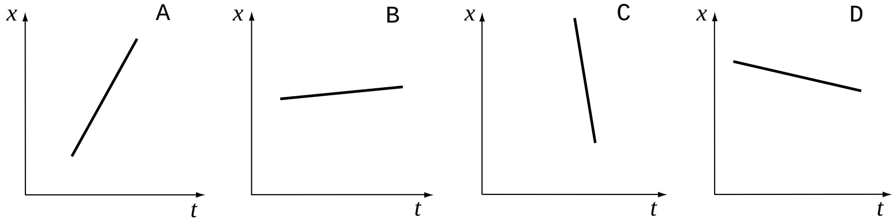

Figure 13 See Question T4.

The point to remember about Question T4 is that in the case of uniform motion it is the steepness or *gradient* of the position–time graph that represents the velocity.

Note *Gradient* is a concept of crucial importance in this module. If you are in any doubt at all about how to evaluate a gradient go back and reread Question R2 (in Subsection 1.3) and its answer. Some authors use the term 'slope' in place of the term 'gradient'.

3.2 Average velocity

If you look again at your graph for Question T1, you will see that in the interval from 0 s to 20 s the graph is curved. This indicates that the velocity is changing throughout that interval. Figure 14 is an expanded version of this part of the graph.

In this case the velocity is certainly not constant, but we can calculate the **average velocity** of the car during any specified interval of time, such as that from t_1 to t_2 which is of duration $\Delta t = (t_2 - t_1)$. To determine the average velocity we use the same method as we employed to find the constant velocity in Subsection 3.1. The average velocity $\langle v_x \rangle$ over the specified time interval is obtained by dividing the change in position coordinate $\Delta x = (x_2 - x_1)$ by the time interval $\Delta t = (t_2 - t_1)$.

$$\langle v_x \rangle = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \qquad (5)$$

☞

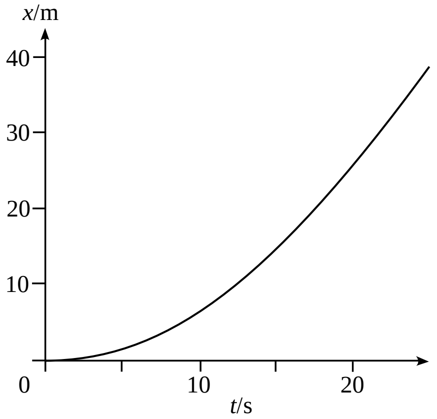

Figure 14 Expanded version of the curved part of the position–time graph for Question T1 (Figure 21).

In Figure 14, the position changes from $x_1 = 2\,\text{m}$ to $x_2 = 26\,\text{m}$ over the interval from $t_1 = 5\,\text{s}$ to $t_2 = 20\,\text{s}$. This means that the change in the car's position is $\Delta x = x_2 - x_1 = 24\,\text{m}$ over a time interval of duration $\Delta t = t_2 - t_1 = 15\,\text{s}$. Using Equation 5 the average velocity over the specified interval is $24\,\text{m}/15\,\text{s} = 1.6\,\text{m}\,\text{s}^{-1}$.

We could have obtained this same result from Figure 14 in a slightly different way. Having identified the two points on the curve that correspond to the beginning and end of the interval we could have drawn a straight line between those points and then determined its *gradient*. The calculation we carried out above was simply one particular way of evaluating that gradient, any other way would have given the same answer. So, once again we see that a velocity — in this case an average velocity — is represented by the *gradient* of a line on a position–time graph.

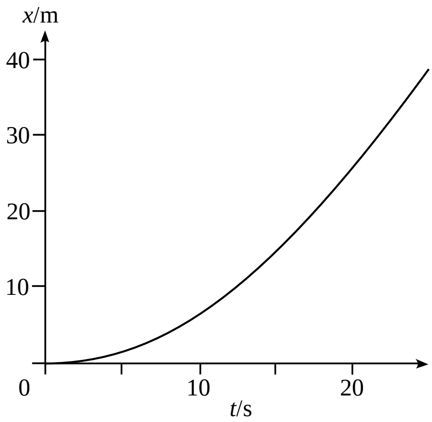

Figure 14 Expanded version of the curved part of the position–time graph for Question T1 (Figure 21).

Question T5

A physics student drives at an average speed of 30 mph to see a friend, who lives a distance of 60 miles away (you may take the problem as being one-dimensional!). On arrival she discovers that he has misunderstood the arrangements and has already left to drive to visit her. She returns home, driving at an average speed of 60 mph on the return trip. What is her average velocity for the whole journey? What is her average speed? If each left originally at the same time and his average speed is 20 mph, who arrives first at her house? ❑

3.3 Instantaneous velocity

The average velocity we calculated in Subsection 3.2 does not give us any information about the car's velocity at any particular instant, say at a time $t = 5\,\mathrm{s}$. Such information is provided by another quantity called the **instantaneous velocity**, which may change from moment to moment. To obtain the instantaneous velocity at $t = 5\,\mathrm{s}$ we need to find the average velocity over smaller and smaller time intervals around the instant $t = 5\,\mathrm{s}$.

