Module M1.2 Numbers, units and physical quantities

Exit module

1 Opening items

1.1 Module introduction

The statements ‘That car was travelling at about 80’, and ‘That car was travelling at 81.6 kilometres per hour’, could both apply to the same situation. The first statement gives information in an approximate form and takes the units of measurement for granted. This is often appropriate in everyday life, but is seldom good enough for scientific purposes; not only is it imprecise, it is also open to misinterpretation since in the UK it would probably be taken to mean 80 *miles* per hour while in other European countries *kilometres* per hour would be presumed. The second statement is much more informative, the units of measurement are clearly stated and the speed is given with some precision; the car was not travelling at 81.5 kilometres per hour, nor at 81.7 kilometres per hour, but at 81.6 kilometres per hour.

This module is about using and interpreting numbers and units in the specification of physical quantities. Some very basic terms with which you may already be familiar are precisely defined, and some widely used conventions that have been adopted by *FLAP* are explicitly spelt out. Throughout the module the main aim is to cause you to think more deeply and more incisively than usual about the exact meaning of numerical statements about physical quantities.

The main teaching text of this module is contained in Sections 2 and 3. The first of these sections is mainly concerned with numbers. Subsections 2.1 and 2.2 introduce *scientific notation* (also known as *powers of ten notation* or *standard form*) which is widely used to represent large and small numbers. Subsection 2.3 refines the definition of scientific notation by using *inequalities* and the *modulus notation* (e.g. $|a|$) for the *absolute value* of a number. The interpretation of quantities specified to a fixed number of *significant figures* or to a given number of *decimal places* is explained in Subsection 2.4, along with the value of using scientific notation to avoid spurious indications of precision. Subsection 2.5 sets out the procedure for *rounding* values to a given number of significant figures and describes the related issues of *rounding errors*, *approximations* and *order of magnitude* estimates. Section 2 ends with a brief introduction to *set notation*, an informal account of the set of *real numbers* and the definition of some of its major subsets such as *integer* and *rational numbers*. Section 3 is concerned with units and deals in sequence with the notational conventions of *SI units*, the correct usage of units in calculations and the investigation of quantities and equations by means of *dimensional analysis.*

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 5. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1

Express the following numbers in scientific notation (this uses 'powers of ten' and is also known as standard form):

(a) 293.45 (b) 1 380 (c) −2 804 (d) 0.005 67

Question F2

Select the numbers from the following list that have (a) the same number of significant figures, and (b) the same order of magnitude:

1.23×10^6 727 8.5×10^6 1.148×10^7 92.874 0.043 2

Question F3

Excited hydrogen atoms emit radiation at certain well defined frequencies. These frequencies are related to changes in the energy of electrons as they jump from one atomic energy level to another. The emissions are described by the equation $f = RcX$ where f is the frequency of the radiation, c the speed of light, X a dimensionless number that depends on the initial and final energy levels of the electron, and R a physical constant called Rydberg's constant. Use the rearrangement $R = f/cX$ to deduce the SI units and dimensions of R.

Study comment Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items* .

1.3 Ready to study?

Study comment In order to study this module you will need to understand the following terms: *power*, *product*, *ratio*, *rearrangement* (*of an equation*), *reciprocal* and *root*. If you are uncertain about any of these terms you can review them now by referring to the *Glossary* which will indicate where in *FLAP* they are developed. In addition, it is assumed that you are familiar with SI units such as metre and kilogram (though SI units in general are briefly reviewed in Section 3) and are accustomed to carrying out simple calculations involving the basic arithmetic operations (×, ÷, +, –), powers and brackets. The following questions will help you to check that you have the required level of familiarity with powers, roots, reciprocals and equation rearrangement.

Question R1

Rewrite each of the following expressions as a single power of x:

(a) $x^3 \times x^4$

(b) $x^5 \times x^{-2}$

(c) $1/x^2$

(d) $1/x^{-6}$

(e) x^8/x^3

(f) $(x^4)^{-6}$

(g) $\sqrt{x^3}\sqrt{1/x^5}$

Question R2

Rearrange the equation $F = 6\pi\eta rv$ so that η is isolated on one side.

2 Numbers and physical quantities

Study comment This section is designed to make you think carefully about the way numbers are written, discussed and interpreted. You should not use a calculator to answer any of the questions or exercises contained in this section unless you are specifically instructed to do so.

2.1 Scientific notation

Many physical quantities have numerical values that are much greater than or less than 1. For example, a non-SI unit of pressure (based on the pressure of the atmosphere) is 1 atm but the equivalent pressure expressed in SI units is 101 325 pascal; similarly, the charge of the electron ($-e$) in SI units is $-0.000\,000\,000\,000\,000\,000\,160\,2$ coulomb. In physics it is customary to write such large or small numbers using **scientific notation** ☞: that is, as a product of a positive or negative *decimal number* (with one non-zero digit before the decimal point) and a suitable *power of ten*. ☞ Using scientific notation, 1 atm = $1.013\,25 \times 10^5$ pascal, and $-e = -1.602 \times 10^{-19}$ coulomb. This subsection explores the numerical aspects of scientific notation and highlights its advantages. Discussion of the role of units in scientific notation is mainly deferred to Section 3.

The **powers of ten** involved in scientific notation take the form 10^n where n can be any positive or negative whole number ☞. Table 1 illustrates some features of powers of ten, and is the subject of questions on the next page.

Question T1

Using the printed entries in Table 1 as a guide, fill in the spaces in the three columns in the upper part of the table, and in all five columns in the lower part. ❏

Table 1 Powers of ten.

Number written in full		Number written using a product of 10		Number written as a power of 10
100 000		$10 \times 10 \times 10 \times 10 \times 10$		10^5
10 000		$10 \times 10 \times 10 \times 10$		10^4
1 000				
100				
10				10^1
1				
0.1	1/10			10^{-1}
0.01	1/100	$1/(10 \times 10)$	$1/10^2$	10^{-2}
0.001				

Question T2

(a) If you look at each row of the *upper* part of the completed Table 1, what pattern links the number of zeros after the 1 in the left-hand column, the number of tens multiplied together, and the power of ten?

(b) If you look at each row of the *lower* part of the completed Table 1, what pattern links the position of the 1 after the decimal point in the left-hand column and the power of ten? ❏

Notice that finding the *reciprocal* of any power of ten is equivalent to changing the sign of the power (e.g. $1/10^2 = 10^{-2}$), and that the relation $10^0 = 1$ forms part of the pattern you have been using.

A **decimal number** is any number expressed in **base** ten notation — the normal system for writing numbers, in which there are ten **digits** (0 to 9) and the position of each digit relative to the decimal point is related to a power of ten. For example, the number 345.6 in base ten notation denotes $3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0 + 6 \times 10^{-1}$. A decimal number does not necessarily include a decimal point; 976, −4.05 and 0.038 are all decimal numbers.

Any number can be expressed as the product of a decimal number and a power of ten. Indeed, this can generally be done in many different ways:

For example, $3\,428 = 3.428 \times 1\,000 = 3.428 \times 10^3$

but equally, $3\,428 = 34.28 \times 100 = 34.28 \times 10^2$

Similarly, $0.056\,7 = 5.67 \times 0.01 = 5.67 \times 10^{-2}$

and $0.056\,7 = 567 \times 0.000\,1 = 567 \times 10^{-4}$

Notice that, in principle, *any* power of ten can be chosen because the decimal number can always be adjusted to ensure that the value of the product remains unchanged. Thus, scientific notation is nothing more than an agreed convention that restricts the form of the decimal number and hence dictates the choice of power.

Question T3

Write the following numbers in full:

(a) 3.2×10^6,   (b) 8.76×10^{-4}.   ❏

Question T4

Write the following numbers in scientific notation:

(a) 98 765,   (b) 0.004 32.   ❏

One advantage of scientific notation is that it is quite compact. Another advantage is the simplicity it gives to calculations that involve only multiplication and division; where powers of ten can be dealt with separately from the decimal numbers and the order in which multiplications are performed does not matter.

For example, $2 \times 10^4 \times 4 \times 10^5 \times 6 \times 10^{-10}$

can be written as $2 \times 4 \times 6 \times 10^4 \times 10^5 \times 10^{-10}$

If we combine the powers of ten we obtain: $2 \times 4 \times 6 \times 10^{-1} = 48 \times 10^{-1} = 4.8$

Even in calculations that cannot so easily be simplified or reordered, it is often worth manipulating the powers of ten to produce a simpler expression before reaching for a calculator, since the fewer numbers you have to key in, the less likely you are to make a mistake.

Question T5

Rewrite the following expressions so that each involves a single power of ten:

(a) $1.40 \times 10^{4} \times 5.50 \times 10^{13} \times 6.20 \times 10^{-15}$

(b) $2 \times 10^{6} \times 3 \times 10^{5}/(8 \times 10^{7})$. ❏

2.2 Scientific notation and calculators

All 'scientific' calculators handle scientific notation, though you may have to select a particular mode such as 'SCI' or 'ENG' on your calculator to activate this feature, or you might have to press a key marked 'EE' or 'EXP' ☞. The following exercises are aimed at helping you to avoid some common mistakes when using powers of ten on a calculator.

✧ Key the number 10 000 000 into your calculator, and make your calculator display it in a more compact form. What do you get?
(If it doesn't happen automatically and the keys mentioned above don't work, try '=' or '×' followed by 'EE' or 'EXP' ☞. If all else fails, read the instructions!)

✧ Clear the calculator, and repeat the previous exercise with 30 000 000.

The 'compact form' that calculators use to display numbers corresponds to scientific notation. Whichever form your calculator uses, you should *write* the numbers as 10^7 and 3×10^7. In particular, resist the temptation to write them as 1 07 and 3 07 (to avoid confusion with 107 and 307) or 1^7 and 3^7.

✧ Why do you think it is important not to write, for example, 3×10^7 as 3^7 when writing down the number shown on your calculator? What does 3^7 really mean?

You have just illustrated one important point about using a calculator:
Always be careful how you write down numbers from the calculator display.

◆ Now try entering first 10^7 and then 3×10^7 directly.

Try keying 1 EE 7 or 1 × EE 7, and then 3 EE 7 or 3 × 1 EE 7. What do you get?

A common mistake is to key 10 EE 7, or 1 × 10 EE 7, when entering 10^7, and 3 × 10 EE 7 (or similar) when entering 3×10^7. This is *WRONG*.

◆ If you key 10 EE 7 (or similar) into a calculator, what number have you in fact entered?

Why do you think this is a common mistake?

The above *incorrect* attempt treats the EE key (or similar) as something that raises the first number (10 in this example) to the power of the second (7 in this example). *But that is not what the EE key does*. Remember that entering 10^7 involves keying in 1 EE 7 (or similar); you can think of the EE key as meaning ‘times ten to the power’.

You have now illustrated another important point about using a calculator:
Be careful to use the EE key correctly when entering powers of ten.

Finally, look at the process of entering negative powers of ten.

✦ Try entering 3×10^{-7} into your calculator first by keying 3 EE 7 +/– or 3 EE +/– 7 (or similar), then clear and try 3 EE – 7.