Mathematically, this process of finding a better and better approximation to the car's instantaneous velocity at a point by shrinking the interval Δt over which the average is taken is said to be a *limiting process*; we speak of finding the **limit** ☞ of the average velocity as the time interval shrinks to zero.

If you look at Figure 15 you will see how the average velocities (which are the gradients of lines passing through the graph at the time of 5 s) come closer to a limiting value as the time intervals from 5 s are made smaller and smaller.

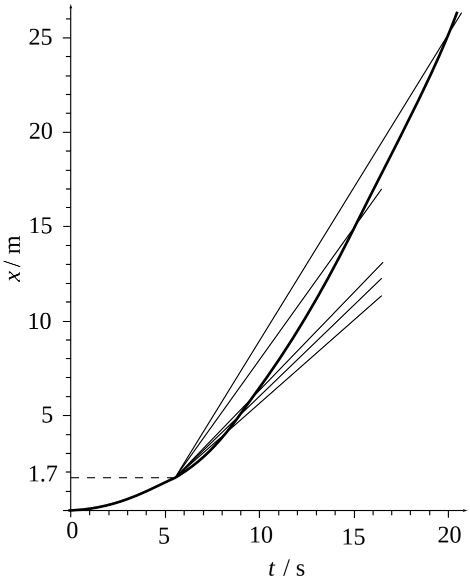

Figure 15 The average velocities (given by the gradients of the lines) approach a limiting value as the time intervals from 5 s get smaller and smaller.

Mathematically, this limiting value is denoted $\lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$, so we can say:

In linear motion, the instantaneous velocity v_x at any particular time is the limit of the average velocity as the time interval around that particular time is made smaller and smaller, i.e.

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

As Figure 16 indicates, the limiting line that is being approached by the lines in Figure 15, as the interval Δt approaches zero, is nothing other than the *tangent* ☞ to the position–time graph at $t = 5\,\text{s}$. So, the limiting velocity is just the gradient of that tangent, and it is this gradient that gives the instantaneous velocity of the car at $t = 5\,\text{s}$. In the branch of mathematics known as **calculus** the limit of the average velocity $\Delta x/\Delta t$, taken over the time interval Δt as the time interval tends to zero, is written as dx/dt. (This is called the **derivative** *of x with respect to t* and should be read as 'dee x by dee t.')

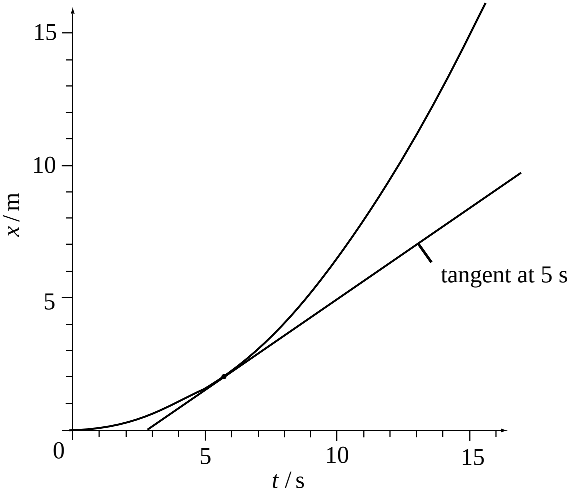

Figure 16 The instantaneous velocity of the car at time $t = 5\,\text{s}$ is the gradient of the tangent at the time, 5 s.

Using this piece of calculus notation we can sum up our discussion of instantaneous velocity in the following way:

> In linear motion, the instantaneous velocity v_x at any particular time is given by the gradient of the tangent to the position–time graph at that time, i.e.
>
> $$v_x = \frac{dx}{dt}$$

If you are already familiar with the techniques of calculus you will know that dx/dt is more than a convenient way of representing the gradient of the tangent to the graph of x against t. In particular, you will be aware that if we have an equation that allows us to work out the value of x that corresponds to any given value of t, i.e. if we know an equation that expresses x as a *function* of t, then we can work out the value of dx/dt at any time algebraically, without having to go to the trouble of plotting any graphs at all. However, whether you are familiar with calculus or not, it is vital to remember the graphical interpretation of dx/dt in terms of the gradient of a tangent — the following question should help you to do that.

Question T6

Table 2 shows position and time measurements for a ball falling from rest under gravity. Plot the points on a position–time graph. Determine the instantaneous velocity dx/dt at 0.20 s by drawing a tangent at 0.20 s and measuring its gradient. Repeat the procedure at 0.60 s. Compare these instantaneous velocities with the average velocities over the intervals from 0.10 to 0.30 s and 0.50 to 0.70 s. ❏

Table 2 See Question T6.

Time t/s	Position x/m
0.00	0.00
0.10	0.05
0.20	0.20
0.30	0.44
0.40	0.78
0.50	1.24
0.60	1.75
0.70	2.40
0.80	3.15

In Section 2 it was pointed out that position–time graphs are just a special class of displacement–time graphs. (Position is just displacement from the origin.) In view of this you may wonder what physical significance can be attached to ds_x/dt — the gradient of a tangent to a general displacement–time graph, in which the displacement s_x may be measured from any specified point. The interpretation is in fact quite straightforward and of some significance. ds_x/dt represents the instantaneous velocity *relative to the reference point from which the displacement is measured*. If the reference point is stationary then $ds_x/dt = dx/dt$ at any time and we learn nothing new. However, if the reference point is located on a moving body (such as the pedestrian in Figure 8) then ds_x/dt at any time will generally differ from dx/dt.

In this latter case in particular it is customary to refer to ds_x/dt at any moment as the (instantaneous) **relative velocity**. ☞ For example, if at some moment the pedestrian in Figure 8 has an instantaneous velocity (dx_1/dt) of $1\,\mathrm{m\,s^{-1}}$, while that of the car (dx_2/dt) is $15\,\mathrm{m\,s^{-1}}$, the velocity of the car relative to the pedestrian (i.e. ds_x/dt) at that moment will be $(15 - 1)\,\mathrm{m\,s^{-1}} = 14\,\mathrm{m\,s^{-1}}$.

> In linear motion, at any time, the instantaneous relative velocity of one body with respect to another is given by the gradient of the tangent to the displacement–time graph, ds_x/dt, at that time, where s_x is the displacement from the first body to the second.

All velocities are really relative velocities. Just as a position x is a special kind of displacement s_x, so an instantaneous velocity dx/dt is a special kind of instantaneous relative velocity ds_x/dt. So, if one day when out driving you are stopped by a policeman and asked 'how fast do you think you were travelling?' you would be quite justified in replying 'relative to what?' However, you would probably be most unwise to do so.

4 Acceleration

The previous section introduced a velocity that changed with time. The *rate* at which the velocity of a body changes with time is known as its **acceleration**. Since velocity is a vector quantity, the rate of change of velocity must also be a vector quantity, the specification of which requires both a magnitude and a direction. In three dimensions an acceleration is therefore usually denoted by $\boldsymbol{a}$ and expressed in terms of its three (scalar) components by an equation of the form

$$\boldsymbol{a} = (a_x, a_y, a_z)$$

As usual we shall avoid dealing with this three-dimensional quantity by considering one-dimensional linear motion in which the acceleration is entirely specified by its scalar component a_x along the x-axis. The component a_x may be positive or negative. A positive value for a_x corresponds to an increase in υ_x with time and a negative value for a_x corresponds to a decrease in υ_x with time. Any acceleration that causes the speed to decrease is called a **deceleration**. In the case of linear motion the magnitude of the acceleration is the positive quantity given by

$$a = |\,a_x| \tag{6}$$

✦ What would be appropriate SI units for the measurement of a or a_x?

4.1 Constant acceleration

Table 3 shows the measured velocity of a falling ball at six different instants of time. Data of this kind can be used to plot a **velocity–time graph**.

Table 3 Velocity of a falling ball.

Time from release t/s	Velocity of ball v_x/m s^{-1}
0.0	0.00
0.2	1.97
0.4	3.93
0.6	6.10
0.8	7.81
1.0	9.80

Question T7

Use the data in Table 3 to plot a velocity–time graph for the falling ball. (Plot the time along the horizontal axis and the velocity along the vertical axis, and draw a smooth line through the points.) ❏

You will have noticed that your graph is a straight line. Equal changes of velocity occur in equal time intervals, and this implies that the ball is falling with **constant acceleration** or **uniform acceleration**. In fact, this graph is an illustration of a well known experimental observation that, provided air resistance can be neglected, a body falling freely from rest near the Earth's surface increases its velocity at a constant rate. The gradient of the line represents the rate of change of velocity, i.e. the acceleration.

Question T8

Measure the gradient of the line you have drawn in answer to Question T7. ❏

Measurements of the magnitude of the acceleration of *any* falling body near the Earth's surface, in the absence of air resistance, give the value 9.8 m s^{-2} (to two significant figures). This quantity is known as the magnitude of the **acceleration due to gravity** at the Earth's surface (or often simply as the acceleration of gravity) and is given the symbol *g*. ☞

4.2 Average acceleration

Figure 17 is a velocity–time graph for the motion of the ball in Questions T7 and T8 over a much longer period of time. After a few seconds the graph becomes markedly curved, indicating that the acceleration is no longer constant. The velocity is still increasing, but at a slower rate. This illustrates the effect of air resistance, which becomes more significant as the velocity increases.

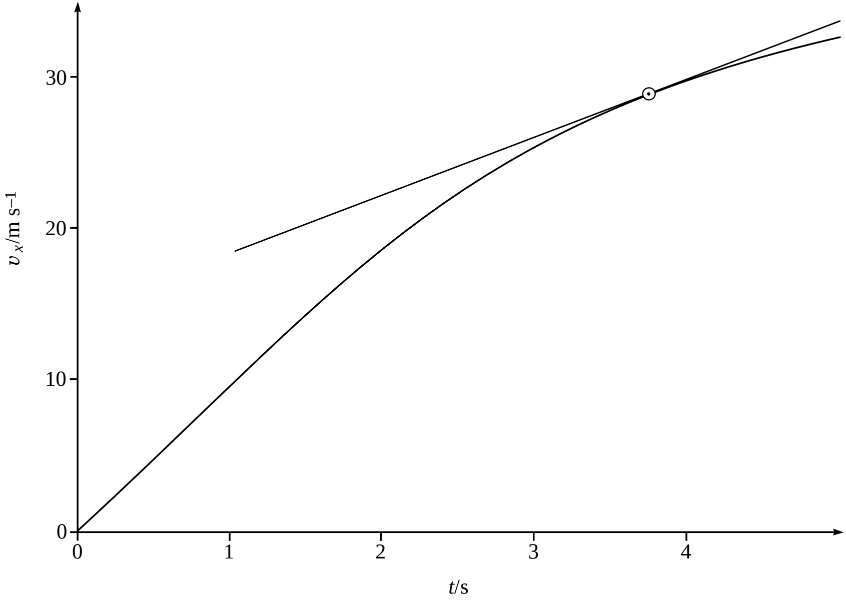

Figure 17 Velocity–time graph of a falling ball.