The +/– key changes the sign of a number, whereas the – (minus) key subtracts the next number you enter. You have illustrated a third important point about using calculators:
Always use the +/– key when entering negative powers of ten.

Question T6

Expressed in scientific notation, what value does your calculator provide for the following?

(a) $1.40 \times 10^{4} \times 5.50 \times 10^{13} \times 6.20 \times 10^{-15}$

(b) $2 \times 10^{6} \times 3 \times 10^{5}/(8 \times 10^{7})$. ❑

2.3 The modulus or absolute value of a number

As defined earlier, a number written in scientific notation is the product of a decimal and an appropriate power of ten, where the decimal may be positive or negative and has a single, non-zero, digit before the decimal point. In other words, if the decimal is positive it must be between 1 and 10 (but not actually equal to 10), and if the decimal is negative it must be between −1 and −10 (but not actually equal to −10). It is both useful and instructive to find a neater way of expressing this condition. This can partly be achieved by introducing the **modulus** or **absolute value** of a number. Given a number r, its *modulus* is denoted $|r|$ and represents the numerical value of r without its sign, e.g. $|3| = 3$ and $|-3| = 3$. More precisely:

> If p is a positive number then $|p| = p$ and $|-p| = p$.

☞

Question T7

Evaluate the following expressions: (a) $|2.3|$, (b) $|-3.1|$,

(c) $|(4-7)/2|$, (d) $|(4-5)(7-9)|$, (e) $|2|/|4|$, (f) $|-6||3|$. ❏

Using the modulus concept we can say that in scientific notation any number is represented as the product of a decimal r and an appropriate power of ten, where $|r|$ is between 1 and 10 (but not actually equal to 10).

This definition can be further refined by making use of an **inequality** ☞. Inequalities are mathematical statements that compare numerical values using the symbols > (read as 'is greater than'), ≥ (read as 'is greater than or equal to'), < (read as 'is less than') and ≤ (read as 'is less than or equal to'). Equipped with these symbols we can say:

> In scientific notation any number can be written as the product of a number r such that $1 \le |r| < 10$, and an appropriate power of ten.

✦ How would you express '$1 \le |r| < 10$' in words?

2.4 Significant figures and decimal places

Note This subsection is concerned with the meaning of numbers. Uncertainty (or 'error') arising from measurement is covered in the *FLAP* modules devoted to measurement and the handling of experimental data.

When a number is used to describe a physical quantity, each of its digits should carry some meaning, in other words, each should be a **significant figure**.For example, stating that a certain length l is 6.00 mm should mean that l *really is* 6.00 mm, rather than 6.01 mm or 5.99 mm. The use of three significant figures in this case implies that the length has been determined to the *nearest* 0.01 mm and is therefore known to be somewhere between 5.995 000 00 ... mm and 6.004 999 99 ... mm ☞. It would be quite wrong, for example, having determined a length as 6.00 mm to the nearest 0.01 mm to write that length as 6.000 mm. The presence of the last zero would imply that the length was in the range 5.999 500 00 ... mm to 6.000 499 99 ... mm, which would not be justified. Thus, the number of significant figures in a value should indicate the *precision* ☞ with which that value is known.

Even if a value is given to the correct level of precision it is not necessarily the case that all of its figures are significant. For example, without changing the information about the length *l*, you could write it with as many zeros as you wished at the front, for example, 06.00 mm or 0 006.00 mm. Such zeros carry no information and are not counted as significant figures. You could also express the same information using different units, metres for example, giving $l = 0.006\,00$ m. In this case the zeros at the start of the number *do* carry important information since they put the rest of the number in its rightful place relative to the decimal point, but they do no more than this. They do not indicate any increase in precision, so they are still not counted as significant figures. Generally then, zeros at the start of a number, even if important, are not counted as significant figures.

Zeros at the end of a number present a different problem. Suppose you went for a walk and measured its distance as 3.4 km to the nearest 0.1 km. If you were asked the length of the walk in metres you might be tempted to say 3 400 m. But that would not be correct; the two zeros on the right appear to be significant and falsely imply that you have measured the length of the walk to the nearest metre. Nonetheless, those zeros are playing an important role; how else could you express the measured length in metres? Fortunately, this question has a simple answer — use *scientific notation*. Just write the length as 3.4×10^3 m, this conveys the essential information without introducing misleading zeros. Avoiding spurious ‘significant figures’ is yet another advantage of using scientific notation. Despite this, data involving whole numbers (such as 3 400 m) are often written with unjustified zeros at the end, so you should treat such figures with care. If someone tells you the world’s population is 6 000 000 000, ask them how they can be so precise.

The *significant figures* in a number are the meaningful digits that indicate its precision. They do not include any zeros to the left of the first non-zero digit. Using scientific notation avoids the need to write down any zeros that are not significant, either to the left or to the right of the significant figures, and thus avoids any ambiguity in writing or interpreting a number.

Decimal numbers are sometimes described in terms of their number of **decimal places** — that is, the number of digits after the decimal point. For example, the number 765.43 is specified to two decimal places. The number of decimal places may also be used to indicate the precision of a value. For example, you might say that the length l = 6.00 mm has been determined 'to two decimal places' but it's important to take the units into account. The same length, determined to the same level of precision, but expressed in metres would have to be given to five decimal places (i.e. 0.006 00 m).

Question T8

Describe each of the following values in terms of its number of significant figures and number of decimal places:

(a) 65.43, (b) 0.003 56, (c) 2 278, (d) 3.04×10^{-5}. ❏

The number of significant figures or decimal places to which a physical quantity is known is generally limited by the precision of a measurement. This applies not only to quantities that are measured directly, but also to those derived from measured quantities via calculations. The following discussion illustrates the effect of combining quantities that are known only to a limited number of significant figures.

Suppose you have measured the length l and breadth b of a rectangle as $l = 3.4\,\text{m}$ and $b = 6.2\,\text{m}$ (i.e. to the nearest 0.1 m). What can you say about its area $A = l \times b$? You might be tempted to say $3.4\,\text{m} \times 6.2\,\text{m} = 21.08\,\text{m}^2$, but that really isn't justified since l and b are only measured to the *nearest* 0.1 m, so their true values might be as much as 0.05 m larger or smaller.

✦ Calculate the area A of the rectangle (a) if $l = 3.350\,0\,\text{m}$ and $b = 6.150\,0\,\text{m}$, and (b) if $l = 3.450\,0\,\text{m}$ and $b = 6.250\,0\,\text{m}$.

In view of these answers, it would be sensible to say that a rectangle of length 3.4 m and breadth 6.2 m has an area of $21\,\text{m}^2$, since that implies its area is between $20.500\,0 \ldots\,\text{m}^2$ and $21.499\,9 \ldots\,\text{m}^2$. It doesn't quite cover all the possibilities, but it is a reasonable compromise given the imprecision of the original measurements. Notice that this (sensible) answer has two significant figures, as did each of the values used to calculate it.

In *FLAP*, most calculations will be carried out using three significant figures. The answers will then also be quoted to three significant figures, though the last figure may not be entirely reliable.

If you use a calculator, you will often find that as many as eight digits are displayed. Do not be fooled into thinking that they are all significant figures. Unless you started with extremely precise initial values, most of the displayed figures will be completely useless. You should look back at the initial values that you used, find the one with the *fewest* significant figures, and then adjust your answer so that it has the same number of significant figures. For example, if your calculator displays a result as 8.362 114 5, but you know that only three figures could be significant (due to your initial input), you should write the answer as 8.36. If, only two figures were significant, you should write 8.4, since 8.362 is closer to 8.4 than it is to 8.3. Generally, if the first 'insignificant' digit is 0, 1, 2, 3 or 4 it can just be dropped, but if it is 5, 6, 7, 8 or 9 then 1 must be added to the last significant digit. This process is called **rounding**. If the last significant digit is left unchanged the number is said to have been **rounded down**, and if the last significant digit is altered the number is said to have been **rounded up**. The errors that can arise in a final answer as result of rounding are called **rounding errors**. Such errors will generally grow as the number of steps in a calculation increases. ☞

In calculations where there are several steps it is generally wise to delay the process of *rounding* until the final step. This will minimize the chance of rounding errors introduced in one step amplifying rounding errors introduced in some other step. However, it is important to realize that allowing your calculator to carry all the given digits in the data through the interim steps of a calculation does not make those figures significant.

Of course, some quantities are known exactly or with very great precision. For example, the speed of light travelling through a vacuum is now *defined* to be exactly $2.997\,924\,58 \times 10^8\,\mathrm{m\,s^{-1}}$, and the mathematical constant π (= 3.142 to three decimal places) has been determined to an enormous number of decimal places. Similarly, the pure numbers that enter into formulae, such as the 2 in the formula $C = 2\pi r$ for the circumference C of a circle of radius r, are usually intended to represent exact values. Such exact or very precise values will not generally limit the precision of a calculation and should not, therefore, be allowed to limit the number of significant figures in a result.

- When multiplying or dividing two numbers, the result should not be quoted to more significant figures than the least precisely determined number. (So, $0.4 \times 1.21 = 0.5$.)
- When adding or subtracting two numbers, the last significant figure in the result should be the last significant figure that appears in *both* the numbers when they are expressed in decimal form without powers of ten. (So, $0.4 + 1.21 = 1.6$ and $0.004 + 1.21 = 1.21$.)
- When calculations are performed with a consistent number of significant figures throughout, *the final result may still not be reliable to that number of significant figures.*

Question T9

The dimensions of a thin metal sheet have been measured as 0.23 m, 6.26 m and 2.88×10^{-3} m. Someone has used these values to calculate the volume of the sheet and has obtained $4.146\,62 \times 10^{-3}\,\text{m}^3$.

(a) How many figures in this answer are significant? What is a sensible value for the volume?

(b) The mass has been measured as 18.17 kg.

If the mass is divided by the volume to calculate the density, how many figures in that answer will be significant? ❏

2.5 Approximations and orders of magnitude

When performing calculations it is often useful to work with **approximations**. For example, if you know that a certain length lies somewhere between 3.1 m and 3.3 m you may wish to approximate it to 3.2 m even though the final '2' is not significant. By making such an approximation you might well obtain a better idea of the true outcome of the calculation than you would by saying that the length was 3 m and reducing the number of significant figures in the final answer. On the other hand the figures in the your final answer will not all be significant and could mislead if taken too seriously. To avoid this danger the symbol ≈ (read as 'is approximately equal to') can be used to indicate that an approximation has been introduced.

Even when quantities are known to a certain level of precision it is sometimes useful to approximate them by 'simpler' values just to get a rough idea of the answer. For example,

$$\frac{3.2 \times 5.9}{4.5} \approx \frac{3.0 \times 6.0}{4.5}$$

Sometimes it is only possible to obtain a very crude approximation to the value of a quantity. The use of such crude approximations is generally indicated by the symbol ~ which can usually be read as 'is very roughly equal to'. Sometimes a more precise meaning is attached to this symbol when it is used to indicate the value of a quantity to the nearest **order of magnitude** — that is, to the nearest power of ten. For example, $2.34 \times 10^7 \sim 10^7$ and $8.765 \times 10^7 \sim 10^8$. Notice that if the first digit of the decimal number is 5 or greater, then the original power of ten must be rounded up to give the order of magnitude.