If we use Figure 17 we can calculate the **average acceleration** over a chosen period of time by means of the method we applied to the calculation of average velocity in the previous section. This leads to the following equation:

$$\langle a_x \rangle = \frac{\Delta v_x}{\Delta t} = \frac{v_{x2} - v_{x1}}{t_2 - t_1} \qquad (7)$$

☞

where $\Delta v_x = v_{x2} - v_{x1}$ is the *change* in v_x over the interval $\Delta t = t_2 - t_1$.

4.3 Instantaneous acceleration

In many cases we wish to know the **instantaneous acceleration** of a moving body, that is the acceleration at a specific time. In Subsection 3.3 we defined an instantaneous velocity in terms of the limit of the average velocity as the time interval shrank to zero. We may use the same ideas now to calculate the instantaneous acceleration. Therefore we can write the following.

In linear motion, the instantaneous acceleration a_x at any particular time is the limit of the average acceleration as the time interval around that particular time is made smaller and smaller, i.e.

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$$

We also saw in Subsection 3.3 that calculus provides a useful shorthand for limits of this kind and that at any particular time such a limit may be interpreted graphically as the gradient of the tangent to an appropriate graph at that particular time. In the present case we are dealing with a velocity–time graph, so the piece of calculus notation we need is dv_x/dt, the derivative of v_x with respect to t. Using this we can say:

In linear motion, the instantaneous acceleration a_x at any particular time is given by the gradient of the tangent to the velocity–time graph at that time, i.e.

$$a_x = \frac{dv_x}{dt}$$

❖ What is the instantaneous acceleration at the point indicated in Figure 17, when the time is 3.75 s? ☞

Compare this with the average acceleration between t = 3.5 s and 4.0 s.

The acceleration you have just found is considerably less than g, because of the effects of air resistance.

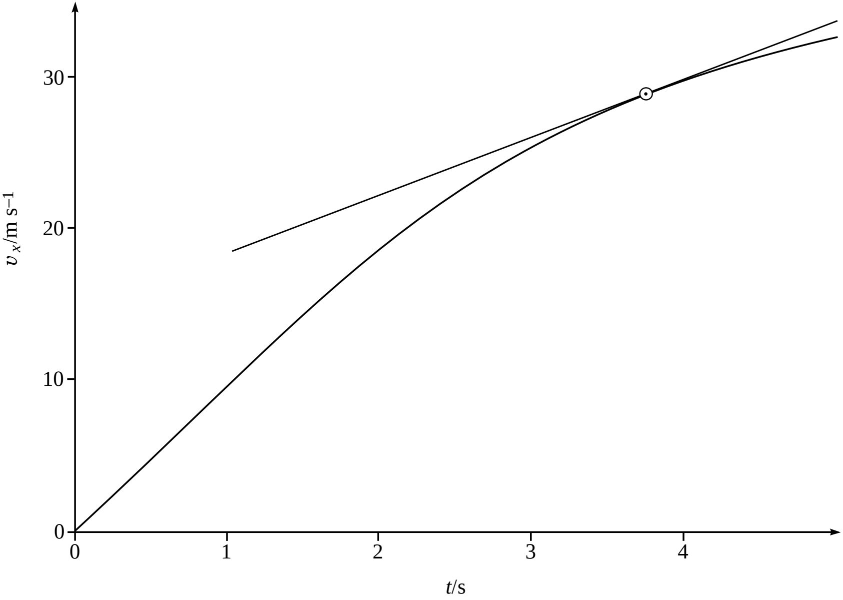

Figure 17 Velocity–time graph of a falling ball.

Question T9

Figure 18 shows the velocity–time graph of a car during a short journey along a straight road. Determine the acceleration dv_x/dt at 10 s, 40 s and 55 s, and then use this information to help you to describe the journey in everyday language. ❏

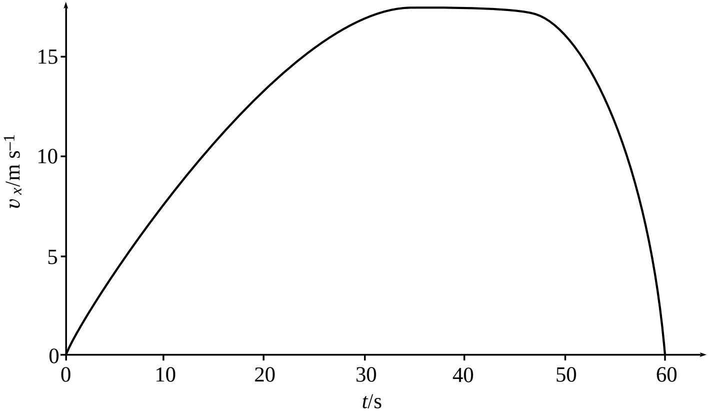

Figure 18 See Question T9.

5 Equations of motion

In previous sections we have seen how the motion of an object such as a car or a ball, can be described and analysed graphically. However, drawing graphs is time consuming and may be inaccurate, so it is highly desirable that methods should be developed for describing and analysing motion algebraically (in terms of equations), without the need for graphs. Such methods certainly exist and are explained in some detail in various *FLAP* modules. The aim of this section is simply to introduce you to such algebraic descriptions in a way that will be clearly related to the graphical descriptions you have already seen. In order to keep the discussion as simple as possible we not only restrict ourselves to motion in one dimension but we further restrict our considerations to moving objects that are sufficiently small and simple that they can be treated as ideal point-like **particles**. Such ideal particles have mass and occupy a definite position at any time, but unlike real objects (such as cars or balls) they cannot rotate, bend or vibrate. Their simplicity makes them ideal subjects for our considerations, and, as it turns out, an excellent starting point from which to consider more realistic objects in other modules.

We will now use the ideas developed in previous sections to derive equations that describe the motion of a particle travelling either with constant velocity or with constant acceleration. We will not concern ourselves here with the more general case of non-uniform acceleration.

5.1 Uniform motion equations

Uniform motion (i.e. constant velocity motion) is the simplest kind of motion. It was treated graphically in Subsection 3.1. In this subsection we are going to consider uniform motion algebraically, but our viewpoint will be slightly different from that which we adopted earlier. The main difference is that the present treatment will concentrate on the changing *displacement* of the particle from some physically defined reference point rather than the changing *position* of the particle relative to an arbitrarily chosen origin. More specifically, *we shall take the position of the particle itself at time $t = 0$ to be the fixed reference point from which the displacement s_x is measured and we shall let u_x represent the constant velocity of the particle relative to that reference point.* As a consequence of this choice of reference point we can be sure that $s_x = 0$ at $t = 0$. Moreover, since the velocity is uniform in this case we can equate u_x with the rate of change of displacement over any part of the motion.

Thus, $$u_x = \frac{\Delta s_x}{\Delta t} = \frac{s_x - 0}{t - 0} = \frac{s_x}{t}$$

i.e. $s_x = u_xt$ (8)

Remember, this applies only if

$v_x = u_x = \text{constant}$ (9)

which also has the consequence

$a_x = 0$ (10)

Equations 8, 9 and 10 are the **uniform motion equations**. They make it possible to work out the displacement from the initial position, the velocity and the acceleration of the particle at any time.

Figure 19a shows uniform motion on a velocity–time graph. This is a very simple graph, nonetheless it illustrates a significant (and easily generalizable) result concerning the *area under the graph* between two given times (see Figure 19b) ☞.

Question T10

Use the uniform motion equations to confirm this claim by writing down an expression for the area under the graph between t_1 and t_2, and then relating that area to the change in displacement over the same interval. ❏

For uniform motion, the area under the velocity–time graph, between any two times t_1 and t_2, is equal to the change in the displacement over that interval.

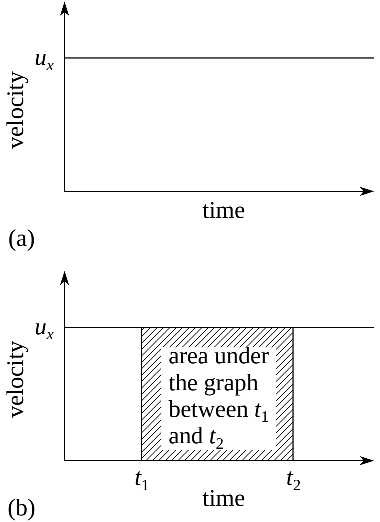

Figure 19 (a) Velocity–time graph of a particle moving with a constant velocity u_x. (b) The area under the velocity–time graph between t_1 and t_2. Note that if the velocity u_x is negative then the area 'under' the graph is also counted as negative.

Although somewhat out of context, it is worth noting at this point that the relationship, that has just been uncovered between the area under a velocity–time graph and the change in displacement, indicates a more general result that applies even when the particle is accelerating and the area under the graph is not rectangular.

In linear motion, the area under a velocity–time graph, between two times t_1 and t_2, is *always* equal to the change in displacement between those times.

5.2 Uniform acceleration equations

Inspired by our success in dealing algebraically with uniform motion let us now move on to a slightly more difficult case, that of a particle moving with constant or uniform acceleration. Suppose such a particle has a velocity u_x at the start of its motion when $t = 0$, and a velocity υ_x at some later time t. These two velocities are known, respectively, as the **initial velocity** and the **final velocity**. If we recall the definition of average acceleration $\langle a_x \rangle$ given in Equation 7, we can write the constant acceleration a_x in terms of these symbols as

$$a_x = \frac{\upsilon_x - u_x}{t}$$

which can be rearranged to give

$$v_x = u_x + a_x t \qquad (11)$$

This is the *first* of the three **uniform acceleration equations** (also known as the **constant acceleration equations**).