Orders of magnitude give a quick way of making a rough comparison between two quantities. For example, the mass of the Earth is 5.98×10^{24} kg, and that of the Moon is 7.35×10^{22} kg, but for a quick comparison it is sufficient to say that the Earth's mass $\sim 10^{25}$ kg and that of the Moon $\sim 10^{23}$ kg, and that the mass of the Earth is two orders of magnitude greater than that of the Moon.

Estimating the order of magnitude of a calculated quantity often provides a useful way of checking that any value found with a calculator is reasonable. It won't guarantee the absence of a mistake, but it will help you to avoid silly mistakes.

✦ Find the approximate value and the order of magnitude of the number y, where $y = 2\pi \times 3.21 \times 10^5 \times 864/(2.04 \times 10^2 \times 77)$.

Question T10

Write down the order of magnitude of the following:
(a) mass of the Sun = 1.99×10^{30} kg;
(b) radius of the Sun = 6.96×10^{8} m;
(c) mass of a proton = $1.672\,6 \times 10^{-27}$ kg;
(d) mass of an electron = 9.11×10^{-31} kg. ❏

Question T11

Without using a calculator, find the order of magnitude of z, where $z = 7.3 \times 10^{3} \times 869.3/(2.3 \times 10^{4} \times 4.07 \times 10^{5})$. ❏

2.6 Sets and types of number

This subsection introduces some mathematical terminology that is useful when describing different types of number. It is worth understanding these differences since the various types of number are interpreted in different ways.

Natural numbers and integers

For the purpose of counting, the only numbers we need are positive, whole, numbers 1, 2, 3 ... and so on. These ‘counting’ numbers are called **natural numbers**. Taken together they form a **set**—a collection of entities defined by some common characteristic. The fact that the natural numbers form a set is sometimes indicated by writing them in braces (curly brackets), though it is more usual to denote the whole set by the single symbol $\mathbb{N}$.

Thus, $\mathbb{N} = \{1, 2, 3, ...\}$

Each number belonging to the set $\mathbb{N}$ is said to be an **element** of $\mathbb{N}$. The fact that a number such as 57 is an element of $\mathbb{N}$ is shown by writing $57 \in \mathbb{N}$ where the symbol $\in$ is read as ‘is an element of’.

Another important set includes negative whole numbers and zero as well as the counting numbers. This is the set of **integers** and is symbolized by $\mathbb{Z}$.

Thus, $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, 3 \dots\}$

The whole of the set $\mathbb{N}$ is included within the set $\mathbb{Z}$, a relationship that is recognized by saying that $\mathbb{N}$ is a **subset** of $\mathbb{Z}$ or by writing $\mathbb{N} \subset \mathbb{Z}$. Since integers are whole numbers it would be quite wrong to regard a given integer, such as 2, as meaning anything other than *exactly* 2. It is in this sense that the 2s in expressions such as $2\pi r$ or $mv^2/2$ are to be interpreted. Thus, *the presence of an integer in a calculation does not affect the number of figures that are significant in the final result (because it really has an infinite number of zeros after the decimal point)*.

Real numbers, rational numbers and irrational numbers

All the numbers used in this module belong to the set of **real numbers**. This set is denoted by the symbol $\mathbb{R}$, and each of its elements can be represented as a positive or negative decimal number ☞. Thus all 'ordinary' numbers such as 15.5, -0.009, 187 and 4.3×10^{-8} are real numbers (even the last of these *could* have been written as a decimal). The set of integers and the set of natural numbers are both subsets of the set of real numbers, and the numerical values of all measurable physical quantities *can* be expressed in terms of real numbers.

You should note that when dealing with pure numbers in a mathematical context, they should normally be interpreted as exact values. This should be contrasted with the customary practice when dealing with the values of physical quantities, where a number generally indicates a range of values. On the whole, you will have to determine how a real number is supposed to be interpreted from the context in which it is used.

The set of real numbers contains two other important subsets; the *rational numbers* and the *irrational numbers*. The set of **rational numbers** includes every real number that can be expressed as a ratio in which one integer is divided by another. The set of **irrational numbers** contains every real number that cannot be expressed as a fraction. Every real number is either rational or irrational.

Real numbers such as 0.5 (= 1/2), –531.25 (= –2 125/4), 53 (= 53/1) and 0.666 ... (= 2/3) are all rational because each can be expressed as a ratio of integers. But there are many other real numbers that are not rational. Several real numbers which are commonly used in physics are irrational, including many square roots. For example, it is easy to show that $\sqrt{2}$ cannot be expressed as a ratio of integers—there are *no* two integers with a ratio that is exactly equal to $\sqrt{2}$, i.e. there are no two integers m and n such that $\sqrt{2} = n/m$. The mathematical constant π (= 3.142 to three decimal places) is also irrational, as is another mathematical constant e (= 2.718 to three decimal places). ☞

One characteristic of irrational numbers is that they extend indefinitely beyond the decimal point, without any repeating patterns of digits. Calculators generally use values of π, e and other irrationals (e.g. $\sqrt{2}$) that extend to at least eight figures, which for most purposes is sufficient to ensure that the presence of such numbers does not adversely affect the precision of any calculation.

Question T12

Which of the following statements are true?