When using the uniform acceleration equations, it is generally a good idea to state clearly:

- the *reference point* from which displacements are to be measured;
- the *direction* in which displacements are positive;
- the *given data*, expressed in terms of the symbols to be used, e.g. a_x, u_x, v_x, t.

It is generally a bad idea to change any of these during a solution.

Question T11

A racing car moves from rest with a uniform acceleration of $9.0\,\mathrm{m\,s^{-2}}$ for the first 5 s. Calculate the velocity of the car after 2.5 s, and then find the time taken for the car to increase its speed from $30\,\mathrm{m\,s^{-1}}$ to $40\,\mathrm{m\,s^{-1}}$. ❏

You have probably seen advertisements for cars stating that they can accelerate from rest to 60 mph in, say, 8 s. Suppose you want to work out how far the car would travel in that time, assuming that the acceleration is uniform. You could do this if you knew the average speed of the car.

✦ What is the average speed of the car over the interval?

✦ What is the distance travelled over the interval?

For our uniformly accelerated particle we can calculate the displacement s_x after a time t in this same way once we have an expression for the average velocity. We will measure the displacement from the initial position of the particle when the time is zero (so the initial displacement is zero) and calculate the final displacement at time t by multiplying the average velocity by t.

✦ What is the expression for the average velocity in this case?

Thus $$s_x = \frac{u_x + v_x}{2}t \qquad (12)$$ ☞

Substitution of the expression for v_x from Equation 11

$$v_x = u_x + a_x t \qquad \text{(Eqn 11)}$$

gives

$$s_x = \frac{u_x + u_x + a_x t}{2}t = \frac{2u_x t + a_x t^2}{2}$$

and hence our *second* uniform acceleration equation is

$$s_x = u_x t + \tfrac{1}{2}a_x t^2 \qquad (13)$$

Try using this equation in the next question.

Question T12

A stone falls from rest with an acceleration of $9.8\,\mathrm{m\,s^{-2}}$. Calculate how far it has fallen after $2.0\,\mathrm{s}$. ❏

Aside The next question is more difficult because it requires you to use both of the uniform acceleration equations derived so far (Equations 11 and 13), in order to eliminate the time t, which is not given in the question.

Question T13

Calculate how fast the stone in Question T12 is moving after it has fallen through 2.0 m. ❏

In answering Question T13 an expression was found for the time at the end of the interval, even though the question did not ask for it. We could have avoided this extra effort if we had an equation which gave the final velocity directly in terms of the acceleration and displacement, the quantities which were given in the question. To derive such an equation we can follow the same route that was used in Question T13, but this time use algebra to derive a general expression for the time t. We can obtain this expression by rearranging Equation 11. Thus

$$t = \frac{v_x - u_x}{a_x}$$

From Equation 12, the displacement is the average velocity multiplied by this time;

so $$s_x = \frac{u_x + v_x}{2}t \qquad \text{(Eqn 12)}$$

and on substitution for t this gives

$$s_x = \frac{{v_x}^2 - {u_x}^2}{2a_x}$$

This may be rearranged to give the third uniform acceleration equation

$${v_x}^2 = {u_x}^2 + 2a_xs_x \qquad (14)$$

Question T14

A car is travelling at an initial velocity of $6.0\,\mathrm{m\,s^{-1}}$. It then accelerates at $3.0\,\mathrm{m\,s^{-2}}$ over a distance of $20\,\mathrm{m}$. Calculate its final velocity. ❏

Finally, it's important to remember that all the equations derived in this subsection apply to situations in which

$$a_x = \text{constant} \qquad (15)$$

Do not make the common mistake of supposing them to work in more general situations.

5.3 Solution to the introductory problem

We will return now to the problem posed in Subsection 1.1. At this point in the module you should be able to solve this problem. Reread the problem and then consider how you would tackle it.

> According to the Highway Code, a car travelling along a straight road at 30 mph (i.e. about $13.3\,\text{m}\,\text{s}^{-1}$, read as 13.3 metres per second) can stop within 23 metres of the point at which the driver sees a hazard. This is known as the stopping distance. If the driver always takes 0.70 s to react to a hazard and apply the brakes, what is the stopping distance at 70 mph (i.e. about $31.1\,\text{m}\,\text{s}^{-1}$) assuming the same deceleration as at 30 mph?

When you have done this, have a look at the solution given below.

We can divide the motion of the car into two stages. During the first stage, corresponding to the driver's reaction time, the car is travelling at a constant velocity. The second stage starts when the brakes are applied and we shall assume the car then has a constant deceleration. The question does not tell us the value of this constant deceleration but since it *is* constant and therefore doesn't depend on the velocity of the car we can work it out from the information we are given about the stopping distance of the car travelling at 30 mph (13.3 $\mathrm{m\,s^{-1}}$).

Within each stage of the motion, displacements will be measured from the position of the car *at the beginning of that stage*. The direction of motion will always be the positive direction for displacements, so all velocities will be positive and deceleration will be a constant *negative* acceleration in this case.

For the car travelling at 13.3 $\mathrm{m\,s^{-1}}$, during the first stage of its motion

$$v_x = u_x = \text{constant} = 13.3\,\mathrm{m\,s^{-1}} \text{ and } t = 0.70\,\mathrm{s}.$$

To find the displacement at the end of this constant velocity stage we can use

$$s_x = u_x t \qquad \text{(Eqn 8)}$$

so $s_x = 13.3\,\mathrm{m\,s^{-1}} \times 0.70\,\mathrm{s} = 9.31\,\mathrm{m}$

Therefore the distance covered by the end of the reaction time is $s = |s_x| = 9.31\,\text{m}$. Since the stopping distance at $13.3\,\text{m}\,\text{s}^{-1}$ is 23.0 m, the distance covered in the second stage must be $(23.0 - 9.31)\,\text{m} = 13.7\,\text{m}$.

For the second stage, while the driver is braking, we thus have

$$u_x = 13.3\,\text{m}\,\text{s}^{-1},\ v_x = 0\,\text{m}\,\text{s}^{-1} \text{ and } s_x = 13.7\,\text{m} \quad \text{(from the new origin)}$$

To find a_x during this constant acceleration stage we can use

$$v_x^2 = u_x^2 + 2a_x s_x \qquad \text{(Eqn 14)}$$

so $$a_x = \frac{v_x^2 - u_x^2}{2s_x} = \frac{(0 - 13.3^2)\,\text{m}^2\,\text{s}^{-2}}{2 \times 13.7\,\text{m}} = -6.46\,\text{m}\,\text{s}^{-2}$$

Now that the acceleration is known, we can apply it to the situation in which the car is travelling at $31.1\,\text{m}\,\text{s}^{-1}$. For the first stage of that motion

$$v_x = u_x = \text{constant} = 31.1\,\text{m}\,\text{s}^{-1} \text{ and } t = 0.70\,\text{s}$$

If we use the constant velocity equation $s_x = u_x t$ we see that

$$s_x = (31.1 \times 0.70)\,\text{m} = 21.8\,\text{m}$$

For the second stage, while the driver is breaking

$u_x = 31.1\,\mathrm{m\,s^{-1}}$, $\upsilon_x = 0\,\mathrm{m\,s^{-1}}$ and $a_x = -6.46\,\mathrm{m\,s^{-2}}$

If we use the constant acceleration equation $\upsilon_x^2 = u_x^2 + 2a_x s_x$ we see that

$$s_x = \frac{\upsilon_x^2 - u_x^2}{2a_x} = \frac{(0 - 31.1^2)\,\mathrm{m^2\,s^{-2}}}{2 \times -6.46\,\mathrm{m\,s^{-2}}} = 74.9\,\mathrm{m}$$

The stopping distance of the car at $31.1\,\mathrm{m\,s^{-1}}$ is the sum of the magnitudes of the displacements in the two stages, so

stopping distance = (21.8 + 74.9) m = 96.7 m

This shows that the stopping distance at 70 mph is more than four times that at 30 mph, which reinforces the need to drive at a safe distance from the car in front.

6 Closing items

6.1 Module summary

1 Space is *three-dimensional*, so three *position coordinates* (x, y, z) are required to locate any point in space relative to the *origin* of a *Cartesian coordinate system*.

2 A *vector* has both magnitude and direction, and may be contrasted with a *scalar* which has no direction. Vectors are often specified in terms of their *components*; these are scalar quantities that may be positive or negative and which are measured along the *axes* of the coordinate system. Vector quantities include position vector $\boldsymbol{r} = (x, y, z)$, displacement $\boldsymbol{s} = (s_x, s_y, s_z)$, velocity $\boldsymbol{v} = (v_x, v_y, v_z)$ and acceleration $\boldsymbol{a} = (a_x, a_y, a_z)$.

3 The *magnitude* of a vector is the 'length' or 'size' of that vector. $|\boldsymbol{r}|$, the magnitude of the position vector $\boldsymbol{r}$ of a point, represents the distance from the origin of the coordinate system to that point. Magnitudes can never be negative.

4 The *displacement* $\boldsymbol{s}$ from one point to another describes the difference in their positions. $|\boldsymbol{s}|$ is the distance between those two points.

5 The *velocity* $\boldsymbol{v}$ of a particle is the rate of change of the position vector of that particle; it tells us how fast the particle is moving and in what direction. $|\boldsymbol{v}|$ is called the *speed* of the particle.

6 The *acceleration* $\boldsymbol{a}$ of a particle is the rate of change of the velocity of that particle.

7 *Linear motion* is motion along a straight line, though not necessarily in one direction along this line. Such motion is one-dimensional since the position of the moving particle can be described in terms of a single position coordinate, x.

8 The linear motion of a particle can be represented on a *position–time graph*, a *displacement–time graph*, or a *velocity–time graph*. The position–time graph shows the displacement from the origin at any time, while the displacement–time graph shows the displacement from some general reference point that may itself be in motion or it may be the particle's position at $t = 0$.

9 If the position–time graph of a moving particle is linear, that particle must be moving with *constant (uniform) velocity*, i.e. with *constant (uniform) speed* and in a fixed direction.