(a) $\mathbb{Z} \subset \mathbb{R}$, (b) $\sqrt{8}$ is irrational, (c) $-4 \in \mathbb{N}$, (d) $2.00 \in \mathbb{Z}$. ❏

Question T13

The equation $T = 2\pi\sqrt{l/g}$ gives the time T that a pendulum of length l takes to complete a full swing (i.e. an oscillation) when the magnitude of the acceleration due to gravity is g. (a) Identify one integer and one irrational number used in the equation. (b) Explain why the number of significant figures in the answer will depend only on the values given for l and g, and not on the other numbers in the equation. ❏

3 Units and physical quantities

3.1 Symbols and units

Units of measurement play an essential role in the specification of most physical quantities. The *Système International d'Unités* (SI), used throughout *FLAP*, is discussed in detail in the modules of the physics strand. However, the seven basic SI units ☞, their standard abbreviations, the terminology of standard multiples, and a number of derived units are summarized in Tables 2, 3 and 4 for easy reference.

Table 2 The seven basic SI units.

Physical quantity	Unit	Symbol for SI unit
length	metre	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	kelvin	K
luminous intensity	candela	cd
amount of substance	mole	mol

Table 3 Standard SI multiples.

Multiple	Prefix	Symbol for prefix
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{0}		
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f

Table 4 A number of derived SI units that are given special names.

Physical quantity	Unit	Symbol for derived SI unit	Definition
energy	joule	J	$\mathrm{kg\,m^2\,s^{-2}}$
force	newton	N	$\mathrm{kg\,m\,s^{-2} = J\,m^{-1}}$
power	watt	W	$\mathrm{kg\,m^2\,s^{-3} = J\,s^{-1}}$
electric charge	coulomb	C	$\mathrm{A\,s}$
electric potential difference	volt	V	$\mathrm{kg\,m^2\,s^{-3}\,A^{-1} = J\,A^{-1}\,s^{-1}}$
electric resistance	ohm	Ω	$\mathrm{kg\,m^2\,s^{-3}\,A^{-2} = V\,A^{-1}}$
electric capacitance	farad	F	$\mathrm{A^2\,s^4\,kg^{-1}\,m^{-2} = A\,s\,V^{-1}}$
magnetic flux	weber	Wb	$\mathrm{kg\,m^2\,s^{-2}\,A^{-1} = V\,s}$
inductance	henry	H	$\mathrm{kg\,m^2\,s^{-2}\,A^{-2} = V\,s\,A^{-1}}$
magnetic field	tesla	T	$\mathrm{kg\,s^{-2}\,A^{-1} = Wb\,m^{-2}}$
frequency	hertz	Hz	$\mathrm{s^{-1}}$
pressure	pascal	Pa	$\mathrm{kg\,m^{-1}\,s^{-2} = N\,m^{-2} = J\,m^{-3}}$

Physical quantities are often represented by algebraic symbols such as E for energy, l for length and so on. Such symbols should generally be presumed to represent the *whole* quantity — that is, the units as well as the numerical value. Thus, it would be correct to speak of ‘a rod of length l’, but it would be incorrect to say ‘a rod of length l m’ since, if l was 2.2 m the latter statement would mean ‘a rod of length 2.2 m m’ which would be misleading and wrong.

Units should generally be treated like algebraic quantities. Thus, if a runner travels a distance $d = 1\,000\,\mathrm{m}$ in a time $t = 200\,\mathrm{s}$ the runner’s average speed υ can be written

$$\upsilon = \frac{d}{t} = \frac{1000\,\mathrm{m}}{200\,\mathrm{s}} = 5.00\,\mathrm{m\,s^{-1}}$$

Notice that in accordance with convention the result of dividing m (for metre) by s (for second) has been written $\mathrm{m\,s^{-1}}$ rather than m/s or m per s. It is generally expected that combinations of units will be displayed in this way, using appropriate powers, as the right-hand column of Table 4 indicates.

Although oblique strokes (/) should be avoided when combining units, they may be used to rewrite a statement such as $\lambda = 567\,\text{nm}$ in the form $\lambda/\text{nm} = 567$ ☞. Notice that the unit is again being treated as an algebraic symbol, and has been used to divide both sides of the original statement. This way of writing a quantity is used mainly in column headings in tables, so that the units do not have to be written next to each entry (thus making the table less cluttered), and in labelling the axes on graphs where it ensures that the values plotted are, technically speaking, pure numbers (which pleases mathematical purists).

❖ Why would it be inappropriate to write λ (nm) in a table column heading or on an axis of a graph?

3.2 Units in equations

Since symbols represent *entire* quantities, you should *always include units when substituting numerical values into equations*. ☞ For example, when using the equation $T = 2\pi\sqrt{l/g}$ ☞ to find the period of oscillation of a pendulum, where $l = 1.2\,\text{m}$ and $g = 9.8\,\text{m}\,\text{s}^{-2}$, it would be wrong to write $T = 2\pi\sqrt{1.2/9.8}$, you should write

$$T = 2\pi\sqrt{\frac{1.2\,\text{m}}{9.8\,\text{m}\,\text{s}^{-2}}} = 2\pi\sqrt{\frac{1.2}{9.8}} \times \sqrt{\frac{\text{m}}{\text{m}\,\text{s}^{-2}}} = 2\pi\sqrt{\frac{1.2}{9.8}}\ \text{s}$$

This is obviously cumbersome and if you are confident about the use of units you might well cut out the intermediate steps, but you still shouldn't omit the units entirely, as though they have magically gone away while you deal with the numbers.

You should look upon units as a help rather than a hindrance. Their consistent inclusion is particularly advantageous in complicated and unfamiliar situations but even in straightforward cases they may help you to avoid elementary errors. For example, suppose you wrongly thought the equation for T was $T = 2\pi\sqrt{g/l}$. By including units in your calculation you would soon see that this equation would give a value for T of $2\pi\sqrt{(9.8\,\text{m}\,\text{s}^{-2})\,/\,(1.2\,\text{m})}$, so its units would be s^{-1}.

Such an outcome should warn you that something has gone wrong. (Notice, though, that the units give no clue as to whether the numerical factor 2π is correct since it is a pure number and has no units.)

Another common error that can be avoided by treating units carefully is that of forgetting to convert a quantity given in units other than base units (a distance in centimetres or kilometres say, or a mass in grams) into the equivalent number of base units for the purposes of calculation.

Including the units in a calculation also allows the units of an unfamiliar quantity to be deduced. For example, Newton's law of gravitation $F_{\rm grav} = Gm_1m_2/r^2$ which relates the magnitude $F_{\rm grav}$ of the gravitational force between two objects to their masses m_1 and m_2 and their separation r, can be used to deduce the units of G, Newton's gravitational constant. We can rearrange the equation to give $G = F_{\rm grav}r^2/(m_1m_2)$. In SI units, $F_{\rm grav}$ could be expressed in newtons, r in metres, and m_1 and m_2 in kilograms. The units of G would then be $\rm N\,m^2/(kg \times kg)$, or, more properly, $\rm N\,m^2\,kg^{-2}$.

Question T14

The equation $F_{\rm el} = q_1q_2/(4\pi\varepsilon_0 r^2)$ relates the magnitude $F_{\rm el}$ of the electrostatic force between two particles of charges q_1 and q_2 to their separation r, where ε_0 is the permittivity of a vacuum.

Use the equation to deduce the SI units of ε_0 when q_1 and q_2 are expressed in coulombs (C), $F_{\rm el}$ in newtons (N) and r in metres (m). ❏

3.3 Dimensional analysis ☞

The discussion of units in Subsection 3.2 was based on the correct presumption that the quantities on both sides of an equation *could* be expressed in terms of the same units. Of course, the units do not *have* to be identical, though they must be equivalent. For example, it is quite true that 2.000 m = 2 000 mm even though the units differ, but it is obviously wrong to say 2.000 m = 2 000 kg. The plausibility of the first equation and the stupidity of the second can both be demonstrated by a process known as **dimensional analysis**.

In the context of dimensional analysis the term **dimension** relates to basic measurable quantities such as mass, length, time, temperature and electric current. These quantities differ fundamentally in that the units used to measure any of them cannot be entirely expressed in terms of the units used to measure the others. This basic incompatibility is recognized by saying that they have *different dimensions*. On the other hand, the fact that a quantity such as area can be measured in the same units as a product of two lengths is expressed by saying that area has the *same dimensions* as length^2.

When carrying out dimensional analysis it is useful to introduce single letters such as M, L and T to represent the dimensions of mass, length and time, respectively, and to enclose a quantity in square brackets when referring to its dimensional nature alone ☞. Thus, we may write

$$[\text{mass}] = \text{M} \quad [\text{length}] = \text{L} \quad [\text{time}] = \text{T}$$

It follows that $[\text{area}] = [\text{length}^2] = \text{L}^2$

Similarly, $[\text{acceleration}] = [\text{length/time}^2] = \text{L T}^{-2}$

and $[\text{force}] = [\text{mass} \times \text{acceleration}] = [\text{mass} \times \text{length/time}^2] = \text{M L T}^{-2}$ ☞

Any physical quantity may be dimensionally analysed in this way, though those which involve thermal quantities such as temperature, or electrical quantities such as electric currents will require the introduction of other basic dimensions in addition to M, L and T.

Question T15

What are the dimensions of the following quantities: (a) density, (b) Newton’s gravitational constant, *G*, (c) speed? ❏

One of the main uses of dimensional analysis is in exposing dimensional inconsistencies in equations. Clearly, if $A = B$ then the dimensions of A and B must be the same ☞, i.e. $[A] = [B]$, it follows that:

> An equation must be false if the quantities being equated, added or subtracted do not have the same dimensions.

For example, the claim that the volume of the outer third of a sphere of radius R is given by $V = \pi R^2/9$ is obviously bogus because it is dimensionally inconsistent; $[V] = \mathrm{L}^3$ while $[\pi R^2/9] = [R^2] = \mathrm{L}^2$.

Note that numerical factors such as $\pi/9$ are **dimensionless**, that is to say they have no dimensions and can be ignored in dimensional analysis (more formally we can write $[\pi/9] = 1$). The same is true of ratios such as length/breadth in which a quantity with certain dimensions is divided by another quantity with the same dimensions. Indeed, such ratios are called **dimensionless ratios**.

Dimensional analysis can also be used to help determine the form that a relationship should take. For example, suppose you have discovered that the period T of a pendulum depends on the length l of the pendulum and the magnitude of the acceleration due to gravity g. You might well arrive at the conclusion that the relationship between these quantities is of the general form

$$T = k \times l^p \times g^q$$

where k is a dimensionless constant and the powers p and q are unknown. If so dimensional analysis implies

$$[T] = [k \times l^p \times g^q] = [k] \times [l^p] \times [g^q]$$

so, $\mathrm{T} = 1 \times \mathrm{L}^p(\mathrm{LT}^{-2})^q = \mathrm{L}^p \times \mathrm{L}^q\,\mathrm{T}^{-2q}$ ☞

i.e. $\mathrm{T} = \mathrm{L}^{p+q} \times \mathrm{T}^{-2q}$

This equation can only hold true if $p + q = 0$ and $-2q = 1$, ☞ and this in turn requires that $q = -1/2$ and $p = 1/2$. Thus, on dimensional grounds

$$T = k \times l^{1/2} \times g^{-1/2} = k\sqrt{l/g}$$

This, of course, is the correct form of the relationship (you met it earlier), though the fact that the dimensionless constant k is actually 2π cannot be determined from dimensional considerations.