10 In linear motion, the *average velocity* $\langle v_x \rangle$ over a specified time interval is

$$\langle v_x \rangle = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \qquad \text{(Eqn 5)}$$

11 In linear motion, the *instantaneous velocity* v_x at any particular time is the limit of the average velocity as the time interval around that particular time is made smaller and smaller. This may be written more compactly in terms of the *derivative* of x with respect to t, dx/dt, which may be interpreted graphically as the gradient of the tangent to the position–time graph at the time in question. Thus,

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

12 In linear motion, at any time, the instantaneous *relative velocity* of one body relative to another is given by the gradient of the tangent to the displacement–time graph ds_x/dt at that time, where s_x is the displacement from the first body to the second.

13 If the velocity–time graph of a moving particle is linear that particle must be moving with *constant (uniform) acceleration*.

14 In linear motion, the *average acceleration* $\langle a_x \rangle$ over a specified interval is

$$\langle a_x \rangle = \frac{\Delta v_x}{\Delta t} = \frac{v_{x2} - v_{x1}}{t_2 - t_1} \qquad \text{(Eqn 7)}$$

15 In linear motion, the *instantaneous acceleration* a_x at a particular time is the limit of the average acceleration as the time interval around that particular time is made smaller and smaller. This may be written more compactly in terms of the *derivative* of v_x with respect to t, dv_x/dt, which may be interpreted graphically as the gradient of the tangent to the velocity–time graph at the time in question. Thus,

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

16 When $v_x = u_x$ = constant, the uniform (linear) motion of a particle can be described algebraically using the *uniform motion equations*:

$$s_x = u_x t \quad \text{(Eqn 8)}$$

$$v_x = u_x = \text{constant} \quad \text{(Eqn 9)}$$

$$a_x = 0 \quad \text{(Eqn 10)}$$

17 When a_x = constant the uniformly accelerated (linear) motion of a particle can be described algebraically using the *uniform acceleration equations*:

$$v_x = u_x + a_x t \quad \text{(Eqn 11)}$$

$$s_x = u_x t + \tfrac{1}{2} a_x t^2 \quad \text{(Eqn 13)}$$

$$v_x^2 = u_x^2 + 2a_x s_x \quad \text{(Eqn 14)}$$

18 The area under a velocity–time graph, between two times t_1 and t_2 is equal to the change in displacement between those times.

6.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Plot position–time, displacement–time and velocity–time graphs from linear motion data.

A3 Describe the linear motion of a body, given its position–time, displacement–time or velocity–time graph, or, conversely, sketch such graphs given a description of the linear motion of the body.

A4 Calculate the average velocity, or instantaneous velocity, or instantaneous relative velocity, as appropriate, from a position–time or displacement–time graph.

A5 Calculate the average acceleration, or instantaneous acceleration, as appropriate, from a velocity–time graph.

A6 Calculate the change in displacement of a particle over a given interval of time from its velocity–time graph.

A7 Derive the uniform motion and constant acceleration equations and use them to solve problems.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

6.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the *Achievements*.

Question E1

(*A2*, *A4*, *A5* and *A6*) Plot a velocity–time graph from the data given in Table 4. From the graph determine: (a) the displacement from the initial position after 30 s, (b) the velocity at t = 10 s, (c) the average acceleration between t = 10 s and t = 20 s.

Table 4 See Question E1.

Time t/s	Velocity v_x/m s^{-1}
0	0
2	5.0
4	10.0
6	14.0
8	12.0
12	8.0
17	4.0
24	2.0
30	1.0

Question E2

(*A2* and *A7*) A stone is released from rest and falls with uniform acceleration under gravity. Calculate its displacement from its initial position over the first 3 s at 0.5s intervals. For ease of calculation the magnitude of the acceleration due to gravity may be taken to be $10\,\mathrm{m\,s^{-2}}$. Use your results to plot a displacement–time graph.

If the stone were released from the top of a cliff and hit the ground at the base of the cliff 2.8 s after it was dropped, what is the height of the cliff?

Question E3

(*A3*) Write a description of the motion of a body, illustrated by the velocity–time graph in Figure 20. (You are not required to calculate displacements.)

Question E4

(*A7*) (a) Derive an equation relating initial velocity, final velocity, acceleration and time for a particle moving with a constant acceleration. (b) A lorry is travelling along a straight road at a constant velocity of $15\,\mathrm{m\,s^{-1}}$ when the driver notices an obstruction in the road 25 m ahead. His reaction time is 0.40 s and the brakes can produce a deceleration of $7.0\,\mathrm{m\,s^{-2}}$. Calculate whether the driver will stop the lorry in time.

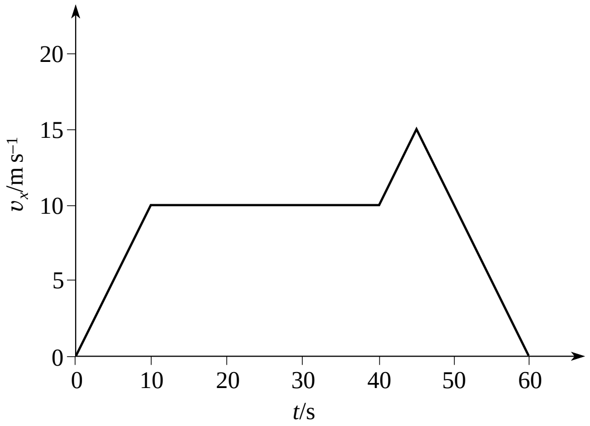

Figure 20 See Question E3.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.