Question T16

If m is a mass, v a speed, h a height and F the magnitude of a force, determine which of the following expressions for the energy E of a system are implausible on dimensional grounds:

(a) mv^2, (b) $mv^2/2$, (c) $mv/4$, (d) $mh/2$, (e) $2Fh$, (f) $Fh + mv^2$,

(g) $mv^2/2 + F^2h/(mv^2)$. ❑

4 Closing items

4.1 Module summary

1 In *scientific notation* a quantity is written as the product of a positive or negative *decimal number* (with one non-zero digit before the decimal point) and an appropriate *power of ten*, followed by the relevant physical units.

2 The *modulus* (or *absolute value*) of a number r, denoted by the symbol $|r|$, is its numerical value without its sign. Thus, if p is a positive number, $|p| = p$ and $|-p| = p$.

3 A number is given to n *significant figures* when it contains n meaningful digits (excluding zeros at the start of the number).

4 In calculations involving numbers known only to n significant figures, only n digits of the answer can be significant and the answer should be *rounded* accordingly. Even so, *rounding errors* may mean that the final answer is not reliable to n significant figures.

5 The fact that x is an *approximation* to y is indicated by writing $x \approx y$. Cruder approximations, particularly *order of magnitude* estimates (to the nearest power of ten), can be indicated by using ~ in place of ≈. Thus $700 \sim 1\,000$ and $0.22 \sim 10^{-1}$.

6 Any number that can be expressed as a decimal is a *real number.* Every real number r is an *element* of the *set of real numbers* $\mathbb{R}$, as is indicated by writing $r \in \mathbb{R}$.

7 Important *subsets* (indicated by the symbol $\subset$) of the set of real numbers include the set of *natural numbers* $\mathbb{N} = \{1, 2, 3, ...\}$, the set of *integers* $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, 3, ...\}$, the set of *rational numbers* (each of which can be written as the ratio of two integers) and the set of *irrational numbers* (each of which cannot be written as a ratio of integers). The numbers π, e, and $\sqrt{2}$ are all irrational.

8 When an algebraic symbol is used to represent a physical quantity it represents both the numerical value *and* the units of that quantity. In calculations units may be treated as though they are algebraic quantities in their own right. (SI units and conventions are summarized in Tables 2, 3 and 4.)

9 *Dimensional analysis* assigns appropriate combinations of basic *dimensions* (mass, length, time, etc.) to each physical quantity and uses such assignments to investigate the plausibility of relationships between those quantities. Such analysis reveals nothing about the values of purely numerical factors or *dimensionless ratios*, but an equation must be false if the quantities being equated do not have the same dimensions or if it suggests adding or subtracting quantities with different dimensions.

4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Use and interpret scientific notation, and the mathematical symbols for modulus, approximation and order of magnitude.

A3 Perform calculations that involve quantities specified to different levels of precision and present appropriately rounded answers that do not contain spurious 'significant' figures.

A4 Carry out elementary approximations and order of magnitude calculations without using a calculator.

A5 Recognize, interpret and use elementary set notation.

A6 Manipulate SI units as part of a calculation involving physical quantities.

A7 Assign dimensions to appropriately defined physical quantities and use such assignments to investigate the plausibility of proposed relationships between such quantities.

Study comment You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions each of which tests one or more of the *Achievements*.

Question E1

(*A2* and *A3*) (a) Express the numbers 390 463 and 0.000 705 8 using scientific notation, (b) round your answers to three significant figures and then (c) give your answers to the nearest order of magnitude.

Question E2

(*A3*) Evaluate the expression $3\,678 \times 2.45 \times 10^{12}/(9.43 \times 10^{-3})$ and express the answer in scientific notation with three significant figures.

Question E3

(*A2*, *A4* and *A6*) Without using a calculator, find the approximate value and order of magnitude of the strength $F_{\rm grav} = Gm_1m_2/r^2$ of the gravitational force on a person of mass $m_1 = 74\,\rm kg$ due to another person of mass $m_2 = 56\,\rm kg$ when the two are separated by a distance $r = 0.6\,\rm m$.

(Note that $G = 6.67 \times 10^{-11}\,\rm N\,m^2\,kg^{-2}$ to three significant figures.)

Question E4

(*A6*) A wire of length l carrying a current I at right-angles to a magnetic field experiences a force of magnitude F where $F = BIl$ and B is a measure of the strength of the magnetic field. Use the fact that, in SI units, I may be measured in amperes (A), l in metres (m) and F in newtons (N) to deduce an appropriate SI unit for B.

Question E5

(*A2*, *A3*, *A6* and *A7*) The equation $T = 2\pi\sqrt{r/g}$ gives the orbital period T of a satellite in circular orbit radius r where g is the (local) magnitude of the acceleration due to gravity. (a) Calculate the period of a satellite in an orbit with $r = 6.6 \times 10^3$ km and $g = 9.8\,\mathrm{m\,s^{-2}}$ and express you answer with an appropriate number of significant figures.

(b) Use dimensional analysis to show that $T = 2\pi\sqrt{r/g}$, is a plausible relationship between T, r and g.

Question E6

(*A2* and *A5*) If a and b are both integers, which of the following are always true?

(a) $|a - b| = |a| - |b|$, (b) $|ab| = |a||b|$, (c) $a \in \mathbb{N}$, (d) $\{a, b\} \subset \mathbb{Z}$, (e) $\mathbb{Z} \in \mathbb{R}$,

(f) $a/b \in \mathbb{R}$.

Study comment This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.